Math 136, Spring 2015, Midterm 1 Review

Terminology: Vector space, subspace, basis, spanning set, linear independence, image (range) , kernel (null space), row space, rank, transpose, inverse.
Main operations: Multiplying matrices and reducing them to echelon form.
Applications: Checking if a set of vectors are independent, checking if a set of vectors span $\mathbf{R}^n$, inverting matrices, computing the column space, row space and the kernel of a map/matrix, finding solutions of linear systems, completing an independent set of vectors to a basis.

Also, you should know how to write the matrix representation of a linear transformation with respect to a basis, check if a given transformation is linear, check if a subset of a vector space is a subspace.

Review Problems

1. Here are some Math 308 exams you can use for practicing computations:
   (a) http://www.math.washington.edu/∼tobiasj/exams/308KAu10Mid.pdf
   Answers at http://www.math.washington.edu/∼tobiasj/exams/308KAu10MidSol.pdf
   (b) http://www.math.washington.edu/∼clenagh/308win15/308MWi13Mid.pdf
   Answers at http://www.math.washington.edu/∼clenagh/308win15/308MWi13MidSol.pdf
   (c) You can also look at the sample problems for the midterm at
   https://www.math.washington.edu/∼morrow/308/308.html.
   The files are in DVI format, it explains at the top of the page. Skip number 4, nonsingular means invertible, symmetric means it equals its transpose.

2. Given a subset D of a vector space V, let $L(D)$ be the set of all vectors of the form $\alpha_1\mathbf{v}_1 + ... + \alpha_n\mathbf{v}_n$ where $\mathbf{v}_1, ..., \mathbf{v}_n$ are in D. In particular, if D generates the vector space V, then $L(D) = V$. For any D, the set $L(D)$ is a vector subspace of V (Proof?)

   Now let $V = \mathbf{R}^3$. Describe $L(D)$ geometrically in these cases:
   (a) $D = \emptyset$. (D is empty)
   (b) $D = \{(0\ 0\ 0)^T\}$.
   (c) $D = \{(1\ 2\ 5)^T\}$.
   (d) $D = \{(1\ 2\ 5)^T, (2\ 0\ 3)^T\}$.
   (e) $D = \{(1\ 2\ 5)^T, (2\ 0\ 3)^T, (4\ 4\ 13)^T\}$.
   (f) $D = \{(1\ 2\ 5)^T, (2\ 0\ 3)^T, (1\ 1\ 3)^T\}$.

3. Find a basis and hence the dimension of the following subspaces of $\mathbf{R}^3$:
   (a) $V_1 = \left\{(x\ y\ z)^T : x + 2y + 3z = 0\right\}$.
   (b) $V_2 = \left\{(x\ y\ z)^T : x + y + z = 0 \text{ and } 2x - y + 3z = 0\right\}$.

4. Suppose A is a 5×7 matrix and that the standard basis vectors $\vec{e_1}$ and $\vec{e_2}$ (in $\mathbf{R}^5$) are in the image of A. What are the possible values for dim(ker A)?

5. Consider the linear map $L_A : \mathbf{R}^5 \to \mathbf{R}^4$ given by the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & -2 & 0 & 2 & -2 \end{pmatrix}$ with respect to the standard bases for both spaces.
   (a) What is the dimension of ranL_A (the image or range of L_A)?
   (b) What is the dimension of kerL_A (the kernel of L_A)?
   (c) Find a basis for kerL_A.
   (d) Find a basis for ranL_A.

6. Let $V = C^\infty(\mathbf{R})$ be the vector space of infinitely differentiable, real-valued functions.

   (a) Show that the set $\{e^x, \sin x, \cos x\}$ is a linearly independent set.

   (b) Let $L : V \to V$ be the map defined by

   $$L : f(x) \mapsto f''(x) + f(x)$$

   Show that L is a linear map.

   (c) Is L surjective? Explain.

   (d) Give a basis for the kernel of L?

7. Let V and W be (possibly infinite dimensional) vector spaces and let $T : V \to W$ be a bijective linear map. Prove that there is a unique linear map $S : W \to V$ such that $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$. Notice there are three things to show: (i) S exists, (ii) S is unique, and (iii) S is linear.

8. Let A be a 10×5 matrix satisfying the condition

   $$A^T \cdot A = I_5 \,,$$

   where I_5 denotes that 5×5 identity matrix. Show that the 10×10 matrix $A \cdot A^T$ has rank 5.

9. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

   (a) Find a basis for the column space of A.

   (b) Find a basis for the row space of A.

10. Let $A_1, A_2 \in \mathbf{R}^5$ be non-zero, column vectors satisfying the condition $A_1^t \cdot A_2 = 0$. Show that the 5×5 matrix

    $$A = A_1 \cdot A_1^t + A_2 \cdot A_2^t$$

    has rank 2.
    Hint: *There is an obvious basis for the column space of A.*

11. Let n be a positive integer and let A be a $10 \times n$ matrix with column vectors $\vec{a}_j$, $j = 1, 2, \ldots, n$ satisfying the identity

    $$\vec{a}_i^T \cdot \vec{a}_j = \delta_{ij}$$

    (a) What is the largest possible value for n? Explain your answer.

    (b) What is the dimension of the row space of A? Explain your answer.

12. A transformation $P : V \to V$ is called a *projection* if $P \circ P = P$.

    (a) Show that the projections (in the geometric sense) you computed in the first homework are projections (in this abstract definition).

    (b) Show that $\mathrm{Im}\, P \cap \ker P = \{\mathbf{0}\}$. The only element common to both is the zero vector.

    (c) Show that any vector $\mathbf{v} \in V$ can be expressed in a unique way as $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in \mathrm{Ker}\, P$ and $\mathbf{w} \in \mathrm{Im}\, P$.

    (d) Show that any transformation $T : V \to V$ can be expressed as a composition $S \circ P$ where S is an isomorphism and P is a projection. Here, you can use part (c) above or take a basis for the kernel and complete it to a basis for V like the problem below.

13. Understanding what a linear map $T : \mathbf{R}^n \to \mathbf{R}^m$does geometrically. This is what I was trying to explain waving my hands in the air on Wednesday and talking about all possible maps $T : \mathbf{R}^2 \to \mathbf{R}^2$. It is made precise here:

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation of rank $0 < r \leq m$. (The rank is not 0 so it is not the map which sends everything to the origin of $\mathbf{R}^m$. It is obvious what that does.) Then, T can be decomposed into a composition of four maps $T = S_m \circ i \circ p \circ S_n$, where

$$S_n : \mathbf{R}^n \to \mathbf{R}^n$$

and

$$S_m : \mathbf{R}^m \to \mathbf{R}^m$$

are isomorphisms,

$$p : \mathbf{R}^n \to \mathbf{R}^r$$

is the projection of $\mathbf{R}^n$ to $\mathbf{R}^r$ given by $p(\mathbf{e}_j) = \mathbf{e}_j$ if $j \leq r$ and $p(\mathbf{e}_j) = 0$ $j > r$ and

$$i : \mathbf{R}^r \to \mathbf{R}^m$$

is "inserting" $\mathbf{R}^r$ into $\mathbf{R}^m$ given by $i(\mathbf{e}_j) = \mathbf{e}_j$.
Note that the standard basis vectors $\mathbf{e}_j$ may be different in different spaces although we use the same notation. For example, $\mathbf{e_1} = (1\,0\,0)$ in $\mathbf{R}^3$ but $\mathbf{e_1} = (1\,0\,0\,0\,0)$ in $\mathbf{R}^5$.

(a) Do some examples of p and i with $r = 1, 2$ and $m, n = 2, 3$ to see what they represent.

(b) Let $\{\mathbf{v}_{r+1}, , \mathbf{v}_n\}$ be a basis of ker T. We can complete it to a basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ of $\mathbf{R}^n$.

(c) Show that the set $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_r)\}$ is a basis for the image of T. Then, we can complete it to a basis for $\mathbf{R}^n$: $\{T(\mathbf{v}_1, ..., T(\mathbf{v}_r), \mathbf{w}_{r+1}, ..., \mathbf{w}_m\}$.

(d) Now, write down the isomorphisms S_n and S_m. We will see later that isomorphisms from $\mathbf{R}^k$ to $\mathbf{R}^k$ are compositions of rotations, reflections and stretches.