

Uniform Circular Motion

The motion of an object moving around a circle at a constant speed can be modeled as follows.

We impose a coordinate system with the origin at the center of the circle.

The object's position can be specified by an angle θ measured counterclockwise from the positive x -axis.

Then

$$\theta = \theta_0 \pm \omega t$$

where θ_0 is the angle of the initial position (i.e., the position at time $t = 0$), ω is the angular speed in radians per unit time, and the plus-or-minus is taken to be plus if the motion is counterclockwise, and minus if the motion is clockwise.

In xy -coordinate, the location of the object is then given by

$$(r \cos(\theta_0 \pm \omega t), r \sin(\theta_0 \pm \omega t))$$

where r is the radius of the circle.

In most problems of this sort, you will need the following:

- θ_0 (where does the thing start?)
- ω in radians (what is the angular speed?)
- the direction of motion (which way is it moving?)
- r (what is the radius of the circle?)
- the time t of interest (when are we trying to find the object's location?)

Some of this information will be given directly to you, while other bits you'll have to calculate.

Of particular importance is the equation

$$v = r\omega.$$

This ties v , r and ω together, so that if you know two of them, you can find the other. In a lot of problems v will be given; it is there so that you can find ω .

Let's look at some examples.

Example 1 Bob starts running around from the northernmost point of a circular track. He takes 1.93 minutes to complete each lap of the track. The track has a radius of 200 m. He runs clockwise. Where is Bob relative to his starting point after running for one hour?

When we read the problem, we want to begin to find as many of the items in the bulleted list as possible. In the first sentence, we are told Bob starts at the northernmost point of the track: this tells us $\theta_0 = \pi/2$.

The second sentence tells us Bob's angular speed by telling us he makes one lap in 1.93 minutes. In one lap, he moves through 2π radians. Hence, his angular speed is

$$\omega = \frac{2\pi}{1.93} = 3.25554 \text{ radians per minute.}$$

His direction we are told is clockwise, so we will go with the "-" choice in our θ formula.

Finally, we are interested in his location after one hour, so $t = 60$ minutes.

Thus, after 60 minutes, we know Bob's location is given by

$$\theta = \theta_0 - \omega t = \frac{\pi}{2} - 3.25554(60) = -193.7614.$$

and his coordinates after one hour are

$$(r \cos \theta, r \sin \theta) = (105.1236, 170.1442).$$

Be sure your calculator is in radian mode when calculating trig values from angles in radians.

Bob's starting point was $(0, 200)$, so Bob's location after one hour can be described as 105.1236 meters east and $200 - 170.1442 = 29.8558$ meters south of his starting point.

Example 2 Maria starts at the southernmost point of a circular track and run clockwise at 4 meters per second. The track has a radius of 50 m.

After running for 10 seconds, how far is she from her starting point? After 1000 seconds?

We collect the important information given in the problem:

- $r = 50m$
- starting position: $\theta_0 = -\frac{\pi}{2}$ (or $\frac{3\pi}{2}$)
- direction of motion: clockwise (-)

- angular speed: $\omega = \frac{v}{r} = \frac{4 \text{ m/s}}{50 \text{ m}} = 0.08 \text{ radians per second}$

Thus, Maria's location after t seconds is given by

$$\theta = -\frac{\pi}{2} - 0.08t.$$

After 10 seconds, her location is given by

$$\theta = 10 - \frac{\pi}{2} - 0.08(10) = -2.370796$$

and her coordinates are

$$(r \cos \theta, r \sin \theta) = (50 \cos -2.370796, 50 \sin -2.370796) = (-35.8678, -34.8353)$$

Her starting point was $(0, -r) = (0, -50)$ and with the distance formula we can find the distance between her starting point and her location after 10 seconds to be 38.9418 meters.

For her location after 1000 seconds, all we do is use $t = 1000$:

$$\theta = \theta_0 - \omega t = -\frac{\pi}{2} - 0.08(1000) = -81.570796$$

and her location is $(r \cos \theta, r \sin \theta) = (49.6944, 5.5195)$. The distance from here to her starting point is 74.5113 meters (by the distance formula).

Example 3 *Maria is running around a circular track. She runs clockwise, and, from her starting time, she takes 8 seconds to reach the westernmost point. She takes 100 seconds to complete each lap, and runs at 3 meters per second. Where is she t second after she starts running?*

Since we know she takes 100 second to complete each lap, we have

$$\omega = \frac{2\pi}{100} \text{ radians per second}$$

and so

$$r = \frac{v}{\omega} = \frac{3}{2\pi/100} = \frac{300}{2\pi} = \frac{150}{\pi} \text{ meters.}$$

We impose a coordinate system with the origin at the center of the track. Then the westernmost point is represented by $\theta = \pi$. Since it takes her 8 seconds to get there, and she runs clockwise, her starting point is represented by

$$\theta_0 = \pi + 8\omega = \frac{29}{25}\pi.$$

Thus, her location after running for t seconds is given by

$$\left(\frac{150}{\pi} \cos \left(\frac{29}{25}\pi - \frac{2\pi}{100}t \right), \frac{150}{\pi} \sin \left(\frac{29}{25}\pi - \frac{2\pi}{100}t \right) \right).$$