Math 126, Sections D and F, Spring 2012, Solutions to Midterm II

1. Answer the following about the vector function

$$\mathbf{r}(t) = \langle 5\cos t, 1 + 3\sin t, 4\sin t\rangle .$$

   (a) (7 points) Compute the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$.

$$\mathbf{r}'(t) = \langle -5\sin t, 3\cos t, 4\cos t\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{25\sin^2 t + 9\cos^2 t + 16\cos^2 t} = 5$$

$$\mathbf{T}(t) = \left\langle -\sin t, \frac{3}{5}\cos t, \frac{4}{5}\cos t\right\rangle$$

$$\mathbf{T}'(t) = \left\langle -\cos t, -\frac{3}{5}\sin t, -\frac{4}{5}\sin t\right\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \mathbf{T}'(t) = \left\langle -\cos t, -\frac{3}{5}\sin t, -\frac{4}{5}\sin t\right\rangle$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \left\langle 0, -\frac{4}{5}, \frac{3}{5}\right\rangle$$

   (b) (2 points) Find the equation of the oscullating plane to this curve at the point when $t = 7$.

$$-\frac{4}{5}(y - (1 + 3\sin 7)) + \frac{3}{5}(z - 4\sin 7) = 0$$

   simplifying to

$$4y + 3z + 4 = 0$$

   (c) (1 point) This is a circle. Tell me in words how you can find its center.
   The normal vectors point towards its center. You can write equations of two lines from two random points and intersect them.
   For any point P on the circle the vector $\vec{PC}$ is parallel to $\mathbf{N}$ where C is the center. You can try to see which point $C = (a, b, c)$ has this property for all points P given by all values of t.
   You can compute the curvature and then use the fact that the radius must be its reciprocal. Then for say $t = 0$, compute the point P_0 and then $\vec{P_0C} = (1/\kappa)\mathbf{N}(0)$ from which you can solve for C.

2. (10 points) Find the minimum and maximum values of the function

$$f(x, y) = 2x^3 - 3x^2 + y^4$$

on the domain $x^2 + y^2 \leq 4$.

First, we find critical points:

$$f_x(x, y) = 6x^2 - 6x = 6x(x - 1) = 0$$

and

$$f_y(x, y) = 4y^3 = 0$$

So the critical points are: $f(0, 0) = 0$ and $f(1, 0) = -1$.
For the boundary, we can use $y^2 = 4 - x^2$, $-2 \leq x \leq 2$. (We can also use $x = 2\cos t$ and $y = 2\sin t$, $0 \leq t \leq 2\pi$ but the presence of y^4 in the equation makes the first choice simpler.)
So, on the boundary,

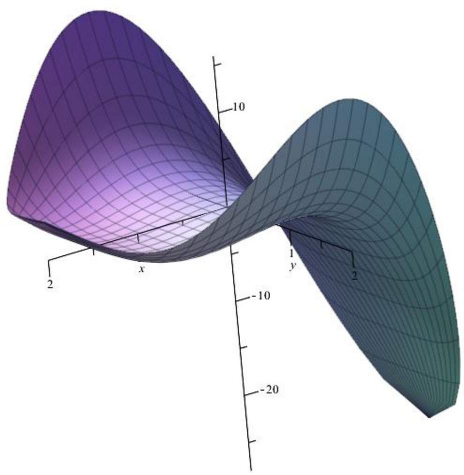
$$g(x) = 2x^3 - 3x^2 + (4 - x^2)^2, = 2 \leq x \leq 2$$

$$g'(x) = 6x^2 - 6x + 2(4 = x^2)(-2x) = 2x(3x - 3 - 8 + 2x^2) = 0$$

From the first factor we get $x = 0$ so $y = \pm 2$. Giving us the values $f(0, 2) = f(0, -2) = 16$. For the second factor, using the quadratic formula $x = \frac{-3\pm\sqrt{97}}{4}$. The one with the minus sign is not between

-2 and 2 so we only get $x = \frac{-3+\sqrt{97}}{4} \approx 1.71$ with function value approximately 2.42.
The endpoints are $x = \pm 2$ with values $f(2,0) = 4$ and $f(-2,0) = -28$. Therefore, the absolute minimum is -28 at $(-2,0)$ and the absolue maximum is 16 at $(0,2)$ and $(0,-2)$.

Here is the graph:



3. Answer the questions about $f(x,y)$ whose contour graph is given below.

   (a) (5 points) Determine whether the following partial derivatives are positive or negative.

   $f_x(P) < 0$ $\quad$ $f_y(P) > 0$ $\quad$ $f_{xx}(P) > 0$ $\quad$ $f_{yy}(P) > 0$ $\quad$ $f_{xy}(P) < 0$

   You can approximate f_x and f_y using the conour lines and the contour values. For the second derivatives, you have to see how the first derivatives change when you move in the x or y directions, depending on the partial derivative your are looking at.

   (b) (3 points) Find and classify critical points you can see in this contour graph. Mark them on the picture and give their approximate coordinates.

   There is a minimum at approximately $(1,1)$, a maximum at approximately $(-1,1)$ and a saddle at appoximately $(0,0)$.

4. (a) (6 points) Changing the order of integration:

$$\int_0^{\pi^2} \int_{\sqrt{x}}^{\pi} \sin\left(\frac{x}{y}\right) dydx = \int_0^{\pi} \int_0^{y^2} \sin\left(\frac{x}{y}\right) dxdy = \int_0^{\pi} -y\cos(y) + y \quad dy$$

$$= -y\sin y\Big|_0^{\pi} + \int_0^{\pi} \sin y \quad dy + \int_0^{\pi} y \quad dy = 2 + \frac{\pi^2}{2}$$

   (b) (6 points) Set up an integral to find the volume of the region bounded by the coordinate planes and the plane $2x + 3y + z = 6$ in the first octant. Evaluate it to find the volume.

$$\int_0^3 \int_0^{\frac{6-2x}{3}} 6 - 2x - 3y \quad dydx = \int_0^3 \frac{(6-2x)^2}{6} dx = 6$$