Lecture 3: The Schwartz space

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Fourier transform and convolution

Theorem

Suppose that $f\,,\,g \in L^1(\mathbb{R}^n)$. Then

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)\,.$$

Proof. By Tonnelli, $e^{-ix\cdot\xi} f(x-y)\,g(y) \in L^1(\mathbb{R}^{2n}, d(x,y))$.

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int e^{-ix\cdot\xi} \int f(x-y)\,g(y)\,dy\,dx \\
&= \int \left(\int e^{-ix\cdot\xi}\, f(x-y)\,dx \right) g(y)\,dy \\
&= \int \left(\int e^{-iz\cdot\xi}\, f(z)\,dz \right) e^{-iy\cdot\xi}\, g(y)\,dy \;=\; \hat{f}(\xi)\,\hat{g}(\xi)
\end{aligned}$$

where $z = x + y$.

The Schwartz space of functions $\mathcal{S}(\mathbb{R}^n)$

Definition

A function $f : \mathbb{R}^n \to \mathbb{C}$ belongs to $\mathcal{S}$ if $f \in C^\infty(\mathbb{R}^n)$, and for all multi-indices α and integers N there is $C_{N,\alpha}$ such that

$$\left|\partial_x^\alpha f(x)\right| \,\le\, C_{N,\alpha}\left(1+|x|\right)^{-N}.$$

- Say that f and all of its derivatives are *rapidly decreasing*.
- Equivalent condition: for all multindices $\alpha, \beta, \;\exists\, C_{\alpha,\beta} < \infty$:

$$\left|x^\alpha \partial_x^\beta f(x)\right| \,\le\, C_{\alpha,\beta}\,.$$

- There is no single norm that characterizes $\mathcal{S}$; instead the "size" of f is characterized by the countable collection of numbers $C_{N,\alpha}$ or $C_{\alpha,\beta}$.

Topology on $\mathcal{S}$

Introduce equivalent countable families of seminorms on $\mathcal{S}$:

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left| x^\alpha\, \partial_x^\beta f(x) \right|, \quad \|f\|_{N,\beta} = \sup_{x\in\mathbb{R}^n} \left| (1+|x|)^N\, \partial_x^\beta f(x) \right|$$

Say that a sequence $f_n \to f$ in $\mathcal{S}$ if $\|f_n - f\|_{\alpha,\beta} \to 0$ for all α, β.

Say that $T : \mathcal{S} \to \mathcal{S}$ is continuous if $Tf_n \to Tf$ whenever $f_n \to f$.

Introduce a metric $d(f,g)$ on $\mathcal{S}$:

$$d(f,g) = \sum_{\alpha,\beta} 2^{-|\alpha|-|\beta|} \frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}$$

$d(f_n, f) \to 0$ if and only if $\|f_n - f\|_{\alpha,\beta} \to 0$ for all α, β.

Theorem

The space $\mathcal{S}$ is complete in the metric $d(f,g)$.

Proof. Suppose $\{f_n\} \subset \mathcal{S}$ is Cauchy: $\lim_{m,n\to\infty} d(f_n, f_m) = 0$.
For each fixed α, β, $\{x^\alpha \partial_x^\beta f_n\}_{n=1}^\infty$ is Cauchy in uniform norm, so

$$\lim_{n\to\infty} x^\alpha \partial_x^\beta f_n = f_{\alpha,\beta} \;\text{ uniformly, some }\; f_{\alpha,\beta} \in C_0(\mathbb{R}^n)\,.$$

Lemma

If $\{g_n\} \subset C^1(\mathbb{R}^n)$ converges uniformly to g, and $\partial_j g_n$ converges uniformly to $g_{(j)}$, then $g \in C^1(\mathbb{R}^n)$, and $\partial_j g = g_{(j)}$.

- Conclude by induction: $f := f_{0,0} \in C^\infty(\mathbb{R}^n)$, and $\partial_x^\beta f_n \to \partial_x^\beta f$ uniformly for every β.
- Easily follows that $x^\alpha \partial_x^\beta f_n \to x^\alpha \partial_x^\beta f$ uniformly, so $f \in \mathcal{S}$, and $d(f_n, f) \to 0$.

Lemma

$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}$ is dense in the metric topology.

Proof. Let $\Phi(x) \in C_c^\infty$ satisfy $\Phi(x) = \begin{cases} 1\,, & |x| \le 1 \\ 0\,, & |x| \ge 2 \end{cases}$

Claim: $\|f - \Phi(R^{-1}\cdot)f\|_{\alpha,\beta} \to 0$ as $R \to \infty$ each $\alpha, \beta,$ if $f \in \mathcal{S}$.

- $(1+|x|)^N\big(1-\Phi(R^{-1}x)\big)|f(x)| \;\le\; R^{-1}\big(1+|x|\big)^{N+1}|f(x)|$

  so: $\|f - \Phi(R^{-1}\cdot)f\|_{N,0} \le R^{-1}\|f\|_{N+1,0}$

- $\partial_x^\beta\Big[\Phi(R^{-1}\cdot)f\Big] = \Phi(R^{-1}\cdot)\partial_x^\beta f + R^{-1}\Phi'(R^{-1}\cdot)\partial_x^{\beta-1} f + R^{-2}\cdots$

  so: $\|f - \Phi(R^{-1}\cdot)f\|_{N,\beta} \le C\,R^{-1}\sum_{\beta'\le\beta}\|f\|_{N+1,\beta'}$

Lemma

The space $\mathcal{S}$ maps continuously into $L^p(\mathbb{R}^n)$ for each $p \in [1,\infty]$, and the image is dense in the L^p norm if $p \in [1,\infty)$.

Proof. Density follows from density of $C_c^\infty(\mathbb{R}^n)$.

For inclusion (if $p < \infty$, the case $p = \infty$ is trivial):

$$\left(\int |f(x)|^p\,dx\right)^{\frac{1}{p}} = \left(\int (1+|x|)^{-p(n+1)} \cdot (1+|x|)^{p(n+1)}|f(x)|^p\,dx\right)^{\frac{1}{p}}$$

$$\leq \|f\|_{n+1,0}\left(\int (1+|x|)^{-p(n+1)}\,dx\right)^{\frac{1}{p}}$$

$$\leq C_n\,\|f\|_{n+1,0}$$