Math 136, Spring 2015, Midterm 2 Review

Terminology: Determinant, trace, eigenvalue, eigenvector, characteristic polynomial, eigenspace, algebraic and geometric multiplicity, inner product, norm, orthogonal and orthonormal vector, orthogonal complement of a subspace, adjoint, transpose, isometry, modulus of an operator, fundamental subspaces of an operator.

Classifying square matrices and operators (linear transformations from one space to itself)**:** Invertible, unitary, orthogonal, unitary equivalent to a triangular matrix, diagonalizable, unitary equivalent to a diagonal matrix, normal, self adjoint, hermitian, positive definite, positive semi-definite. How are these related. For example, every self adjoint operator is necessarily normal. Are there normal operators that are not self adjoint?

Computations and Applications: Determinants, inverses, Gram-Schmidt orthogonalization, solving systems of linear differential equations, identifying quadratic curves on the plane, orthogonal projection onto a subspace, approximating solutions to systems of linear equations.

Review Problems

1. Let V be the vector space of continuous functions on the interval $[0, \pi]$, that vanish at 0 and π; and let $(\,,\,)$ be the scalar product defined by

$$(f, g) = \int_0^\pi f(x)g(x)\,dx\,.$$

   Let $g_k(x) = \sin(kx)$, for $k = 1, 2, 3, \ldots$, and let $W_n \subset V$ be the subspace generated by the set $\{g_k : k = 1, 2, \ldots, n\}$.

   (a) Show that $\{g_k : k = 1, 2, \ldots\}$ is an orthogonal set.

   (b) Let $f(x) = x(\pi - x)$. Let f_n denote the orthogonal projection of f onto W_n. Show that

$$f_{2n+1}(x) = \frac{8}{\pi}\sum_{k=0}^{n}\frac{\sin((2k+1)x)}{(2k+1)^3}\,.$$

2. Let A be an $n \times n$ skew-symmetric matrix ($A = -A^T$).

   (a) Show that if n is odd then A must be singular. ($\det A = 0$).

   (b) Show by example that A need not be singular if n is even.

3. Let A be a 3×3 symmetric matrix. You are told that $\lambda_1 = 3$, $\lambda_2 = -3$, and $\lambda_3 = 0$ are eigenvalues of A and that

$$B_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad B_2 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad B_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix},$$

   are the corresponding (unit length) eigenvectors. Find A.

4. Let $A = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix}$ and $L_A : \mathbf{R}^5 \to \mathbf{R}^2$ where $L_A(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbf{R}^5$. Find orthogonal bases for the four fundamental subspaces of L_A.

5. Recall that (by definition) an inner product on $\mathbf{R}^2$ is a map of the form

$$(\,,\,) : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$$

   that satisfies the following properties for all $\vec{x}, \vec{y}, \vec{z} \in \mathbf{R}^2$ and $a \in \mathbf{R}$:

   (i) $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$
   (ii) $(a\vec{x}, \vec{y}) = a(\vec{x}, \vec{y})$
   (iii)$(\vec{x} + \vec{y}, \vec{z}) = (\vec{x}, \vec{z}) + (\vec{y}, \vec{z})$
   (iv) $(\vec{x}, \vec{x}) \geq 0$ and if $(\vec{x}, \vec{x}) = 0$ then $\vec{x} = \vec{0}$.

Consider the map defined by

$$(\vec{x}, \vec{y}) = \vec{x}^T \cdot \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix} \cdot \vec{y}\,.$$

It is not difficult to see properties (i),(ii), and (iii) are satisfied no matter what value b has. Show that (iv) is satisfied if and only if $|b| < 1$.

6. Let $A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Find a matrix B such that $B^2 = A$. Do not simply give the answer, show your work!

7. Which of the matrices below are diagonalizable? In each case, give a one or two sentence explanation for your answer.

$$(a)\ \begin{pmatrix} 3 & 1 & 4 & 5 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 4 & 5 \\ 1 & 1 & 5 & 7 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

8. What is the shape of the set of all solutions to the equation

$$4x^2 - 2xy + 4y^2 = 1?$$

9. Find an orthonormal basis for the space of solutions of the equation $x - y + z = 0$.

10. Solve the initial value problem

$$\frac{dx_1}{dt} + 2x_1 - x_2 = 0 \qquad\qquad x_1(0) = 6$$
$$\frac{dx_2}{dt} - x_1 + 2x_2 = 0 \qquad\qquad x_2(0) = 7$$

as follows:

(i) Write the problem in vector form:

$$\frac{d}{dt}X + A \cdot X = O \qquad\qquad X(0) = \begin{pmatrix} 6 \\ 7 \end{pmatrix}.$$

(ii) Find an orthonormal basis of eigenvectors of the matrix A.

(iii) Find two independent solutions $X = X_1(t)$ and $X = X_2(t)$ of $dX/dt + A \cdot X = O$ with

$$X_1(t) = f_1(t)B_1 \text{ and } X_2(t) = f_2(t)B_2\,,$$

where $\{B_1, B_2\}$ is an orthonormal basis of $\mathbf{R}^2$, and f_1 and f_2 are real valued functions.

(iv) Using (ii), write the general solution in the form

$$X(t) = B \cdot \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where the columns of B are B_1 and B_2.

(v) Show that $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B^t \cdot \begin{pmatrix} 6 \\ 7 \end{pmatrix}$.

(vi) Put this all together to write the solution in the form

$$X(t) = B \cdot \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \cdot B^t \cdot X_0$$

11. Solve the initial value problem

$$\begin{aligned} x_1' &= x_2 + x_3, \qquad & x_1(0) &= 3 \\ x_2' &= x_1 + x_3, \qquad & x_2(0) &= 0 \\ x_3' &= x_1 + x_2, \qquad & x_3(0) &= 0 \end{aligned}$$

by finding an orthonormal basis of eigenvectors of the appropriate matrix.

Hint: $t^3 - 3t - 2 = (t-2)(t+1)^2$

12. Let $A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ be a non-zero 3×3 skew symmetric matrix.

(i) Show that the characteristic polynomial of A is of the form

$$p_A(t) = t^3 + (a^2 + b^2 + c^2)t = t(t - \lambda_1)(t - \lambda_2) = t(t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2)\,,$$

and from this conclude that the spectrum of A is of the form $\{0, \lambda_1, \lambda_2\}$, with

$$\lambda_1 + \lambda_2 = 0 \text{ and } \lambda_1\lambda_2 = a^2 + b^2 + c^2\,,.$$

Conclude that $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, with $\omega = \sqrt{a^2 + b^2 + c^2}$

(ii) Let $B_1 = \frac{1}{\omega}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Show that $A \cdot B_1 = O$.

(iii) Let $B_2 + iB_3$ satisfy

$$A \cdot (B_2 + iB_3) = i\omega(B_2 + iB_3) \implies A \cdot B_2 = -\omega\, B_3 \text{ and } A \cdot B_3 = \omega B_2$$

Prove that $\|B_2|| = \|B_3\|$ and $\langle B_1, B_2\rangle = \langle B_1, B_3\rangle = \langle B_3, B_3\rangle = 0$. Hence, (after rescaling B_2 and B_3 if necessary) $\{B_1, B_2, B_3\}$ is an orthogonal basis for $\mathbf{R}^3$ with respect to the standard scalar product.

Hint: *Show that $X^t \cdot A \cdot X = 0$ for any vector; then show that for $i \neq j$, the scalar product $\langle B_i, B_j\rangle$ can be expressed as a multiple of X^tAX for $X = B_i$ or $X = B_j$.*

(iv) Use these facts to prove that

$$R_t = \exp(tA) = B \cdot \exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega t \\ 0 & -\omega t & 0 \end{pmatrix} \cdot B^t = B \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega t) & \sin(\omega t) \\ 0 & -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \cdot B^t\,.$$

In particular, R_t is a clockwise rotation by ωt radians about the vector B_1:

$$R_t \cdot B_1 = B_1, \quad R_t(B_2) = \cos(\omega t)B_2 - \sin(\omega t)B_3\; R_t(B_3) = \sin(\omega t)B_2 + \cos(\omega t)B_3\,.$$