Math 136, Spring 2015, Homework 5

For practice do problems 1-6 in Section 1 and problems 1-12 in Section 2 of Chapter 4.

To hand in

1. Problem 9 in Section 2.

2. Consider the linear map $L_A : \mathbf{R}^n \to \mathbf{R}^n$ given by multiplication by the real $n \times n$ matrix A.

   Suppose that $\lambda = \alpha + i\beta \in$, $\beta \neq 0$, is a complex eigenvalue of L_A, with complex eigenvector $Z = X_1 + iX_2$, where X_1 and X_2 are column vectors in $\mathbf{R}^n$ (not both zero). Thus,

   $$A \cdot Z = (A \cdot X_1) + i(A \cdot X_2) = (\alpha X_1 - \beta X_2) + i(\alpha X_2 + \beta X_1)\,.$$

   (a) Prove that the vectors X_1 and X_2 are linearly independent and, therefore, span the 2-dimensional subspace $W = \operatorname{span}(X_1, X_2) \subset \mathbf{R}^n$.

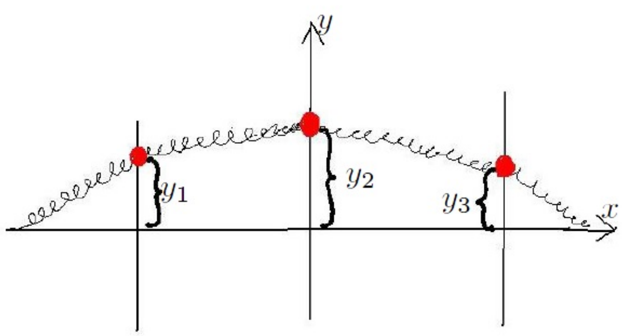
   (b) Show that L_A restricts to define a linear map $L_W : W \to W$ and that the matrix of L_W with respect to the basis $\{X_1, X_2\}$ of W is

   $$|\lambda| \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

   where $\lambda = |\lambda| e^{i\theta}$ (the polar form of λ).

   (c) What is the geometrical interpretation of the result of part (b)?

3. Three objects of mass m connected by springs are free to move up and down as shown in the figure below.



   Let y_k, $k = 1, 2, 3$ denote the height of the k-th object above the x-axis. Let F_k denote the vertical component of the net force on the k-th object. One can show for y_k all small,

   $$F_1 = -2Ky_1 + Ky_2\,, \quad F_2 = Ky_1 - 2Ky_2 + Ky_3\,, \quad F_3 = Ky_2 - 2Ky_3\,.$$

   Let $\omega = \sqrt{K/m}$.

   (a) Show that Newton's equations of motion can be written in the matrix form

   $$\frac{d^2Y}{dt^2} + A \cdot Y = O$$

   where

   $$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ and } A = \omega^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

   (b) Find the general solution by diagonalizing the appropriate matrix.

   (c) Interpret the eigenvectors you find.

   (d) Suppose that at time $t = 0$, $y_1 = 1$, $y_2 = 0$, $y_3 = -1$, and the velocity of every mass is zero. Find $Y(t)$ for all t.