Math 120 Precalculus

A first course in Problem Solving

Math 120 Precalculus:
A First Course in Problem Solving

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Author Note

For most of you, this course will be unlike any mathematics course you have previously encountered. Why is this?

Learning a new language.

Colleges and universities have been designed to help us discover, share and apply knowledge. As a student, the preparation required to carry out this three part mission varies widely, depending upon the chosen field of study. One fundamental prerequisite is fluency in a "basic language"; this provides a common framework in which to exchange ideas, carefully formulate problems and actively work toward their solutions. In modern science and engineering, college mathematics has become this "basic language", beginning with *precalculus*, moving into *calculus* and progressing into more advanced courses. The difficulty is that college mathematics will involve genuinely new ideas and the mystery of this unknown can be sort of intimidating. However, everyone in this course has the intelligence to succeed!

How does this course compare to high school Precalculus?

There are key differences between the way teaching and learning takes place in high schools and universities. Our goal is much more than just getting you to reproduce what was done in the classroom. Here are some key points to keep in mind:

- The pace of this course will be faster than a high school class in precalculus. Above that, we aim for greater command of the material, especially the ability to extend what we have learned to new situations.
- This course aims to help you build the stamina required to solve challenging and lengthy multi-step problems.
- As a rule of thumb, this course should on average take 15 hours of effort per week. That means that in addition to the 5 classroom hours per week, you would spend 10 hours extra on the class. This is only an average and my experience has shown that 12-15 hours of study per week

(outside class) is a more typical estimate. In other words, for many students, this course is the equivalent of a half-time job!

- Because the course material is developed in a highly cumulative manner, we recommend that your study time be spread out evenly over the week, rather than in huge isolated blocks. An analogy with athletics is useful: If you are preparing to run a marathon, you must train daily; if you want to improve your time, you must continually push your comfort zone.

Prerequisites.

This course assumes prior exposure to the "mathematics" in Chapters 1 and 2. This material is fully developed, in case you need to brush up on a particular topic. If you have never encountered the concept of a function, graphs of functions, linear functions or quadratic functions, this course will probably seem too advanced. You are not assumed to have taken a course which focuses on mathematical problem solving; that is the purpose of this course.

Internet.

There is a great deal of archived information specific to this course that can be accessed via the World Wide Web at the URL address

`http://www.math.washington.edu/~colling/HSMath120/home.html`

Why are we using this text?

Prior to 1990, the performance of a student in precalculus at the University of Washington was not a predictor of success in calculus. For this reason, the mathematics department set out to create a new course with a specific set of goals in mind:

- A review of the essential mathematics needed to succeed in calculus.
- An emphasis on problem solving, the idea being to gain both experience and confidence in working with a particular set of mathematical tools.

This text was created to achieve these goals and the 1999-2000 academic year marks the fifth year in which it has been used.

Notation, etc.

This book is full of worked out examples. We use the the notation "*Solution.*" to indicate where the reasoning for a problem begins; the symbol □ is used to indicate the end of the solution to a problem.

If you run across an exercise which has a "star" next to the number, like "12*", this is a warning that the problem might be more challenging than usual. Sometimes it is starred because of the amount of time necessary and other times because it is substantially more difficult mathematically.

There is a table of contents (TOC) that is useful in helping you find a topic treated earlier in the course. It is also a good rough outline when it comes time to study for the final examination. The book also includes an index at the end.

Finally, there are two appendices at the end of the text. In appendix A, we present answers to selected problems. In appendix B, we collect a variety of formulas used throughout the course (e.g. the circumference of a circle, etc.).

The Preface.

How to succeed in Math 120.

Most people learn mathematics by doing mathematics. That is, you learn it by active participation; it is very unusual for someone to learn the material by simply watching their instructor perform on Monday/Wednesday/Friday. For this reason, the homework is THE heart of the course and more than anything else, study time is the key to success in Math 120. I advise 15 hours of study per week, OUTSIDE class. Also, during the first week, the number of study hours will probably be even higher as you adjust to the viewpoint of the course and brush up on algebra skills.

Here is one model I usually suggest for the course: Prior to a given class, make sure you have looked over the reading assigned. If you can't finish it, at least look it over and get some idea of the topic to be discussed. Having looked over the material ahead of time, you will get FAR MORE out of the lecture. Then, after lecture, you will be ready to launch into the homework. If you follow this model, it will minimize the number of times you leave class in a daze. Finally, when things look bleak, it is always good to keep in mind that thousands of students have been through this course and gone on to take calculus.

Acknowledgments.

The efforts of several people have led to numerous changes, corrections and improvements. We want to specifically thank Patrick Averbeck, Jim Baxter, Cindy Burton, Jerry Folland, Dan Fox, Grant Galbraith, Peter Garfield, Joel Grus, Fred Kuczmarski, Julie Harris, Michael Harrison, Ron Irving, Ian Jannetty, Mark Johnson, Michael Keynes, Don Marshall, Linda Martin, Pat Perkins, Ken Plochinski, Eric Rimbey, Tim Roberts, Aaron Schlafly, David Schneider, Marilyn Stor, Steve Tanner, Paul Tseng, and Rebecca Tyson. I am grateful to everyone for their hard work and dedication to making this a better product for our students. Also, special thanks to Adrienne (aka Adri-N), Allyson, Angela, John, Lee, Marcel, Mary, Nora and Tiff for you know what...

The Minority Science and Engineering Program (MSEP) of the College of Engineering supports the development of this textbook. It is also authoring additional materials, namely, a student study guide and an instructor guide. MSEP actively uses these all of these materials in its summer mathematics program for freshman pre-engineers. We want to thank MSEP for its contributions to this textbook.

We also want to thank Jennifer Mazuca and LeAnne R. Lay for the cover design we are using this year.

Comments.

Send comments, corrections, and ideas to *colling@math.washington.edu* or *kdp@engr.washington.edu*.

Preface

Have you ever noticed this peculiar feature of mathematics: When you don't know what is going on, it is really hard, difficult, and frustrating. But, when you know what is going on, mathematics seems incredibly easy, and you wonder why you had trouble with it in the first place!

Here is another feature of learning mathematics: When you are struggling with a mathematical problem, there are times when the answer seems to *pop* out at you. At first, nothing is there, then very suddenly, in a flash, the answer is all there, and you sit wondering why you didn't "see" the solution sooner. We have a special name for this: It's the "A-Ha!" experience. Often the difficulty you have in studying mathematics is that the *rate* at which you are having an A-Ha! experience might be so low that you get discouraged or, even worse, you give up studying mathematics altogether. One purpose of this course is to introduce you to some strategies that can help you increase the rate of your mathematical A-Ha! experiences.

What is a story problem?

When we ask students if they like story problems, more often than not, we hear statements like: "I hate story problems!" So, what is it about these kinds of problems that causes such a negative reaction? Well, the first thing you can say about story problems is that they are mostly made up of *words*. This means you have to make a big effort to *read* and *understand* the words of the problem. If you don't like to read, story problems will be troublesome.

The second thing that stands out with story problems is that they force you to *think* about how things work. You have to give deep thought to how things in the problem *relate* to each other. This in turn means that story problems force you to connect many steps in the solution process. You are no longer given a list of *formulas* to work using memorized steps. So, in the end, the story problem is a multi-step process such that the "A-Ha!" comes only after lots of intense effort.

All of this means you have to spend *time* working on story problems. It is impossible to sit down and spend only a minute or two working each problem. With story problems, you have to spend much more time working toward a solution, and at the university, it is common to spend an hour

or more working each problem. So another aspect of working these kinds of problems is that they demand a lot of *work* from you, the problem solver.

We can conclude this: *What works is work!* Unfortunately, there is no *easy* way to solve all story problems. There *are*, however, techniques that you can use to help you work efficiently. In this course, you will be presented with a wide range of mathematical tools, techniques, and strategies that will prepare you for university level problem solving.

What are the BIG errors?

Before we look at how to make your problem solving more efficient, let's look at some typical situations that make problem solving *inefficient*. If you want to be ready for university level mathematics, we are sure you have heard somewhere: "You must be prepared!" This means you need to have certain *well-developed mathematical skills* before you reach the university. We would like to share with you the three major sources of errors students make when working problems, especially when they are working exam problems. Every time we sit down and review solutions with a student who has just taken an exam, and who has lost a lot of points in that exam, we find errors falling pretty much into three categories, and these errors are the major cause of inefficient mathematical problem solving.

The first type of error that loses points is *algebra*. This is an error of not knowing all of the algebraic rules. This type of error also includes mistakes in the selection and use of mathematical symbols. Often, during the problem solving process, you are required to introduce mathematical *symbols*. But, without these symbols, you cannot make any further progress. Think of it this way: *Without symbols, you cannot do any mathematics involving equations!*

The second error we see in problem solving has to do with *visualization*. In this case, we're talking about more than the graphics you can get from a calculator. Graphing and curve sketching are very important skills. But, in doing story problems, you might find it almost impossible to create a solution without first drawing a *picture*[1] of your problem. Thus, by not drawing a good picture of the problem, students get stuck in their exams, often missing the solution to a problem entirely.

Finally, the third big source of error is not *knowing mathematical definitions*. Actually, this is a huge topic, so we will only touch on some of the main features of this kind of error. The key thing here is that by not knowing mathematical definitions, it becomes very hard to know what to *do* next in a multi-step solution to a story problem.

Here is what it all boils down to: *Mathematical definitions, for the most part, provide little cookbook procedures for computing or measuring something.* For example, if you did not know the mathematical definition of "speed," you would not know that to measure speed, you first measure your *distance* and you simultaneously measure the *time* it takes to cover that distance. Notice this means you have *two* measuring instruments working at the same time. The second thing you must do, according to the definition of speed, is *divide* the distance you measured by the time you measured. The *result* of your division is a number that you will call *speed*. The definition is a step-by-step procedure that everyone agrees to when talking about "speed." So, it's easy to understand that if you are trying to solve a story problem requiring a speed computation and you did not know the definition or you could not remember the definition of speed, you are going to be "stuck" and no further progress will be possible!

[1] Whenever we talk about a *picture* of your problem, we mean not just the drawing itself. In this case, the picture must include the drawing and the *labels* which clearly signify the quantities related to your problem.

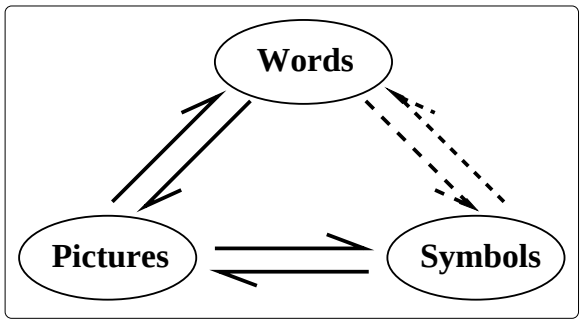
What does all of this mean for you? As you study your mathematics, make sure you are the best you can be in these three areas: Algebra, Visualization, and Definitions. Do a little algebra every day. Always draw a picture to go with all your problems. And, know your mathematical definitions without hesitation. Do this and you will see a very large portion of your math errors disappear!

Problem Solving Strategies.

This topic would require another book to fully develop. So, for now, we would like to present some problem solving ideas you can start using right away.

Let's look now at a common scenario: A student reads a story problem then exclaims, maybe with a little frustration: "If I only had the formula, I could solve this problem!" Does this sound familiar? What is going on here, and why is this student frustrated? Suppose you are this student. What are you actually trying to *do?* Let's break it down. First, you are reading some descriptive information in ***words*** and you need to *translate* this word information into ***symbols***. If you had the symbolic information, you would be in a position to mathematically solve your problem right away.

Unfortunately, you cannot solve anything without first translating your words into symbols. And, *going directly from words to symbols is usually very difficult!* So, here we are looking for some alternative approach for translating words into symbols. The picture below is the answer to this problem solving dilemma.



A lot is going on in this figure. Let's consider some of the main features of this diagram. First, it is suggesting that you are dealing with information in three different *forms:* Words, Pictures, and Symbols. The arrows in this diagram suggest that in *any* problem solving situation, you are actually translating information from one form to another. The arrows also suggest that there are alternative *paths* you can take to get from one form to another! This is a very, very important point: the idea that there is more than one way to get from words to symbols.

Let's rewind this discussion: You're reading a story problem. But, now, before giving any thought to what your formula is, that is, before worrying about your symbolic information, *you grab a blank sheet of paper and start drawing a* ***picture*** *of your problem.* And, to your picture you add symbols denoting the quantities you need in your problem. At this point in your problem solving, you are not trying to write any equations; you are only trying to *see* what your problem *looks* like. You are also concentrating on another extremely important step: Deciding what symbols to use in your problem!

Now you have a good picture of your problem. It shows not only what the problem looks like, but symbolically shows all the problem's variables and constants. You can start using this information to mathematically model your problem. The process of creating a mathematical model is actually

nothing more than the arrow in the diagram going from ***pictures*** to ***symbols.*** Mathematical modeling is the jump you make from the visual information you have created to information contained in your formulas.

Let's summarize the problem solving process. You start with a description of a problem that is presented to you mainly in the form of words. Instead of trying to jump directly from words to symbols, you jump from words to pictures. Once you have a good picture, you jump from pictures to symbols. And, all the time, you are relying on mathematical definitions as you interpret the words of your problem; on visualization techniques as you draw pictures related to your problem; and, on your algebra skills as you are formulating the equations you need to solve your problem.

There is one final thing to notice about the diagram in this section. All of this discussion so far deals with the situation where your direction is from

$$\text{Words} \Longrightarrow \text{Pictures} \Longrightarrow \text{Symbols}.$$

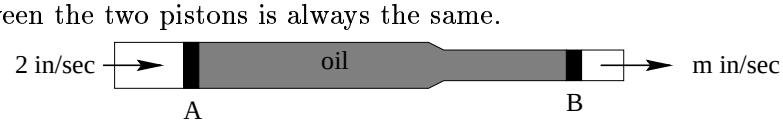
But when you study the diagram you see that the arrows go *both* ways! So, we will leave you with this to think about: What does it mean, within the context of problem solving, when you have

$$\text{Symbols} \Longrightarrow \text{Pictures} \Longrightarrow \text{Words}\,?$$

An Example.

Here is a worked example that is taken from a typical homework assignment for Section 1.1 of this book. See if you can recognize the multitude of steps needed to arrive at the equations that allow us to compute a solution. That is, try to identify the specific way in which information is being transformed during the problem solving process.

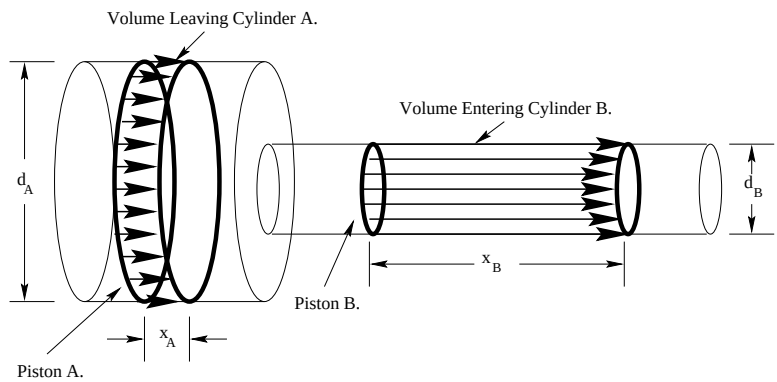
This problem illustrates the principle used to make a good "squirt gun". A cylindrical tube has diameter 1 inch, then reduces to diameter d. The tube is filled with oil and piston A moves to the right 2 in/sec, as indicated. This will cause piston B to move to the right m in/sec. Assume the oil does not compress; that means the volume of the oil between the two pistons is always the same.



1. If the diameter of the narrow part of the tube is $\frac{1}{2}$ inch, what is the speed of piston B?
2. If B moves 11 in/sec, what is the diameter of the narrow part of the tube?

Solution.

The first thing to do with any story problem is to draw a picture of the problem. In this case, you might re-sketch the picture so that it looks 3-dimensional. As you draw, add in *mathematical symbols* signifying quantities in the problem.



The next thing is to clearly define the variables in your problem:

1. Let V_A and V_B stand for the change in volumes as piston A moves to the right.
2. Let d_A and d_B represent the diameters of each cylinder.
3. Let r_A and r_B represent the radii of each cylinder.
4. Let s_A and s_B stand for the speeds of each piston.
5. Let x_A and x_B stand for the distance traveled by each piston.

Now that you have some symbols to work with, you can write the given data down this way:

1. $s_A = 2\,\frac{\text{inches}}{\text{second}}$.
2. $d_A = 1.0\,\text{inch}$.

After you have studied this problem for a while, you would write down some useful relationships:

1. The volume of any cylinder is

$$V = \pi r^2 h$$

where r is the radius of the cylinder, and h is its height or length. From this, you can derive the volume of a cylinder in terms of its diameter, d:

$$V = \frac{\pi}{4} d^2 h.$$

2. "Distance" = "Rate" × "Time". In terms of this problem, you would write

$$x = st,$$

where x is the distance your piston moves, and s is the speed of the piston's motion.

Now you are in a position to create a mathematical model that describes what is going on:

1. From the two relationships above, you can derive the volume equations for each cylinder so that the diameters and speed of the pistons are included:

$$V_A = \frac{\pi d_A^2 x_A}{4} \Longrightarrow V_A = \frac{\pi d_A^2 (s_A t)}{4}$$

and

$$V_B = \frac{\pi d_B^2 x_B}{4} \Longrightarrow V_B = \frac{\pi d_B^2 (s_B t)}{4}$$

2. Since the oil does not compress, at each instant when piston A is moving, you must have $V_A = V_B$, thus:
$$\frac{\pi d_A^2 (s_A t)}{4} = \frac{\pi d_B^2 (s_B t)}{4}.$$
After canceling π, t, and 4, you end up with a *mathematical model* describing this problem that you can use to answer all sorts of interesting questions:
$$d_A^2 s_A = d_B^2 s_B.$$

Using your model, you can compute the following solutions:

1. Given: $d_B = \frac{1}{2}\,\text{in}$, $d_A = 1.0\,\text{in}$, and $s_A = 2\,\frac{\text{in}}{\text{sec}}$, find s_B. From your model, you derive:
$$d_A^2 s_A = d_B^2 s_B \Longrightarrow s_B = \frac{d_A^2 \cdot s_A}{d_B^2}$$
from which you can compute
$$m = s_B = 8\,\frac{\text{in}}{\text{sec}}.$$
2. Likewise, you can use your model to compute
$$d_B = \sqrt{\frac{2}{11}}\,\text{in},$$
exactly.

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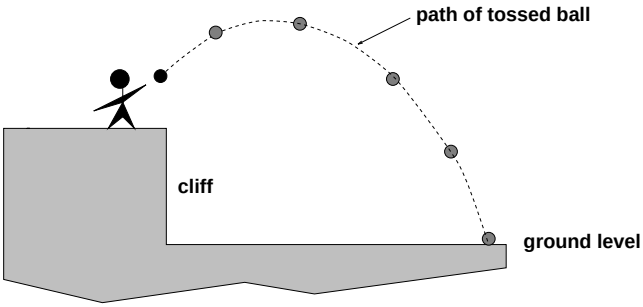
Chapter 1

Warming Up

The basic theme of this book is to study *precalculus* within the context of problem solving. This presents a challenge, since skill in problem solving is as much an art or craft as it is a science. As a consequence, the process of learning involves an active apprenticeship rather than a passive reading of a text. We are going to start out by assembling a basic toolkit of examples and techniques that are essential in everything that follows. The main ideas discussed in the next couple of chapters will surely be familiar; our perspective on their use and importance may be new.

The process of going from equations to pictures involves the key concept of a *graph*, while the reverse process of going from pictures (or raw data) to equations is called *modeling*. Fortunately, the study of graphing and modeling need not take place in a theoretical vacuum. For example, imagine you have tossed a ball from the edge of a cliff. A number of natural questions arise:



> *Where and when does the ball reach it's maximum height? Where and when does the ball hit the ground? Where is the ball located after t seconds?*

We can attack these questions from two directions. If we knew some basic *physics*, then we would have equations for the motion of the ball. Going from these equations to the actual curved path of the ball becomes a *graphing* problem; answering the questions requires that we really understand the relationship between the symbolic equations and the curved path. Alternatively, we could approach these questions without knowing any *physics*. The idea would be to collect some data, keeping track of the height and horizontal location of the ball at various times, then find equations whose graphs will "best" reproduce the collected data points; this would be a *modeling* approach to the problem. Modeling is typically harder than graphing, since it requires good intuition and a lot of experience.

1.1 Units and Rates

A marathon runner passes the one-mile marker of the race with a clocked speed of 18 feet/second. If a marathon is 26.2 miles in length and this speed is maintained for the entire race, what will be the runner's total time?

This simple problem illustrates a key feature of modeling with mathematics: Numbers don't occur in isolation; a number typically comes with some type of *unit* attached. To answer the question, we'll need to recall a formula which precisely relates "total distance traveled" to "speed" and "elapsed time". But, we must be **VERY CAREFUL** to use consistent units. We are given *speed* units which involve distance in "feet" and the length of the race involves distance units of "miles". We need to make a judgment call and decide on a single type of distance unit to use throughout the problem; either choice is OK. Let's use "feet", then here is the fact we need to recall:

$$\boxed{\text{total distance traveled (ft)} = (\text{constant speed (ft/sec)}) \times (\text{elapsed time (sec)}) \qquad (1.1.1)}$$

To apply the formula, let t represent the elapsed time in seconds and first carry out a "conversion of units" using the conversion factor "5280 ft/mile". Recall, we can manipulate the units just like numbers, canceling common units on the top and bottom of a fraction:

$$26.2\,\text{mile} \times (5280\,\text{ft/mile}) \quad = \quad (26.2)(5280)\frac{\cancel{\text{mile}}\cdot\text{ft}}{\cancel{\text{mile}}} = (26.2)(5280)\,\text{ft}.$$

The formula in the box can now be applied:

$$\begin{aligned}(26.2)(5280)\,\text{ft} &= 18\,\text{ft/sec} \times t \\ \left(\frac{(26.2)(5280)}{18}\right)\frac{\cancel{\text{ft}}}{\cancel{\text{ft}}/\text{sec}} &= t \\ 7685.33\,\frac{1}{1/\text{sec}} &= 7685.33\,\text{sec} = t\end{aligned}$$

So, the runner would complete the race in $t = 7685.33$ seconds. If we wanted this answer in more sensible units, we would go through yet another units conversion:

$$t \quad = \quad 7685.33\,\text{sec} \times (1\text{min}/60\text{sec}) \times (1\text{hr}/60\text{min}) = \left(\frac{7685.33}{60^2}\right)\text{hr} = 2.1348\,\text{hr}.$$

The finish clock will display elapsed time in units of "hours : minutes : seconds". Two further conversions (see Exercise 1) lead to our runner having a time of 2:08:05.33; this is a world class time!

Manipulation of units becomes especially important when we are working with the *density* of a substance, which is defined by

$$\text{density} \stackrel{\text{def}}{=} \frac{\text{mass}}{\text{volume}}.$$

For example, pure water has a density of 1 g/cm^3. Notice, given any two of the quantities "density, volume or mass", we can solve for the remaining unknown using the formula. For example, if 857 g of an unknown substance has a volume of 2.1 liters, then the density would be

$$d = \frac{\text{mass}}{\text{volume}} = \frac{857\text{g}}{2.1\text{L}} = \left(\frac{857}{2.1}\right) \times (\text{g/L}) \times (1\text{L}/1000\text{cm}^3) = 0.408\text{g/cm}^3.$$

Here is a more involved example:

Example 1.1.2: *A sphere of solid gold has a mass of 100* kg *and the density of gold is 19.3* g/cm^3. *What is the radius of the sphere?*

Solution. To answer this, let r be the unknown radius of the sphere in units of cm. From the formulas in Appendix C, the volume of the sphere is $V = \frac{4}{3}\pi r^3$ cm^3. If $d = \frac{\text{mass}}{\text{volume}}$ is the density of gold, we now can write down an equation and carry out the required unit manipulations:

$$\begin{aligned} 19.3\,\text{g/cm}^3 &= \left(\frac{100\,\text{kg}}{\frac{4}{3}\pi r^3}\right) \cdot (1000\,\text{g/kg}) \\ r^3 &= \left(\frac{100}{19.3(4/3)\pi}\right) \frac{\text{kg}(1000\,\text{g/kg})}{\text{g/cm}^3} = 1237\,\text{cm}^3 \end{aligned}$$

So, the sphere has radius $r = 10.73$ cm. □

Total Change = Rate × Time.

We live in a world where things are changing as time goes by: the temperature during the day, the cost of tuition, the distance you will travel after leaving this class, and so on. The ability to precisely describe how a quantity is changing becomes especially important when making any kind of experimental measurements. For this reason, let's start with a clear and careful definition. If a quantity is changing with respect to time (like temperature, distance or cost), we can keep track of this using what is called a *rate* (also sometimes called a *rate of change*); this is defined as follows:

$$\text{rate} \stackrel{\text{def}}{=} \frac{\text{change in the quantity}}{\text{change in time}}$$

This sort of thing comes up so frequently, there is some special shorthand notation commonly used: We let the Greek letter Δ (pronounced "delta") be shorthand for the phrase "change in". With this agreement, we can rewrite our rate definition in this way:

$$\text{rate} \stackrel{\text{def}}{=} \frac{\Delta\,\text{quantity}}{\Delta\,\text{time}}$$

But, now the question becomes: How do we calculate a rate? If you think about it, to calculate "Δ quantity" in the rate definition requires that we compare two quantities at two different times and see how they differ (i.e. how they have changed). The two times of comparison are usually called the *final time* and the *initial time*. We really need to be precise about this, so here is what we mean:

$$\begin{aligned}\Delta\,\text{quantity} &= (\text{value of quantity at final time}) - (\text{value of quantity at initial time})\\ \Delta\,\text{time} &= (\text{final time}) - (\text{initial time}).\end{aligned}$$

For example, suppose that on June 4 we measure that the temperature at 8:00 am is 65^oF and at 10:00 am it is 71^oF. So, the final time is 10:00 am, the initial time is 8:00 am and the temperature is changing according to the

$$\begin{aligned}\text{rate} &= \frac{\Delta\,\text{quantity}}{\Delta\,\text{time}} = \frac{\text{final value of quantity} - \text{initial value of quantity}}{\text{final time} - \text{initial time}}\\ &= \frac{71 - 65\,\text{degrees}}{10:00 - 8:00\,\text{hours}} = 3\,\text{deg/hr}.\end{aligned}$$

As a second example, suppose on June 5 the temperature at 8:00 am is 71° and at 10:00 am it is 65°. So, the final time is 10:00 am, the initial time is 8:00 am and the temperature is changing according to the

$$\begin{aligned}\text{rate} &= \frac{\Delta\,\text{quantity}}{\Delta\,\text{time}} = \frac{\text{final value of quantity} - \text{initial value of quantity}}{\text{final time} - \text{initial time}}\\ &= \frac{65 - 71\,\text{degrees}}{10:00 - 8:00\,\text{hours}} = -3\,\text{deg/hr}.\end{aligned}$$

These two examples illustrate a rate can be either a positive or a negative number. More importantly, it highlights that we really need to be careful when making a rate computation. In both examples, the initial and final times are the same and the two temperatures involved are the same, BUT whether they occur at the initial or final time is interchanged. If we accidently mix this up, we will end up being off by a minus sign.

There are many situations where the rate is the same for all time periods. In a case like this, we say we have a *constant rate*. For example, imagine you are driving down the freeway at a constant speed of 60 mi/hr. The fact that the speedometer needle indicates a steady speed of 60 mi/hr means the rate your distance is changing is constant.

In cases when we have a constant rate, we often want to find the *total amount of change* in the quantity over a specific time period. The key principle in the background is this:

$$\boxed{\text{total change} = \text{constant rate} \times \text{time} \qquad (1.1.3)}$$

It is important to mention that this formula only works when we have a constant rate, but that will be the only situation we encounter in this course. One of the main goals of calculus is to develop a version of (1.1.3) that works for non-constant rates. Here is another example; others will occur throughout the text.

Example 1.1.4: *A water pipe mounted to the ceiling has a leak. It is dripping onto the floor below and creates a circular puddle of water. The surface area of this puddle is increasing at a constant rate of 4 cm^2/hour. Find the surface area and dimensions of the puddle after 84 minutes.*

Solution. The quantity changing is "surface area" and we are given a "rate" and "time". Using the formula in (1.1.3), at time $t = 84$ minutes,

$$\text{Total Surface Area} \quad = \quad \text{rate} \times \text{time} = (4\text{cm}^2/\text{hr}) \times (\frac{84}{60}\,\text{hr}) = 5.6\,\text{cm}^2$$

The formula for the area of a circular region of radius r is given on the back cover of this text. Using this, the puddle has radius $r = \sqrt{\frac{5.6}{\pi}} = 1.335$ cm at time $t = 84$ minutes. □

The Modeling Process.

Modeling is a method used in disciplines ranging from architecture to zoology. Modeling techniques will crop up anytime we are problem solving and consciously trying to both "describe" and "predict". Inevitably, mathematics is introduced to add structure to the model, but the clean equations and formulas only arise after some (or typically a lot) of preliminary work.

A model can be thought of as a caricature in that it will pick out certain features (like a nose or a face) and focus on those at the expense of others. It takes a lot of experience to know which models are "good" and "bad", in the sense of isolating the right features. In the beginning, modeling will lead to frustration and confusion, but by the end of this course our comfort level will dramatically increase. Let's look at an illustration of the problem solving process.

How much time do you anticipate studying precalculus each week ?

One possible response is simply to say "a little" or "way too much!". You might not think these answers are the result of modeling, but they are. They are a consequence of modeling the total amount of study time in terms of categories such as "a little", "some", "lots", "way too much", etc. By drawing on your past experiences with math classes and using this crude model you arrived at a preliminary answer to the question.

Let's put a little more effort into the problem and try to come up with a numerical estimate. If T is the number of hours spent on precalculus a given week, it is certainly the case that:

T = (hours in class) + (hours reading text) + (hours doing homework)

Our time in class each week is known to be 5 hours. However, the other two terms require a little more thought. For example, if we can comfortably read and digest a page of text in (on average) 15 minutes and there are r pages of text to read during the week, then

$$\text{(hours reading text)} = \tfrac{15}{60}r \text{ hours.}$$

As for homework, if a typical homework problem takes (on average) 25 minutes and there are h homework problems for the week, then

$$\text{(hours doing homework)} = \tfrac{25}{60}h \text{ hours.}$$

We now have a mathematical model for the weekly time commitment to precalculus:

$$T = 5 + \frac{15}{60}r + \frac{25}{60}h \text{ hours.}$$

Is this a good model? Well, it is certainly more informative than our original crude model in terms of categories like "a little" or "lots". But, the real plus of this model is that it clearly isolates the features being used to make our estimated time commitment and it can be easily modified as the amount of reading or homework changes. So, this is a pretty good model. However, it isn't perfect; some homework problems will take a lot more than 25 minutes!

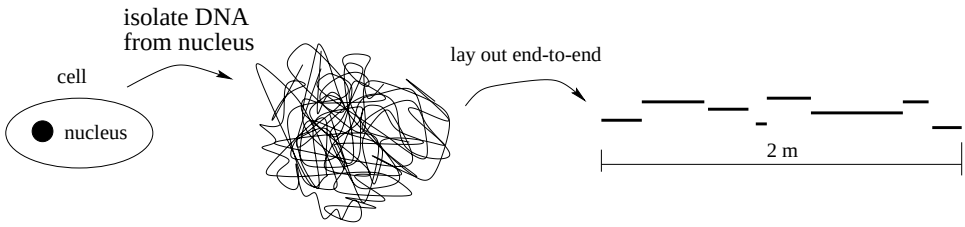
Problems

1. Marathon runners keep track of their speed using units of *pace* = minutes/mile.
   (a) Lee has a speed of 16 ft/sec; what is his pace?
   (b) Allyson has a pace of 6 min/mile; what is her speed?
   (c) Adrienne and Dave are both running a race. Adrienne has a pace of 5.7 min/mile and Dave is running 10.3 mph. Who is running faster?
2. Which is a better deal: A 10 inch diameter pizza for \$8 or a 15 inch diameter pizza for \$16?
3. The famous *theory of relativity* predicts that a lot of weird things will happen when you approach the speed of light $c = 3 \times 10^8$ m/sec. For example, here is a formula that relates the mass m_o (in kg) of an object at rest and it's mass when it is moving at a speed v:

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

   (a) Suppose the object moving is Dave, who weighs $m_o = 66$ kg at rest. What is Dave's mass at 90% of the speed of light? At 99% of the speed of light? At 99.9% of the speed of light?
   (b) How fast should Dave be moving to have a mass of 500 kg?
4. During a typical evening in Seattle, *Pagliacci* receives phone orders for pizza delivery at a constant rate: 18 orders in a typical 4 minute period. How many pies are sold in 4 hours? Assume *Pagliacci* starts taking orders at 5pm and the profit is a constant rate of: \$11 on 10 pies. When will phone order profit exceed \$1000?

5. A typical cell in the human body contains molecules of *deoxyribonucleic acid*, referred to as *DNA* for short. In the cell, this DNA is all twisted together in a tight little packet. But, imagine unwinding (straightening out) all of the DNA from a single typical cell and laying it "end-to-end"; then the sum total length will be approximately 2 meters.



Assume the human body has 10^{14} cells containing DNA. How many times would the sum total length DNA in your body wrap around the equator of the earth?

6. During the 1950's, Seattle was dumping an average of 20 million gallons of sewage into Lake Washington each day.
   (a) How much sewage went into Lake Washington in a week? In a year?
   (b) In order to illustrate the amounts involved, imagine a rectangular prism whose base is the size of a football field (100 yards$\times$ 50 yards) with height h yards. What are the dimensions of such a rectangular prism containing the sewage dumped into Lake Washington in a single day?

   (Note: Dumping into Lake Washington has stopped; now it goes into the Puget Sound.)

7. Dave has inherited an apple orchard on which 60 trees are planted. Under these conditions, each tree yields 12 bushels of apples. According to the local WSU extension agent, each time Dave removes a tree the yield per tree will go up 0.45 bushels. Let x be the number of trees in the orchard and N the yield per tree.
   (a) Find a formula for N in terms of the unknown x. (Hint: Make a table of data with one column representing various values of x and the other column the corresponding values of N. After you complete the first few rows of the table, you need to discover the pattern.)
   (b) What possible reason(s) might explain why the yield goes up when you remove trees?

8. Congress is debating a proposed law to reduce tax rates. If the current tax rate is $r\,\%$, then the proposed rate after x years is given by this formula:

$$\frac{r}{1+\dfrac{1}{1+\dfrac{1}{x}}}.$$

   Rewrite this formula as a simple fraction. Use your formula to calculate the new tax rate after 1, 2, 5 and 20 years. Congress claims that this law would ultimately cut peoples tax rates by 75 %. Do you believe this claim?

1.2 Imposing Coordinates

You find yourself visiting Spangle, WA and dinner time is approaching. A friend has recommended Tiff's Diner, an excellent restaurant; how will you find it?

Of course, the solution to this simple problem amounts to locating a "point" on a two-dimensional map. This idea will be important in many problem solving situations, so we will quickly review the key ideas.

The coordinate system.

If we are careful, we can develop the flow of ideas underlying two-dimensional coordinate systems in such a way that it easily generalizes to three-dimensions. Suppose we start with a blank piece of paper and mark two points; let's label these two points "P" and "Q". This presents the basic problem of finding a foolproof method to reconstruct the picture.

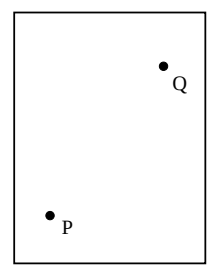
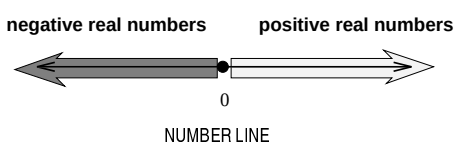
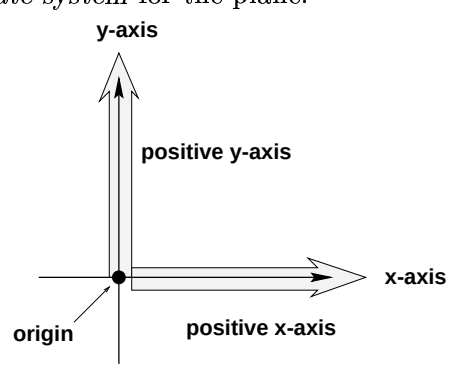


Figure 1.2.1: Two points in a plane

The basic idea is to introduce a *coordinate system* for the plane (analogous to the city map grid of streets), allowing us to catalog points in the plane using pairs of real numbers (analogous to the addresses of locations in the city).

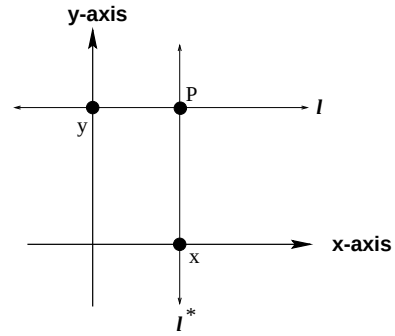
Here are the details. Start by drawing two perpendicular lines, called the *horizontal axis* and the *vertical axis*, each of which looks like a copy of the real number line. We refer to the intersection point of these two lines as the *origin*. Given P in the plane, the plan is to use these two axes to obtain a pair of real numbers (x, y). With this in mind, the horizontal axis is often called the x-axis and the vertical axis is often called the y-axis. Remember, a typical real number line (like the x-axis or the y-axis) is divided into three parts: the positive numbers, the negative numbers and 0 (see below). This allows us to specify positive and negative portions of the x-axis and y-axis. Unless we say otherwise, we will always adopt the convention that the *positive x-axis* consists of those numbers to the right of the origin on the x-axis and the *positive y-axis* consists of those numbers above the origin on the y-axis. We have just described the *xy-coordinate system* for the plane:





Going from P to a pair of real numbers.

Imagine a coordinate system had been drawn on our piece of paper in Figure 1.2.1 above. Let's review the procedure of going from a point P to a pair of real numbers:



1. First, draw two new lines passing through P, one parallel to the x-axis and the other parallel to the y-axis; call these ℓ and ℓ^*, as pictured.
2. Notice that ℓ will cross the y-axis exactly once; the point on the y-axis where these two lines cross will be called "y". Likewise, the line ℓ^* will cross the x-axis exactly once; the point on the x-axis where these two lines cross will be called "x".
3. If you begin with two different points, like P and Q in Figure 1.2.1, you will see that the two <u>pairs</u> of points you obtain will be different; i.e. if Q gives you the pair (x^*, y^*), then either $x \neq x^*$ or $y \neq y^*$. This shows that two different points in the plane give two different pairs of real numbers and describes the process of going from the point P to the pair of real numbers (x, y).

The great thing about the procedure we just described is that it is reversible! In other words, suppose you start with a pair of real numbers, say (x, y). Locate the number x on the x-axis and the number y on the y-axis. Now draw two lines: a line ℓ parallel to the x-axis passing through the number y on the y-axis and a line ℓ^* parallel to the y-axis passing through the number x on the x-axis. The two lines ℓ and ℓ^* will intersect in exactly one point in the plane, call it P. This describes how to go from a given pair of real numbers to a point in the plane. In addition, if you start with two different pairs of real numbers, then the corresponding two points in the plane are going to be different. In the future, we will constantly be going back and forth between points in the plane and pairs of real numbers using these ideas.

Coordinate System 1.2.1: *Every point P in the xy-plane corresponds to a unique pair of real numbers (x, y), where x is a number on the horizontal x-axis and y is a number on the vertical y-axis; for this reason, we commonly use the notation "$P = (x, y)$".*

Having specified positive and negative directions on the horizontal and vertical axes, we can now divide our two dimensional plane into four *quadrants*. The first quadrant corresponds to all the points where both coordinates are positive, the second quadrant consists of points with the first coordinate negative and the second coordinate positive, etc. Every point in the plane will lie in one of these four quadrants or on one of the two axes. This quadrant terminology is useful to give a rough sense of location, just as we use the terminology "Northeast, Northwest, Southwest and Southeast" when discussing locations on a map.

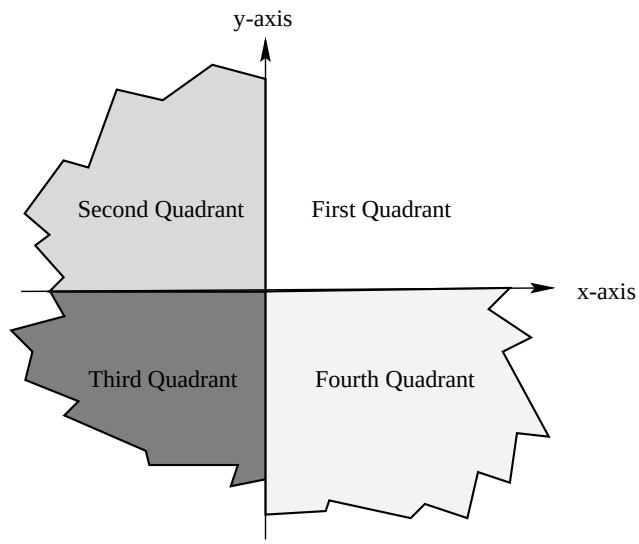


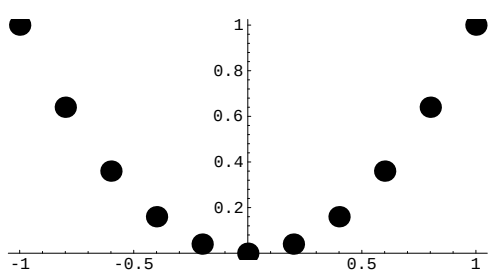
Figure 1.2.2: Quadrants in the xy-plane

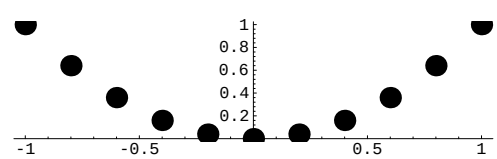
Three Features of a Coordinates System.

A coordinate system involves *scaling*, *labeling* and *units* on each of the axes.

Scaling.

Sketch two xy coordinate systems. In the first, make the scale on each axis the same. In the second, assume "one unit" on the x axis has the same length as "two units" on the y axis. Plot the points (-1,1), (-4/5,16/25), (-3/5,9/25), (-2/5,4/25), (-1/5,1/25), (0,0), (1/5,1/25), (2/5,4/25), (3/5,9/25), (4/5,16/25), (1,1).





Both pictures illustrate how the points lie on a parabola in the xy-coordinate system, but the *aspect ratio* has changed; the aspect ratio is defined by this fraction:

$$\text{aspect ratio} \stackrel{\text{def}}{=} \frac{\text{length of one unit on the vertical axis}}{\text{length of one unit on the horizontal axis}}.$$

In the left-hand picture above, the aspect ratio 1, whereas the right-hand picture has aspect ratio $\frac{1}{2}$. In problem solving, you will often need to make a rough assumption about the relative axis scaling. This will depend entirely on the information given in the problem. Most graphing devices will allow you to specify the aspect ratio.

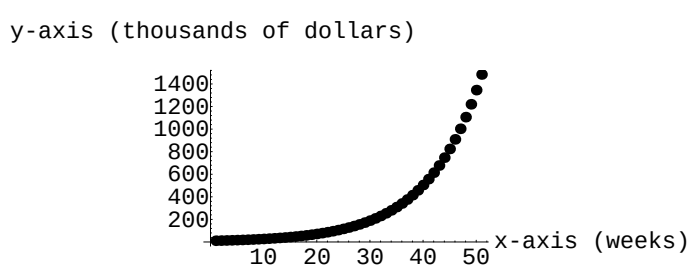
Axes units and labels.

Sometimes we are led to coordinate systems where each of the two axes involve different types of units and labels. Here is a sample, which illustrates the power of using pictures.

> *As the marketing director of Turboweb software, you have been asked to deliver a brief message at the annual stockholders meeting on the performance of your product. Your staff has assembled this tabular collection of data; how can you convey the content of this table most clearly?*

TURBOWEB SALES (in $1000's)									
week	sales	week	sales	week	sales	week	sales	week	sales
1	11.0517	11	30.0417	21	81.6617	31	221.98	41	603.403
2	12.214	12	33.2012	22	90.2501	32	245.325	42	666.863
3	13.4986	13	36.693	23	99.7418	33	271.126	43	736.998
4	14.9182	14	40.552	24	110.232	34	299.641	44	814.509
5	16.4872	15	44.8169	25	121.825	35	331.155	45	900.171
6	18.2212	16	49.5303	26	134.637	36	365.982	46	994.843
7	20.1375	17	54.7395	27	148.797	37	404.473	47	1099.47
8	22.2554	18	60.4965	28	164.446	38	447.012	48	1215.1
9	24.596	19	66.8589	29	181.741	39	494.024	49	1342.9
10	27.1828	20	73.8906	30	200.855	40	545.982	50	1484.131

One idea is to simply flash an overhead slide of this data to the audience; this can be deadly! A better idea is to use a visual aid. Suppose we let the variable x represent the *week* and the variable y represent the *gross sales* (in thousands of dollars) in week x. We can then plot the points (x, y) in the xy-coordinate system.



Notice, the units on the two axes are very different: y-axis units are "thousands of dollars" and x-axis units are "weeks". In addition, the aspect ratio of this coordinate system is not 1. The beauty of this picture is the visual impact it gives your audience. From the coordinate plot we can get a sense of how the sales figures are dramatically increasing. In fact, this plot is good evidence you deserve a big raise!

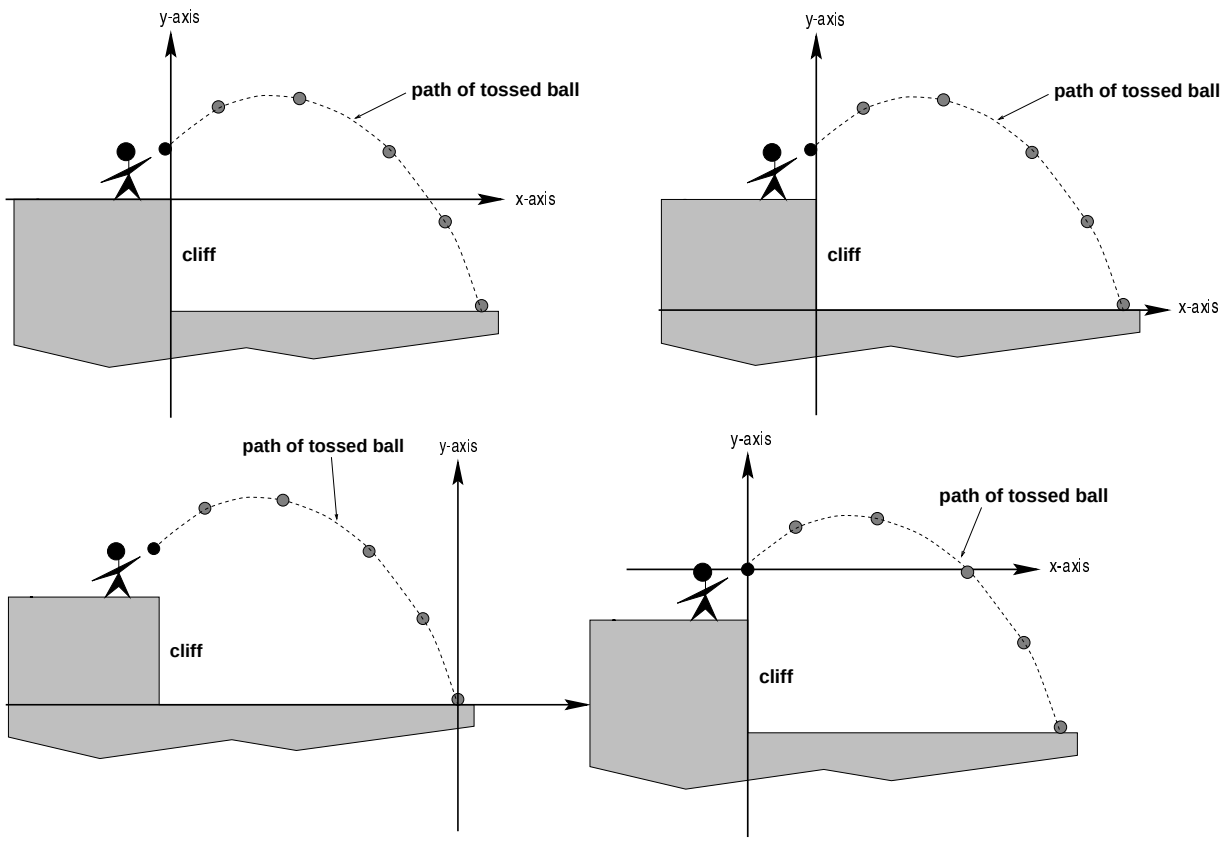
Mathematical modeling is all about relating concrete phenomena and symbolic equations, so we want to embrace the idea of visualization. Most typically, visualization will involve plotting a collection of points in the plane. This can be achieved by providing a "list" or a "prescription" for

plotting the points. The material we review in the next couple of sections makes the transition from symbolic mathematics to visual pictures go more smoothly.

A Key Step in all Modeling Problems.

Return to the tossed ball scenario on p.1. How do we decide where to draw a coordinate system in the picture ?

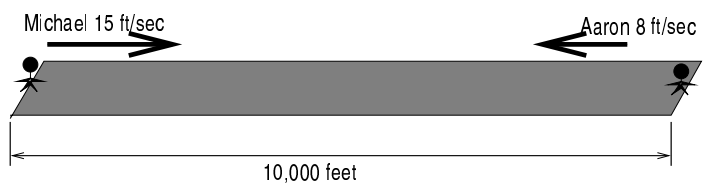
As pictured below, here are four natural choices of xy-coordinate system:



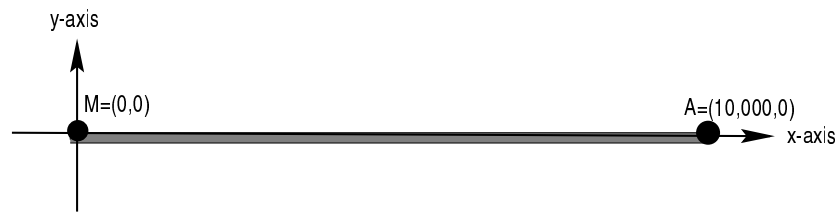
Notice, a choice of coordinate system amounts to specifying the origin and these four choices amount to specifying the origin to be either the top of the cliff, the bottom of the cliff, the landing point of the ball or the launch point of the ball. So, which choice do we make? The answer is that any of these choices will work, but one choice may be more natural than another. For example, the upper right-hand picture is probably the most natural choice: in this coordinate system, the motion of the ball takes place in the first quadrant, so the x and y coordinates of any point on the path of the ball will be non-negative.

The initial problem solving/modeling step of deciding on a choice of xy-coordinate system is called *imposing a coordinate system*: There will often be many possible choices; it takes problem solving experience to develop intuition for a "natural" choice. This is a key step in all modeling problems.

Example 1.2.2: *Michael and Aaron are running toward each other beginning at opposite ends of a 10,000 ft. airport runway, as pictured. Where and when will these guys collide?*

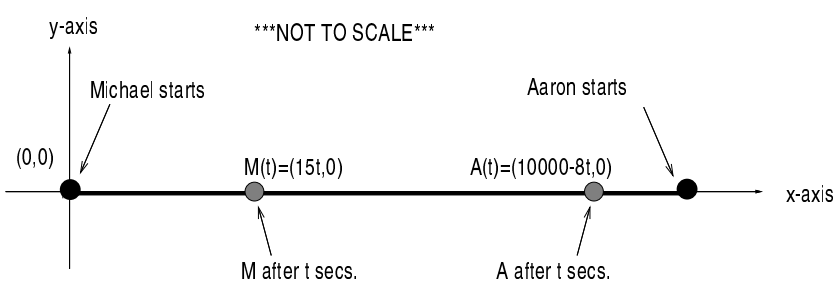


Solution. This problem requires that we find the "time" and "location" of the collision. Our first step is to impose a coordinate system:



We choose the coordinate system so that Michael is initially located at the point $M = (0, 0)$ (the origin) and Aaron is initially located at the point $A = (10000, 0)$. To find the coordinates of Michael after t seconds, we need to think about how distance and time are related.

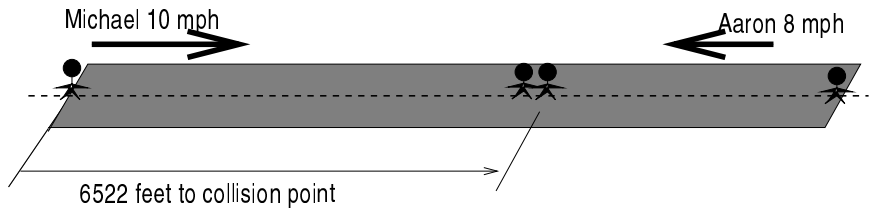
Since Michael is moving at the rate of 15 ft/second, then after one second he is located 15 feet right of the origin; i.e. at the point $(15, 0)$. After 2 seconds, Michael has moved 15 feet the first second plus 15 feet the second second, for a total of 30 feet; so he is located at the point $(30, 0)$, etc. Conclude Michael has traveled $15t$ feet **to the right** after t seconds; i.e. his location is the point $M(t) = (15t, 0)$. Similarly, Aaron is located 8 ft. left of his starting location after 1 second (at the point $(9992, 0)$), etc. Conclude Aaron has traveled $8t$ ft. **to the left** after t seconds; i.e. his location is the point $A(t) = (10000 - 8t, 0)$.



The key observation required to solve the problem is that the point of collision occurs when the coordinates of Michael and Aaron are equal. Because we are moving along the horizontal axis, this amounts to finding where and when the x-coordinates of the $M(t)$ and $A(t)$ agree. This is a straightforward algebra problem:

$$\begin{aligned} 15t &= 10000 - 8t \\ 23t &= 10000 \\ t &= 434.78 \end{aligned} \tag{1.2.3}$$

To the nearest tenth of a second, the runners collide after 434.8 seconds. Plugging into either expression for the position: $M(434.8) = (15(434.8), 0) = (6522, 0)$.



□

Distance.

We end this section with a discussion of direction and distance in the plane. To set the stage, think about the following analogy:

> *You are in a plane flying from Denver to New York. How far will you fly? To what extent will you travel north? To what extent will you travel east?*

Consider two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the xy coordinate system, where *we assume that the units on each axis are the same*; for example, both in units of "feet". Imagine standing at the location P and walking to the location Q along a straight line segment connecting the two points; see the left-hand figure below. Now ask yourself this question: *To what overall extent have the x and y coordinates changed?*

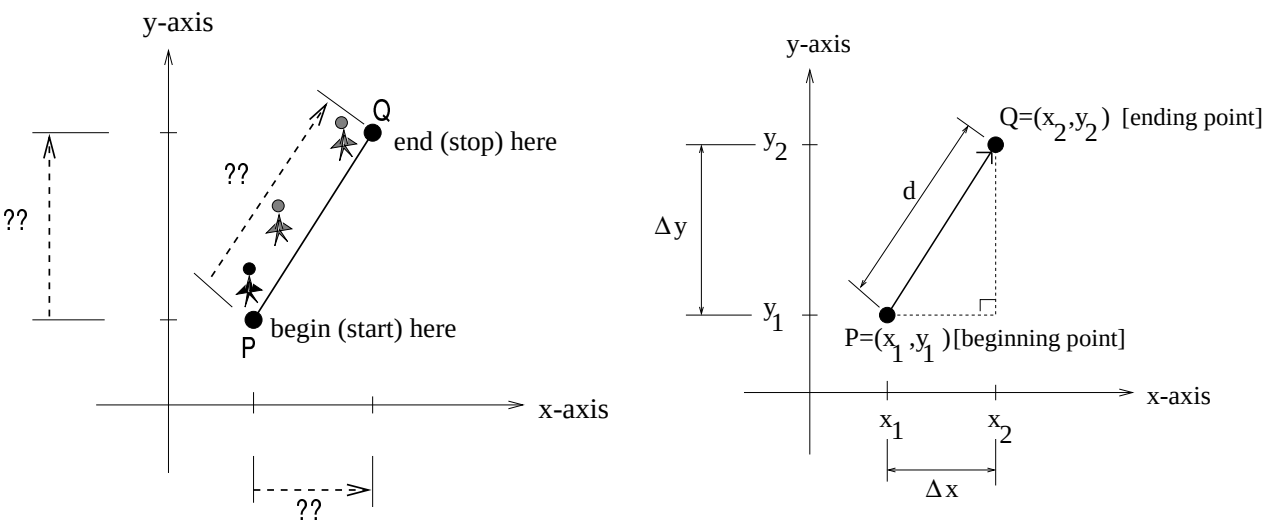


Figure 1.2.3: The meaning of Δx and Δy.

To answer this, we introduce visual and notational aides into this figure. We have inserted an "arrow" pointing from the starting position P to the ending position Q; see the right-hand figure above. To simplify things, introduce the notation Δx to keep track of the change in the x-coordinate and Δy to keep track of the change in the y-coordinate, as we move from P to Q. Each of these quantities can now be computed:

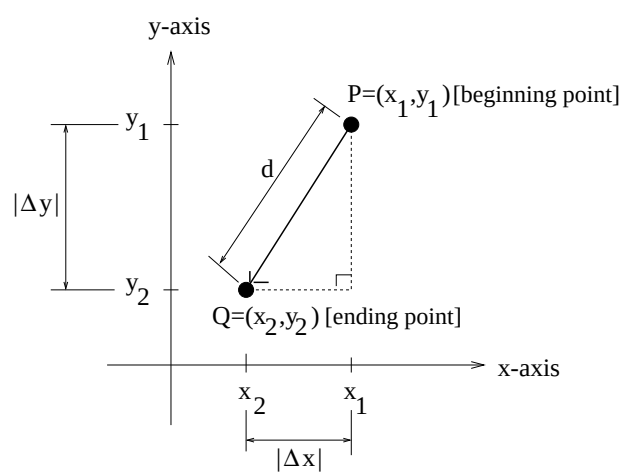
$$\begin{aligned}
\Delta x &= \text{change in } x\text{-coordinate going from } P \text{ to } Q \\
&= (x\text{-coord of ending point}) - (x\text{-coord of beginning point}) \\
&= x_2 - x_1 \\
\Delta y &= \text{change in } y\text{-coordinate going from } P \text{ to } Q \\
&= (y\text{-coord of ending point}) - (y\text{-coord of beginning point}) \\
&= y_2 - y_1.
\end{aligned} \tag{1.2.4}$$

We can interpret Δx and Δy using the right triangle pictured and the *Pythagorean Theorem*:

$$d^2 = (\Delta x)^2 + (\Delta y)^2; \text{ i.e., } d = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

which tells us the distance d from P to Q. In other words, d is the distance we would walk if we had walked along that line segment connecting the two points. As an example, if $P = (1, 1)$ and $Q = (5, 4)$, then $\Delta x = 5 - 1 = 4$, $\Delta y = 4 - 1 = 3$ and $d = 5$.

There is a subtle idea behind the way we defined Δx and Δy: You need to specify the "beginning" and "ending" points used to do the calculation in (1.2.4). What happens if we had reversed the choices in Figure 1.2.3?



Then the quantities Δx and Δy will both be negative and the lengths of the sides of the right triangle are computed by taking the *absolute value* of Δx and Δy. As far as a distance calculation is concerned, the previous formula still works because of this algebra equality:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(|\Delta x|)^2 + (|\Delta y|)^2}.$$

We will sometimes refer to Δx and Δy as *directed distances* in the x and y directions. The notion of directed distance becomes important in our discussion of *lines* in §1.4 and *vectors* in Chapter 4; it is also very important in *calculus*. For example, if $P = (5, 4)$ and $Q = (1, 1)$, then $\Delta x = 1 - 5 = -4$, $\Delta y = 1 - 4 = -3$ and $d = 5$.

Distance Formula 1.2.5: *If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in the plane, then the straight line distance between the points (in the same units as the two axes) is given by the formula $d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.*

!!! CAUTION !!!

If your algebra is a little rusty, a very common mistake crops up when using the distance formula. For example,

$$\sqrt{3^2 + 4^2} \neq \sqrt{3^2} + \sqrt{4^2} = 3 + 4 = 7.$$

Notice, the correct answer is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

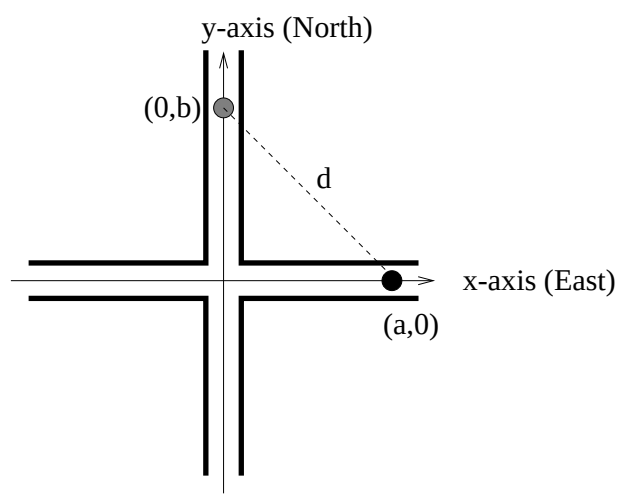
!!! CAUTION !!!

Example 1.2.6: *Two cars depart from a four way intersection at the same time, one heading East and the other heading North. Both cars are traveling at the constant speed of 30 ft/sec. Find the distance (in miles) between the two cars after 1 hour 12 minutes. In addition, determine when the two cars would be exactly 1 mile apart.*

Solution. Begin with a picture of the situation. We have indicated the locations of the two vehicles after t seconds and the distance d between them at time t. By the distance formula, the distance between them is

$$d = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}.$$

This formula is a first step; the difficulty is that we have traded the mystery distance d for two new unknown numbers a and b. To find the coordinate a for the Eastbound car, we know the car is moving at the rate of 30 ft/sec, so it will travel $30t$ feet after t seconds; i.e. $a = 30t$. Similarly, we find that $b = 30t$.



Substituting into the formula for d we arrive at

$$d = \sqrt{(30t)^2 + (30t)^2} = \sqrt{2t^2(30)^2} = 30t\sqrt{2}.$$

First, we need to convert 1 hour and 12 minutes into seconds so that our formula can be used:

$$\begin{aligned} 1 \text{ hr } 12 \text{ min} &= 1 + 12/60 \text{ hr} = 1.2 \text{ hr} \\ &= (1.2\text{hr})\left(\frac{60\text{min}}{\text{hr}}\right)\left(\frac{60\text{sec}}{\text{min}}\right) = 4320 \text{ sec.} \end{aligned}$$

Substituting $t = 4320$ sec and recalling that 1 mile = 5280 feet, we arrive at

$$d = 129600\sqrt{2}\,\text{feet} = 183282 \text{ feet} = 34.71 \text{ miles.}$$

For the second question, we specify the distance being 1 mile and want to find when this occurs. The idea is to set d equal to 1 mile and solve for t. However, we need to be careful, since the units for d are feet:

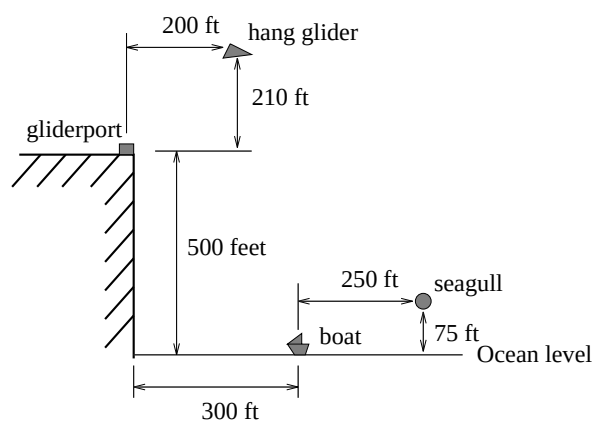
$$\begin{aligned} 30t\sqrt{2} &= d = 5280 \\ t &= \frac{5280}{30\sqrt{2}} = 124.45 \text{ seconds} = 2 \text{ minutes } 4 \text{ seconds.} \end{aligned}$$

□

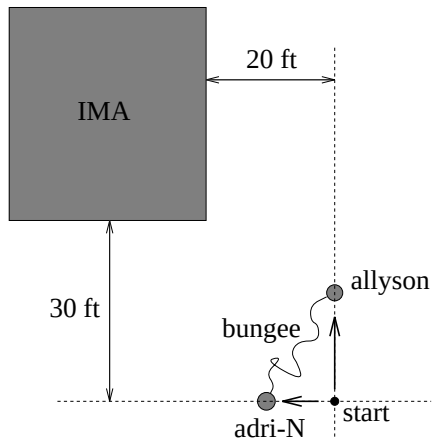
Problems

1. Suppose two cars depart from a four way intersection at the same time, one heading north and the other heading west. The car heading north travels at the steady speed of 30 ft/sec and the car heading west travels at the steady speed of 58 ft/sec.
   (a) Find an expression for the distance between the two cars after t seconds.
   (b) Find the distance in miles between the two cars after 3 hours 47 minutes.
   (c) When are the two cars 1 mile apart?

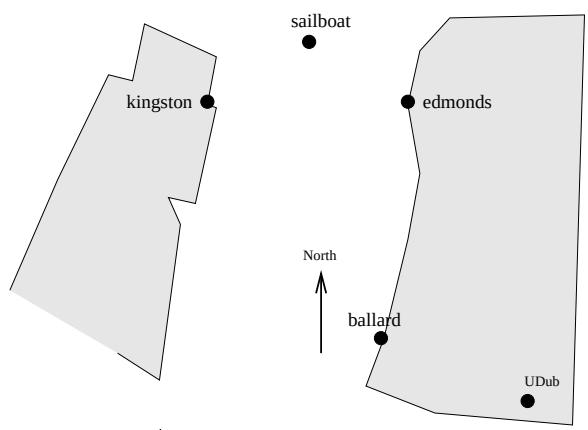
2. A hang glider launches from a gliderport in La Jolla. The launch point is located at the edge of a 500 ft. high cliff over the Pacific Ocean. Impose three different coordinate systems: one with origin at the gliderport, one with the origin at the hang glider and the third with origin at the boat location. Answer these questions for each coordinate system separately.
   (a) What are the coordinates of the hang glider?
   (b) What are the coordinates of the seagull?
   (c) What are the coordinates of the boat?
   (d) What are the coordinates of the gliderport?



3. Allyson and Adrienne have decided to connect their ankles with a *bungee cord*; one end is tied to each persons ankle. The cord is 30 feet long, but can stretch up to 90 feet. They both start from the same location. Allyson moves 10 ft/sec and Adrienne moves 8 ft/sec in the directions indicated.
   (a) Where are the two girls located after 2 seconds?
   (b) After 2 seconds, will the slack in the bungee cord be used up?
   (c) Determine when the bungee cord first becomes tight; i.e. there is no slack in the line. Where are the girls located when this occurs?
   (d) When will the bungee cord first touch the corner of the building? (Hint: Use a fact about "similar triangles".)



4. Erik's disabled sailboat is floating at a stationary location 3 miles East and 2 mile North of Kingston. A ferry leaves Kingston heading due East toward Edmonds at 12 mph. At the same time, Erik leaves the sailboat in a dinghy heading due South 10 ft/sec (hoping to intercept the ferry). Edmonds is 6 miles due East of Kingston.



   (a) Compute Erik's speed in mph and Ferry speed in ft/sec.
   (b) Impose a coordinate system and complete this table of data concerning locations of Erik and the ferry. Insert into the picture the locations of the ferry and Erik after 7 minutes.

	0 second	30 second	7 minutes	t hours .
Ferry coordinates				
Erik's coordinates				
Distance between them				

   (c) Explain why Erik misses the ferry.
   (d) After 10 minutes, a Coast Guard boat leaves Kingston heading due East at a speed of 25 ft/sec. Will the Coast Guard boat catch the ferry before it reaches Edmonds? Explain.

5. Using an xy-coordinate system, plot the points $A = (1, 1)$, $B = (\frac{3}{2}, 1)$, $C = (2, 1)$, $D = (-1, -1)$, $E = (-1, -\frac{1}{2})$, and $F = (-1, 0)$.
   (a) Use the distance formula to find:
      (i) The distance from A to D.
      (ii) The distance from B to E.
      (iii) The distance from C to F.
   (b) For each number t, define a point $P(t)$ by the formula $P(t) = (1 + t, 1)$. Plot the points $P(0)$, $P(0.5)$, $P(1)$, $P(-1)$, $P(2)$.
   (c) Likewise, for each number t, define a point $Q(t)$ by the formula $Q(t) = (-1, -1 + t)$. Plot the points $Q(0), Q(0.5), Q(1)$, $Q(-1), Q(2)$.
   (d) Use the distance formula to compute the distance between $P(t)$ and $Q(t)$. Your formula will involve t.
   (e) Find a value of t so that the distance between $P(t)$ and $Q(t)$ is 6. Where are the two points located for this value of t?

1.3 Three Simple Curves

Before we discuss graphing, we first want to become acquainted with the sorts of pictures that will arise. This is surprisingly easy to accomplish: Impose an xy-coordinate system on a blank sheet of paper. Take a sharp pencil and begin moving it around on the paper. The resulting picture is what we will call a *curve*. For example, here is a sample of the sort of "artwork" we are trying to visualize.

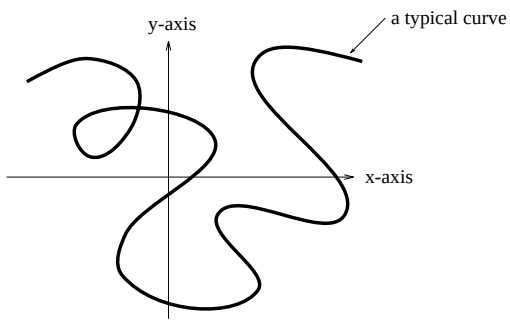


Figure 1.3.1: A typical curve

A number of examples in the text will involve basic curves in the plane. When confronted with a curve in the plane, the fundamental question we always try to answer is this:

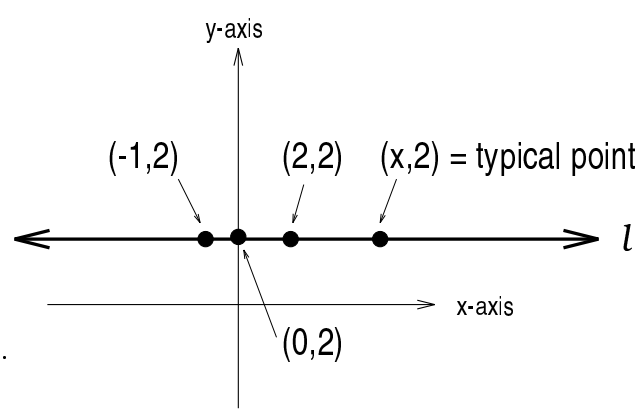
> *Can we give a condition (think of it as a "test") which will tell us precisely when a point in the plane lies on a curve?*

Typically, the kind of condition we will give involves an equation in two variables (like x and y). We consider the three simplest situations in this chapter: horizontal lines, vertical lines and circles.

The Simplest Lines.

Undoubtedly, the simplest curves in the plane are the horizontal and vertical lines.

For example, sketch a line parallel to the x-axis passing through 2 on the y-axis; the result is a horizontal line ℓ, as pictured. This means the line ℓ passes through the point $(0, 2)$ in our coordinate system. A concise symbolic prescription for ALL of the points on ℓ can be given using "set notation":



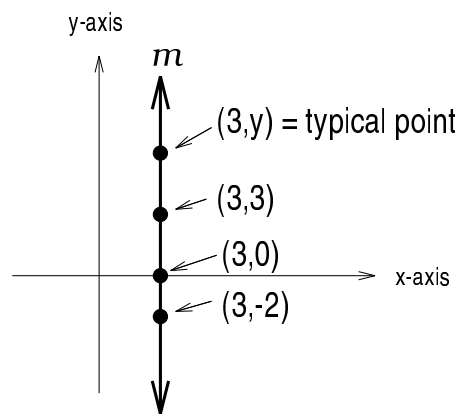
$$\ell \quad = \quad \{\text{all points } (x, 2) \text{ where } x = \text{ any real number}\}.$$

We read the right-hand side of this expression as "*...the set of all points* $(x, 2)$ *where* x*= any real number...*". Notice, the points (x, y) on the line ℓ are EXACTLY the ones that lead to *solutions* of the equation $y = 2$; i.e. take any point on this line, plug the coordinates into the equation $y = 2$ and

you get a true statement. Because the equation does not involve the variable x and only constrains y to equal 2, we see that x can take on any real value. In short, we see that plotting all of the solutions (x, y) to the equation $y = 2$ gives the line ℓ. We usually refer to the set of all solutions of the equation $y = 2$ as the *graph* of the equation $y = 2$.

As a second example, sketch the vertical line m passing through 3 on the x-axis; this means the line m passes through the point $(3, 0)$ in our coordinate system. A concise symbolic prescription for ALL of the points on m can be given using "set notation":

$$m \quad = \quad \{\text{all points}\,(3, y) \text{ where}\, y = \text{ any real number}\}.$$



Notice, the points (x, y) on the line m are EXACTLY the ones that lead to *solutions* of the equation $x = 3$; i.e. take any point on this line, plug the coordinates into the equation $x = 3$ and you get a true statement. Because the equation does not involve the variable y and only specifies that $x = 3$, we see that y can take on any real number value. In short, we see that plotting all of the solutions (x, y) to the equation $x = 3$ gives the line m. We usually refer to the set of all solutions of the equation $x = 3$ as the *graph* of the equation $x = 3$.

These two simple examples highlight our first clear connection between a geometric figure and an equation; the link is achieved by plotting all of the solutions (x, y) of the equation in the xy-coordinate system. These observations work for any horizontal or vertical line.

Horizontal and Vertical Lines 1.3.1: *A horizontal line ℓ passing through k on the y-axis is precisely a plot of all solutions (x, y) of the equation $y = k$; i.e. ℓ is the graph of $y = k$. A vertical line m passing through h on the x-axis is precisely a plot of all solutions (x, y) of the equation $x = h$; i.e. m is the graph of $x = h$.*

Circles.

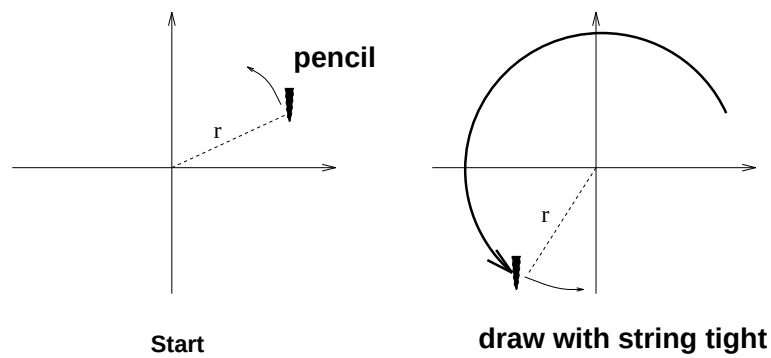
Another common curve in the plane is a circle. Let's see how to relate a circle and an equation involving the variables x and y. As a special case of the distance formula (1.2.5), suppose $P = (0, 0)$ is the origin and $Q = (x, y)$ is any old point in the plane; then

$$\text{distance from } P \text{ to } Q \quad = \quad \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

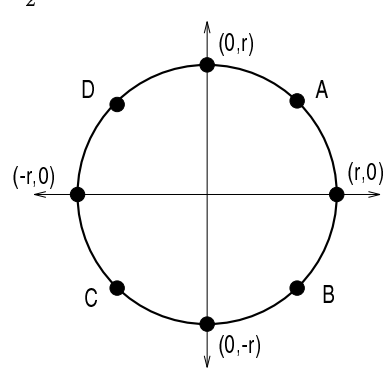
This calculation tells us that a point (x, y) is of distance r from the origin if and only if $r = \sqrt{x^2 + y^2}$; or, squaring each side, that $x^2 + y^2 = r^2$. This shows

$$\{(x, y)|\text{distance } (x, y) \text{ to origin is } r\} = \{(x, y)|x^2 + y^2 = r^2\}. \tag{1.3.2}$$

What is the left-hand side of (1.3.2)? To picture all points in the plane of distance r from the origin, fasten a pencil to one end of a non-elastic string (a string that will not stretch) of length r and tack the other end to the origin. Holding the string tight, the pencil point will locate a point of distance r from the origin. We could visualize all such points by simply moving the pencil around the origin, all the while keeping the string tight.



What is the right-hand side of (1.3.2)? A point (x, y) in the right-hand set is a solution to the equation $x^2 + y^2 = r^2$; i.e. if we plug in the coordinates we get a true statement. For example, here we plot eight solutions $(r, 0), (-r, 0), (0, r), (0, -r), A = (\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}), B = (\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}), C = (\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}), D = (\frac{-r}{\sqrt{2}}, \frac{r}{\sqrt{2}})$, of the equation. To see that the last point is a solution, here is the sample calculation: $(\frac{-r}{\sqrt{2}})^2 + (\frac{r}{\sqrt{2}})^2 = \frac{r^2}{2} + \frac{r^2}{2} = r^2$.

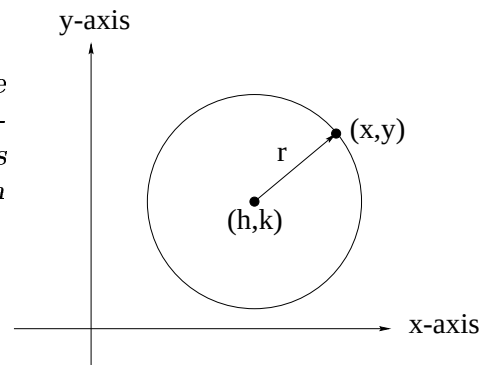


Since the two sides of (1.3.2) are equal, drawing the circle of radius r is the same as plotting all of the solutions of the equation $x^2 + y^2 = r^2$. The same reasoning can be used to show that drawing a circle of radius r centered at a point (h, k) is the same as plotting all of the solutions of the equation: $(x - h)^2 + (y - k)^2 = r^2$. We usually refer to the set of all solutions of the equation as the *graph* of the equation.

Circles 1.3.3: *Let (h, k) be a given point in the xy-plane and $r > 0$ a given positive real number. The circle of radius r centered at (h, k) is precisely all of the solutions (x, y) of the equation*

$$(x - h)^2 + (y - k)^2 = r^2;$$

i.e. the circle is the graph of this equation.



We refer to the equation in the box as the *standard form* of the equation of a circle; from it you can immediately read off the center and the radius of the circle it is describing.

!!! CAUTION !!!

Be very careful with the minus signs "−" in the standard form for a circle equation. For example, the equation

$$(x + 3)^2 + (y - 1)^2 = 7$$

is NOT in standard form. We can rewrite it in standard form:

$$(x - (-3))^2 + (y - 1)^2 = (\sqrt{7})^2;$$

so, this equation describes a circle of radius $\sqrt{7}$ centered at $(-3, 1)$.

!!! CAUTION !!!

Examples 1.3.4: (i) The circle of radius 1 centered at the origin is the graph of the equation $x^2 + y^2 = 1$. This circle is called the *unit circle* and will be used extensively.

(ii) A circle of radius 3 centered at the point $(h, k) = (1, -1)$ is the graph of the equation $(x-1)^2+(y-(-1))^2 = 3^2$; or, equivalently $(x-1)^2+(y+1)^2 = 3^2$; or, equivalently $x^2+y^2-2x+2y = 7$.

(iii) The circle of radius $\sqrt{5}$ centered at $(2, -3)$ does not pass through the origin; this is because $(0, 0)$ is **not** a solution of the equation $(x - 2)^2 + (y + 3)^2 = 5$.

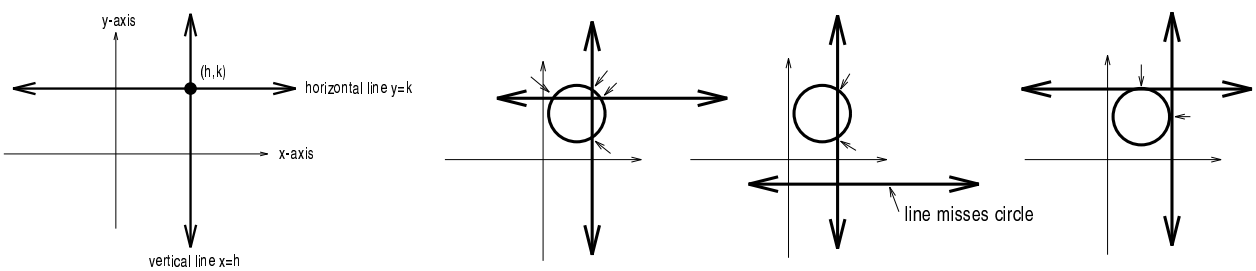
Intersecting Curves I.

In many problem solving situations, we will have two curves in the plane and need to determine where the curves intersect one another. Before we discuss a general procedure, let's make sure we really understand the meaning of the word "intersect". From latin, the word "inter" means "within or in between" and the word "sectus" means "to cut". So, the intersection of two curves is the place where the curves "cut into" each other; in other words, where the two curves cross one another.

If the pictures of two curves are given to us up front, we often will be able to visually decide whether or not they intersect. This is one good reason for drawing a picture of any physical problem

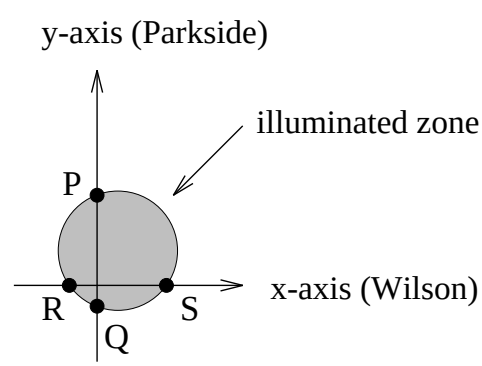
we are trying to solve. We will need a small bag of tricks used for finding intersections of curves. We begin with intersections involving the curves studied in this section.

Two different horizontal lines (or two different vertical lines) will never intersect. However, a horizontal line always intersects a vertical line exactly once. Given a circle and a horizontal or vertical line, we may or may not have an intersection. Staring at these pictures, you can convince yourself a given horizontal or vertical line will intersect a circle in either two points, one point or no points. This analysis is all pictorial; how do you find the explicit coordinates of an intersection point? Let's look at a sample problem to isolate the procedure used.



Example 1.3.5: *Glo-Tek Industries has designed a new halogen street light fixture for the city of Seattle. According to the product literature, when placed on a 50' light pole, the resulting useful illuminated area is a circular disc 120 feet in diameter. Assume the light pole is located 20 feet east and 40 north of the intersection of Parkside Ave. (a north/south street) and Wilson St. (an east/west street). What portion of each street is illuminated?*

Solution. The illuminated area is a circular disc whose diameter and center are both known. Consequently, we really need to study the intersection of this circle with the two streets. Begin by imposing the pictured coordinate system; we will use units of feet for each axis. The illuminated region will be a circular disc centered at the point $(20, 40)$ in the coordinate system; the radius of the disc will be $r = 60$ feet.



We need to find the points of intersection P, Q, R and S of the circle with the x-axis and the y-axis. The equation for the circle with $r = 60$ and center $(h, k) = (20, 40)$ is

$$(x - 20)^2 + (y - 40)^2 = 3600.$$

To find the circular disc intersection with the y-axis, we have a system of two equations to work with:

$$\begin{aligned} (x - 20)^2 + (y - 40)^2 &= 3600; \\ x &= 0. \end{aligned}$$

To find the intersection points we *simultaneously* solve both equations. To do this, we replace $x = 0$ in the first equation (i.e. we impose the conditions of the second equation on the first equation) and arrive at

$$\begin{aligned}
(0-20)^2 + (y-40)^2 &= 3600; \\
400 + (y-40)^2 &= 3600; \\
(y-40)^2 &= 3200; \\
(y-40) &= \pm\sqrt{3200} \\
y &= 40 \pm \sqrt{3200} = 96.57, -16.57.
\end{aligned}$$

Notice, we have two solutions. This means that the circle and y-axis intersect at the points $P = (0, 96.57)$ and $Q = (0, -16.57)$. Similarly, to find the circular disc intersection with the x-axis, we have a system of two equations to work with:

$$\begin{aligned}
(x-20)^2 + (y-40)^2 &= 3600; \\
y &= 0.
\end{aligned}$$

Replace $y = 0$ in the first equation (i.e. we impose the conditions of the second equation on the first equation) and arrive at

$$\begin{aligned}
(x-20)^2 + (0-40)^2 &= 3600; \\
(x-20)^2 &= 2000; \\
(x-20) &= \pm\sqrt{2000}; \\
x &= 20 \pm \sqrt{2000} = 64.72, -24.72.
\end{aligned}$$

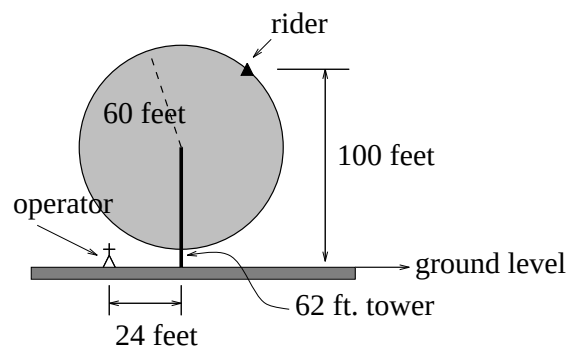
Conclude the circle and x-axis intersect at the points $S = (64.72, 0)$ and $R = (-24.72, 0)$. □

The procedure we used in the solution of the previous example gives us a general approach to finding the intersection points of circles with horizontal and vertical lines; this will be important in the exercises.

Problems

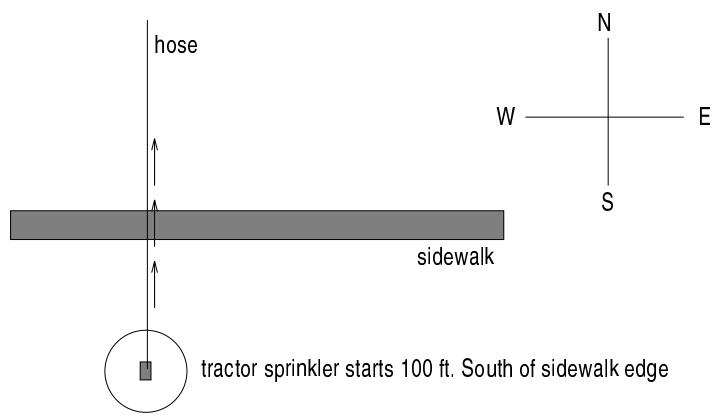
1. This exercise emphasizes the "mechanical aspects" of circles and their equations.
   (a) Find an equation whose graph is a circle of radius 3 centered at $(-3, 4)$.
   (b) Find an equation whose graph is a circle of diameter $\frac{1}{2}$ centered at the point $(3, -\frac{11}{3})$.
   (c) Find four different equations whose graphs are circles of radius 2 through (1,1).
   (d) Consider the equation $(x-1)^2 + (y+1)^2 = 4$. Which of the following points lie on the graph of this equation: (1,1), (1,-1), (1,-3), $(1+\sqrt{3}, 0), (0, -1-\sqrt{3})$, (0,0).

2. An amusement park Ferris Wheel has a radius of 60 feet. The center of the wheel is mounted on a tower 62 feet above the ground (see picture). For these questions, the wheel is not turning.



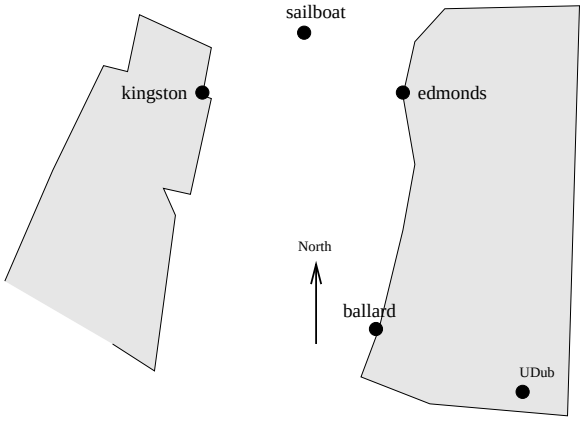
(a) Impose a coordinate system.
(b) Suppose a rider is located at the point in the picture, 100 feet above the ground. If the rider drops an ice cream cone straight down, where will it land on the ground?
(c) The ride operator is standing 24 feet to one side of the support tower on the level ground at the location in the picture. Determine the location of a rider on the Ferris Wheel so that a dropped ice cream cone lands on the operator. (Note: There are two answers.)

3.* A *crawling tractor sprinkler* is located as pictured below, 100 feet South of a sidewalk. Once the water is turned on, the sprinkler waters a circular disc of radius 20 feet and moves North along the hose at the rate of $\frac{1}{2}$ inch/second. The hose is perpendicular to the 10 ft. wide sidewalk.



(a) Impose a coordinate system. Describe the initial coordinates of the sprinkler and find equations of the lines forming the North and South boundaries of the sidewalk.
(b) When will the water first strike the sidewalk?
(c) When will the water from the sprinkler fall completely North of the sidewalk.
(d) Find the total amount of time water from the sprinkler falls on the sidewalk.
(e) Sketch a picture of the situation after 33 minutes. Draw an accurate picture of the watered portion of the sidewalk.
(f) Find the area of GRASS watered after one hour.

4. Erik's disabled sailboat is floating stationary 3 miles East and 2 mile North of Kingston. A Ferry leaves Kingston heading toward Edmonds at 12 mph. After 20 minutes the ferry turns heading due South. Ballard is 8 miles South and 1 mile West of Edmonds. Impose coordinates with Ballard the origin. Edmonds is 6 miles due East of Kingston.



   (a) Find the equations for the lines along which the ferry is moving and draw in these lines.
   (b) The sailboat has a radar scope that will detect any object within 3 miles of the sailboat. Looking down from above, as in the picture, the radar region looks like a circular disk. The *boundary* is the "edge" or circle around this disc, the *interior* is the inside of the disk, and the *exterior* is everything outside of the disk (i.e. outside of the circle). Give a mathematical (equation) description of the boundary, interior and exterior of the radar zone. Sketch an accurate picture of the radar zone by determining where the line connecting Kingston and Edmonds would cross the radar zone.
   (c) When does the ferry enter the radar zone?
   (d) How would you determine where and when the ferry exits the radar zone?
   (e) How long does the Ferry spend inside the radar zone?

1.4 Linear Modeling

Sometimes, we will begin a section by looking at a specific problem which highlights the topic to be studied; this section offers the first such vista. View these as illustrations of precalculus in action, rather than confusing examples. Don't panic, the essential algebraic skills will be reviewed once the motivation is in place.

The Earning Power Problem.

The government likes to gather all kinds of data. For example, here is some data on the average annual income for full-time workers; this was taken from the 1990 *Statistical Abstract of the U.S.*:

YEAR	WOMEN (dollars)
1970	\$5616
1987	\$18,531

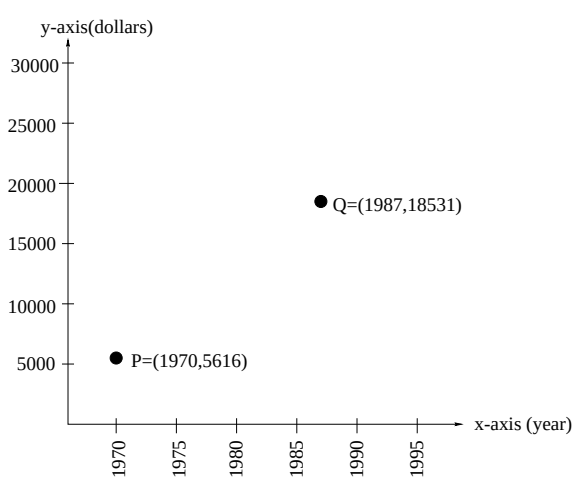
YEAR	MEN (dollars)
1970	\$9521
1987	\$28,313

Table 1.4.1: Earning power data

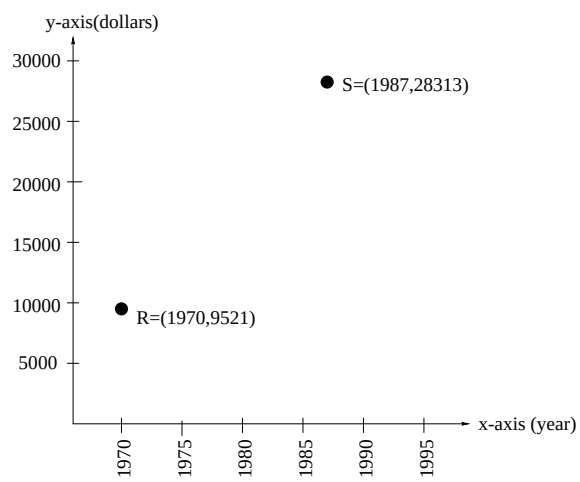
Given this information, a natural question would be:

How can we predict the future earning power of women and men?

One way to answer this question would be to use the data in the table to construct two different *mathematical models* that predict the future (or past) earning power for women or men. In order to do that, we would need to make some kind of initial assumption about the type of mathematical model expected. Let's begin by drawing two identical xy-coordinate systems, where the x-axis has units of "year" and the y-axis has units of "dollars". In each coordinate system, the data in our table gives us two points to plot: In the case of women, the data table gives us the points $P = (1970, 5616)$ and $Q = (1987, 18531)$. Likewise, for the men, the data table gives us the points $R = (1970, 9521)$ and $S = (1987, 28313)$.



Data for Women's Earning Power



Data for Men's Earning Power

To study the future earning power of men and women, we are going to make an *assumption:*

> *For women, if the earning power in year x is \$$y$, then the point (x, y) lies on the line connecting P and Q. Likewise, for men, if the earning power in year x is \$$y$, then the point (x, y) lies on the line connecting R and S.*

In the real world, the validity of this kind of assumption would involve a lot of statistical analysis. This kind of assumption leads us to what is called a *linear model*, since we are demanding that the data points predicted by the model (i.e. the points (x, y)) lie on a line in a coordinate system. Now that we have made this assumption, our job is to find a way to mathematically describe when a point (x, y) lies on one of the two lines pictured below:

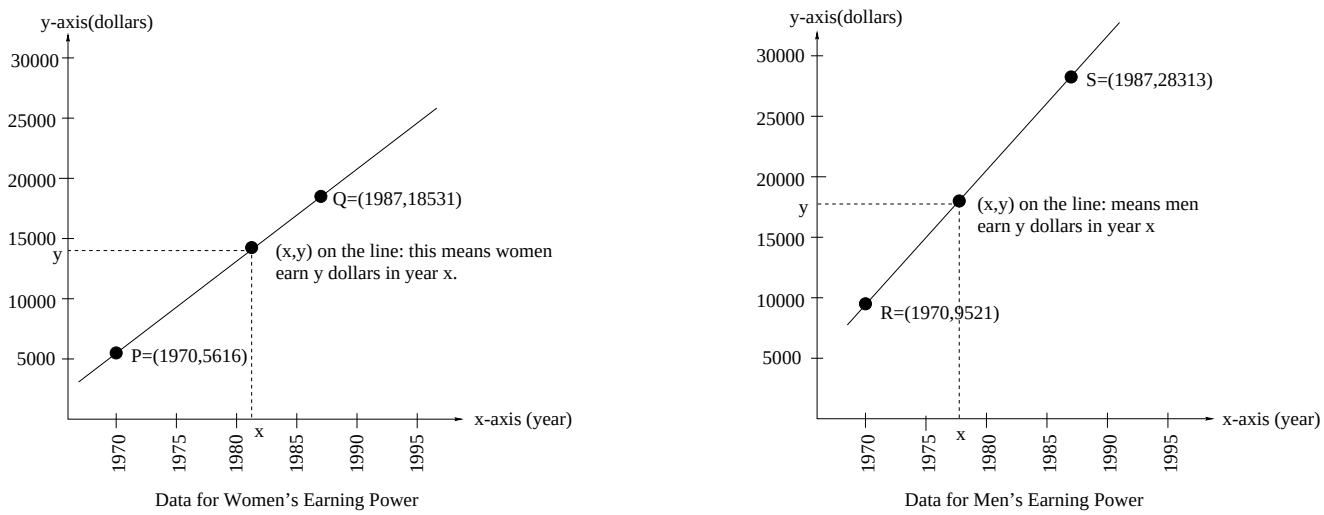


Figure 1.4.1: Linear Models for Women's and Men's Earning Power

Our goal in the next subsection is to review the mathematics necessary to show that the lines in Figure 1.4.1 are the so-called *graphs* of these two equations:

$$\begin{aligned} y &= \left(\frac{28313 - 9521}{1987 - 1970}\right)(x - 1970) + 9521 = \frac{18792}{17}(x - 1970) + 9521, \qquad \text{(Men's Model)} \\ y &= \left(\frac{18531 - 5616}{1987 - 1970}\right)(x - 1970) + 5616 = \frac{12915}{17}(x - 1970) + 5616 \qquad \text{(Women's Model)} \end{aligned} \tag{1.4.1}$$

Relating Lines and Equations.

A systematic approach to studying equations and their graphs would begin with the simple cases, gradually working toward the more complicated. Thinking visually, the simplest curves in the plane would be straight *lines*. As we discussed in §1.3, a point on the vertical line in the left-hand picture below will always have the same x-coordinate; we referred to this line as the graph of the equation

$x = h$. Likewise, a point on the horizontal line in the middle picture will always have the same y-coordinate; we referred to this line as the graph of the equation $y = k$. The right-hand picture is different, in the sense that neither the x nor the y coordinate is constant; i.e. as you move a point along the line, both coordinates of the point are changing. In the case of this line, it is reasonable to guess that it is the graph of some equation involving both x and y. The question is: What is the equation?

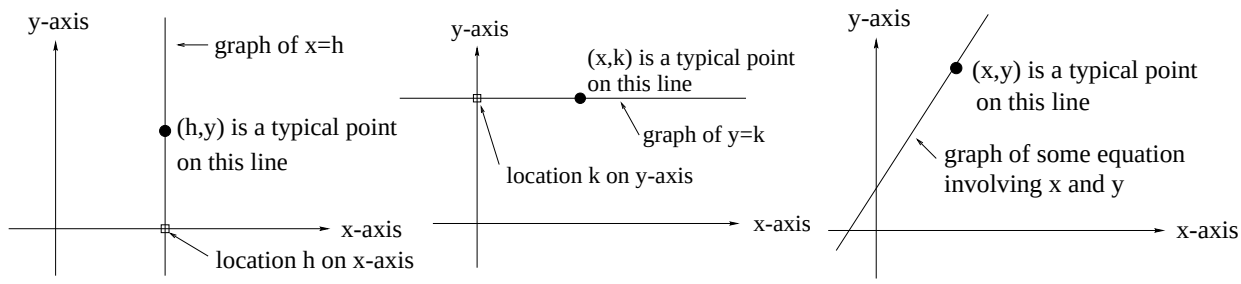


Figure 1.4.2: Lines in the plane

Here is the key geometric fact needed to model lines by mathematical equations:

"... *a line is completely determined by two different points lying on it* ..."	*(1.4.2)*

This fact tells us that if you are given two different points on a line, you can reconstruct the line in a coordinate system by simply lining a ruler up with the two points. In our discussion, we will need to pay special attention to the difference between vertical and non-vertical lines.

Non-vertical Lines.

Assume in this section that ℓ is a non-vertical line in the plane; for example, the right-hand line in Figure 1.4.2. If we are given two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on a line ℓ, then (1.2.4) defined two quantities we can calculate:

$$\Delta x = \text{change in } x \text{ going from } P \text{ to } Q = x_2 - x_1.$$

$$\Delta y = \text{change in } y \text{ going from } P \text{ to } Q = y_2 - y_1.$$

We define the *slope* of the line ℓ to be the ratio of Δy by Δx, which is usually denoted by m:

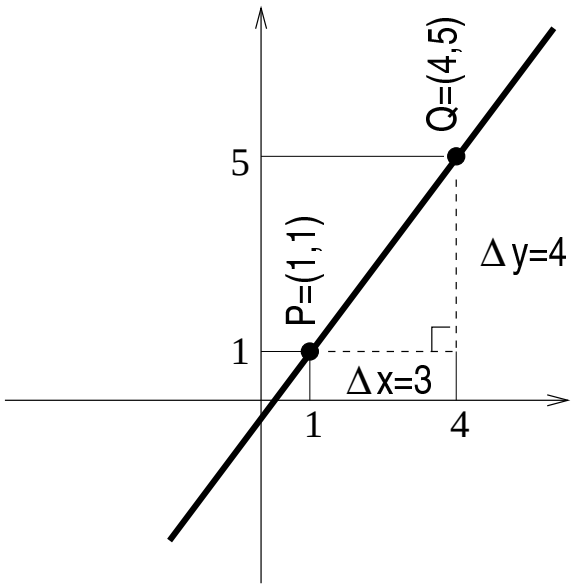
$$m = \text{slope of } \ell \stackrel{\text{def}}{=} \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} \tag{1.4.3}$$

Notice, we are using the fact that the line is non-vertical to know that this ratio is always defined; i.e. we will never have $\Delta x = 0$ (which would lead to illegal division by zero). There is some additional terminology that goes along with the definition of the slope. The term Δy is sometimes called the *rise of* ℓ and Δx is called the *run of* ℓ. For this reason, people often refer to the slope of a line ℓ as "the rise over the run", meaning

$$\text{slope of } \ell \stackrel{\text{def}}{=} \frac{\text{rise of } \ell}{\text{run of } \ell} = \frac{\Delta y}{\Delta x}.$$

In addition, notice that the calculation of Δy involves taking the difference of two numbers; likewise, the calculation of Δx involves taking the difference of two numbers. For this reason, the slope of a line ℓ is sometimes called a *difference quotient*.

For example, suppose $P = (1, 1)$ and $Q = (4, 5)$ lie on a line ℓ. In this case, the rise$=\Delta y = 4$ and the run$=\Delta x = 3$, so $m = \frac{4}{3}$ is the slope of ℓ.



When computing Δx, pay special attention that it is the x-coordinate of the destination point Q minus the x-coordinate of the starting point P; likewise, when computing Δy, it is the y-coordinate of the destination point Q minus the y-coordinate of the starting point P. We can reverse this order in **both** calculations and get the same slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-\Delta y}{-\Delta x}.$$

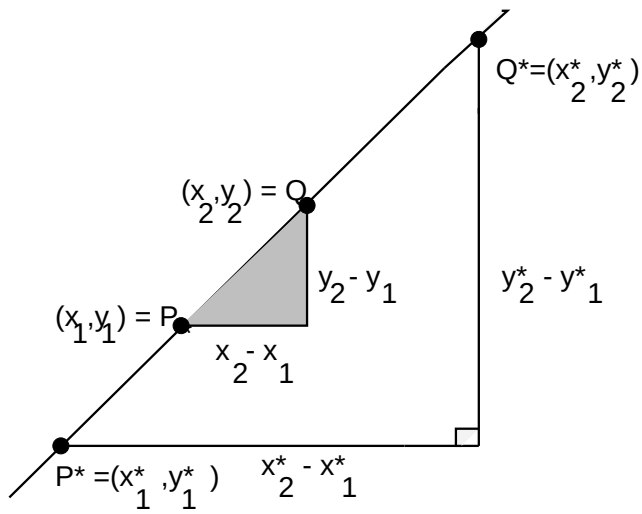
!!! CAUTION !!!

We CANNOT reverse the order in just one of the calculations and get the same slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_2 - y_1}{x_1 - x_2} \text{ and } m = \frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_1 - y_2}{x_2 - x_1}.$$

!!! CAUTION !!!

It is very important to notice that the calculation of the slope of a line does not depend on the choice of the two points P and Q. This is a real windfall, since we are then always at liberty to pick our favorite two points on the line to determine the slope. The reason for this freedom of choice is pretty easy to see by looking at a picture. If we were to choose two other points $P^* = (x_1^*, y_1^*)$ and $Q^* = (x_2^*, y_2^*)$ on ℓ, then we would get two similar right triangles:



Basic properties of similar triangles tell us ratios of lengths of common sides are equal, so that

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2^* - y_1^*}{x_2^* - x_1^*};$$

but this just says the calculation of the slope is the same for any pair of distinct points on ℓ. For example, lets redo the slope calculation when $P^* = P = (x_1, y_1)$ and $Q^* = (x, y)$ represents an arbitrary point on the line. Then the two ratios of lengths of common sides give us the equation

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1}, \\ y - y_1 &= m(x - x_1). \end{aligned}$$

This can be rewritten as

$$y = m(x - x_1) + y_1 \quad \text{or} \quad y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1. \tag{1.4.4}$$

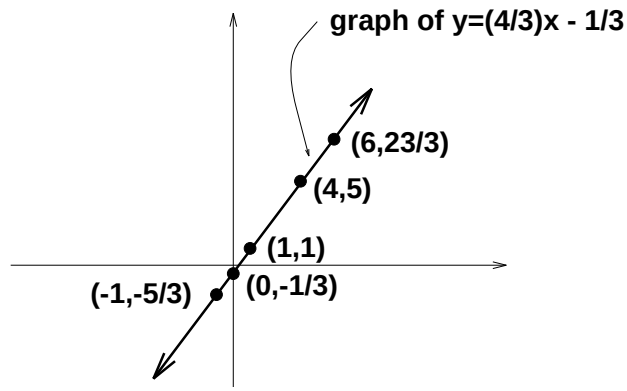
The left equation is usually called the *point slope formula* for the line ℓ (since the data required to write the equation amounts to a point (x_1, y_1) on the line and the slope m), whereas the right equation is called the *two point formula* for the line ℓ (since the data required amounts to the coordinates of the points P and Q). In any event, we now see that

$$\boxed{(x, y) \text{ lies on } \ell \text{ if and only if } (x, y) \text{ is a solution to } y = m(x - x_1) + y_1. \qquad (1.4.5)}$$

We can plot the collection of ALL solutions to the equation in (1.4.5), which we refer to as the *graph* of the equation. As a subset of the xy-coordinate system,

$$\boxed{\ell = \{ \text{ all points } (x, m(x - x_1) + y_1) \text{ where } x \text{ is any real number}\}. \qquad (1.4.6)}$$

Example 1.4.7: Consider the line ℓ through the two points $P = (1,1)$ and $Q = (4,5)$. Then the slope of ℓ is $m = 4/3$ and ℓ consists of all pairs of points (x,y) such that the coordinates x and y satisfy the equation $y = \frac{4}{3}(x-1)+1$. Letting $x = 0, 1, 6$ and -1, we conclude that the following four points lie on the line ℓ: $(0, \frac{4}{3}(0-1)+1) = (0, \frac{-1}{3})$, $(1, \frac{4}{3}(1-1)+1) = (1,1)$, $(6, \frac{4}{3}(6-1)+1) = (6, \frac{23}{3})$ and $(-1, \frac{4}{3}(-1-1)+1) = (-1, \frac{-5}{3})$. By the same reasoning, the point (0,0) does not lie on the line ℓ.

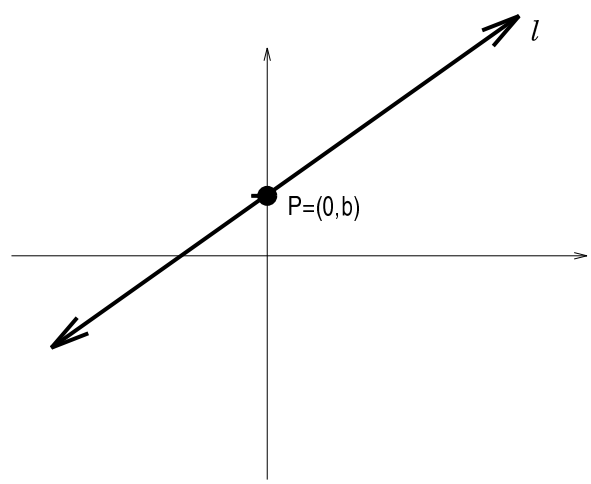


As a set of points in the plane, we have

$$\ell \;=\; \{\text{ all pairs } (x, \frac{4}{3}(x-1)+1) \mid x \text{ is any real number}\}.$$

Returning to the general situation, we can obtain a third general equation for a non-vertical line. To emphasize what is going on here, plug the specific value $x = 0$ into the equation in (1.4.4) and obtain the point $P = (0,b)$ on the line, where $b = m(0-x_1)+y_1 = -mx_1 + y_1$. But, (1.4.4) can be written

$$\begin{aligned} y &= m(x-x_1)+y_1 \\ &= mx - mx_1 + y_1 \\ &= mx + b. \end{aligned}$$



The point P is distinguished; it is precisely the point where the line ℓ crosses the y-axis; usually called the y-intercept. The *slope intercept equation of the line* is the form

$$y = mx + b, \tag{1.4.8}$$

where the slope of the line is m and b is the y-intercept of the line.

Summary 1.4.9: Non-vertical Lines. *Let ℓ be a non-vertical line in the xy-plane. There are three ways to obtain an equation whose graph is ℓ; depending on the data provided for ℓ:*

(i) *If $P = (x_1, y_1), Q = (x_2, y_2)$ are two different points on the line, then the* <u>*two-point formula*</u> $y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$ *gives an equation whose graph is ℓ.*

(ii) *If $P = (x_1, y_1)$ is a point on the line and m is the slope of ℓ, then the* <u>*point-slope formula*</u> $y = m(x - x_1) + y_1$ *gives an equation whose graph is ℓ.*

(iii) *If the line ℓ intersects the y-axis at the point $(0, b)$ and m is the slope of the line ℓ, then the* <u>*slope-intercept formula*</u> $y = mx + b$ *gives an equation whose graph is ℓ.*

General Lines.

We have handled non-vertical lines in (1.4.9). Combined with (1.3.1), this describes any line in the plane as the graph of some equation involving x and y. Notice that the equation always has the form

$$Ax + By + C = 0,$$

for some constants A, B, C. Equations like this are called *linear equations*. In general, non-vertical lines will be of the most interest to us, since these are the lines which can be viewed as the graphs of *functions*; we will discuss this in §2.1.

Lines and Rate of Change.

If we draw a non-vertical line in the xy coordinate system, then it's slope will be the *rate of change* of y with respect to x:

$$\text{slope} \quad = \quad \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} \stackrel{\text{def}}{=} \text{rate of change of } y \text{ with respect to } x.$$

We should emphasize, this rate of change is a constant; in other words, this rate is the same no matter where we do the slope computation on the line. The point-slope formula for a line can now be interpreted as follows:

A line is determined by a point on the line and the rate of change of y with respect to x.

An interesting thing to notice is how the units for x and y figure into the rate of change calculation. For example, suppose that we have the equation $y = 10,000x + 200,000$ which relates the value y of a house (in dollars) to the number of years x you own it. For example, after 5 years, $x = 5$ and the value of the house would be $y = 10,000(5) + 200,000 = \$250,000$. In this case, the equation $y = 10,000x + 200,000$ is linear and already written in slope-intercept form, so the slope can be read off as $m = 10,000$. If we carry along the units in the calculation of $\frac{\Delta y}{\Delta x}$, then the numerator involves "dollar" units and the denominator "years" units. That means that carrying along units, the slope is actually $m = 10,000$ dollars/year. In other words, the value of the house is changing at a rate of 10,000 dollars/year.

At the other extreme, if the units for both x and y are the same, then the units cancel out in the rate of change calculation; in other words, the slope is a unit-less quantity, simply a number. This sort of thing will come up in the mathematics you see in chemistry and physics.

One important type of rate encountered is the *speed* of a moving object. Suppose an object moves along a straight line at a constant speed m, as pictured below.

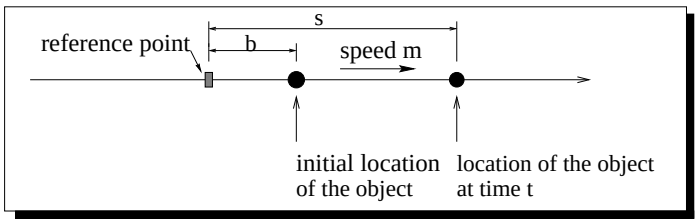


Figure 1.4.3: Motion along a line

If we specify a reference point, we can let b be the starting location of the moving object, which is usually called the *initial location* of the object. We can write down an equation relating the initial location b, the time t, the constant speed m and the location s at time t:

$s =$ (location of object at time t)

$s =$ (initial location of object) + (distance object travels in time t)

$s = b + mt$,

where t is in the same time units used to define the rate m. Notice, both b and m would be constants given to us, so this is a linear equation involving the variables s and t. We can graph the equation in the ts-coordinate system:

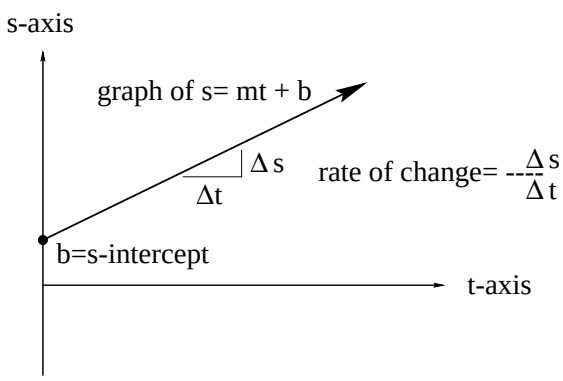


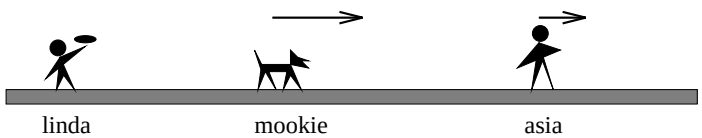
Figure 1.4.4: The graph of $s = mt + b$

It is important to distinguish between this picture (the graph of $s = mt + b$) and the path of the object in Figure 1.4.3; these are two different pictures. The graph of the equation should be thought of as a visual aid attached to the equation $s = mt + b$. The general idea is that using this visual aid can help answer various questions involving the equation, which in turn will tell us things about the motion of the object in Figure 1.4.3.

Two other comments related to this discussion are important. First, concerning notation, the speed m is often symbolized by v to denote constant *velocity* and b is written as s_o (the subscript "o"

meaning "time zero"). With these changes, the equation becomes $s = s_o + vt$, which is the form in which it would be written in a typical physics text. As a second note, if you return to Figure 1.4.4, you will notice we only drew in the positive t axis. This was because t represented time, which is always a non-negative quantity.

Example 1.4.10: *Linda, Asia and Mookie are all playing frisbee. Mookie is 10 meters in front of Linda and always runs 5 m/sec. Asia is 34 meters in front of Linda and always runs 4 m/sec. Linda yells "go!" and both Mookie and Asia start running directly away from Linda to catch a tossed frisbee. Find linear equations for the distances between Linda, Mookie and Asia after t seconds.*



Solution. Let s_M be the distance between Linda and Mookie and s_A the distance between Linda and Asia, after t seconds. An application of the above formula tells us

$s_M =$ (initial distance between Linda and Mookie) + (distance Mookie runs in t seconds)

$s_M = 10 + 5t.$

Likewise,

$s_A =$ (initial distance between Linda and Asia) + (distance Asia runs in t seconds)

$s_A = 34 + 4t.$

If s_{MA} is the distance between Mookie and Asia after t seconds, we compute

$s_{MA} = s_A - s_M =$ (34 +4t) - (10 +5t) = 24 - t meters.

In all cases, the distances we computed are given by linear equations of the form $s = b + mt$, for appropriate b and rate m. □

Back to the Earning Power Problem.

We now return to the motivating problem at the start of this section. Recall the plot right-hand plot of Figure 1.4.1. We can model the men's earning power using the first and last data points, using the ideas we have discussed about linear equations. To do this, we should specify a "beginning point" and an "ending point" (recall Figure 1.2.3) and calculate the slope:

$$R_{begin} = (1970, 9521) \text{ and } S_{end} = (1987, 28313).$$

We find that

$$\Delta y = 28313 - 9521 = 18792 \qquad \Delta x = 1987 - 1970 = 17$$

$$m = \frac{\Delta y}{\Delta x} = \frac{18792}{17}.$$

If we apply the "point-slope formula" for the equation of a line, we arrive at the equation:

$$y = \frac{18792}{17}(x - 1970) + 9521. \tag{1.4.11}$$

The graph of this line will pass through the two points R and S in Figure 1.4.1. We can describe ALL points on the graph of this equation using (1.4.6); here is how we would describe the portion relevant to the years between 1970 and 2000:

$$\text{men's earning curve} = \left\{\text{all points } \left(x, \frac{18792}{17}(x - 1970) + 9521\right) \text{ where } 1970 \leq x \leq 2000\right\}$$

We sketch the graph below, indicating two new points T and U.

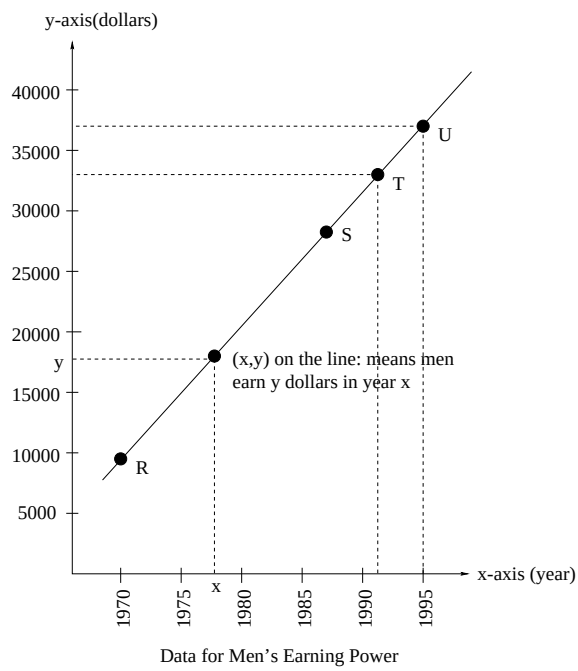


Figure 1.4.5: Linear model of Men's Earning Power

We can use the model in (1.4.11) to make predictions of two different sorts: (i) predict earnings at some date, or (ii) predict when a desired value for earnings will occur. For example, let's graphically discuss the earnings in 1995:

- Draw a vertical line $x = 1995$ up to the graph and label the intersection point U.
- Draw a horizontal line ℓ through U. The line ℓ crosses the y axis at the point

$$\frac{18792}{17}(\underline{1995} - 1970) + 9521 = 37156.$$

- The coordinates of the point $U = (1995, 37156)$.

Conclude that \$37,156 is the Men's Earning Power in 1995. For another example, suppose we wanted to know when men's earning power will equal \$33,000? This means we seek a data point T on the

men's earning curve whose y-coordinate is 33,000. By (1.4.11), T has the form

$$T = \left(x, \frac{18792}{17}(x - 1970) + 9521\right).$$

We want this to be a data point of the form $(x, 33000)$. Setting these two points equal and equating the second coordinates leads to an algebra problem:

$$\begin{aligned} \frac{18792}{17}(x - 1970) + 9521 &= 33000 \\ x &= 1991.24. \end{aligned}$$

This means men's earning power will be \$33,000 at the end of the first quarter of 1991. Graphically, we interpret this reasoning as follows:

- Draw a horizontal line $y = 33000$ and label the intersection point T on the model.
- Draw a vertical line ℓ through T. The line ℓ crosses the x axis at the point 1991.24.
- The coordinates of the point $T = (1991.24, 33000)$.

In the exercises, you will be asked to show that the women's earning power model is given by the equation

$$y = \frac{12915}{17}(x - 1970) + 5616.$$

Using the two linear models for the earning power of men and women, are women gaining on men? You will also be asked to think about this question in the exercises.

What's Needed to Build a Linear Model?.

As we progress through this text, a number of different "types" of mathematical models will be discussed. We will want to think about the information needed to construct that particular kind of mathematical model. Why would we care? For example, in a laboratory context, if we knew a situation being studied was given by a linear model, this would effect the amount of data collected. In the case of linear models, we can now make this useful statement:

A linear model is completely determined by:

1. *One data point and a slope (a rate of change), or* (1.4.12)
2. *Two data points, or*
3. *An intercept and a slope (a rate of change).*

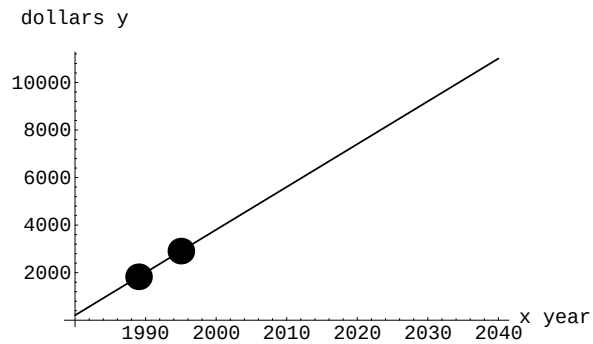
Linear Application Problems.

Example 1.4.13 *The yearly resident tuition at the University of Washington was \$1827 in 1989 and \$2907 in 1995. Assume that the tuition growth at the UW follows a linear model. What will be the tuition in the year 2000? When will yearly tuition at the University of Washington be \$10,000?*

Solution. If we consider a coordinate system where the x-axis represents the year and the y-axis represents dollars, we are given two data points: $P = (1989, 1827)$ and $Q = (1995, 2907)$. Using the two-point formula for the equation of line through P and Q, we obtain the equation

$$y = 180(x - 1989) + 1827;$$

the graph of this equation gives a line through the given points as pictured below:



If we let $x = 2000$, we get $y = \$3807$, which tells us the tuition in the year 2000. On the other hand, if we set the equation above equal to $\$10,000$, we can solve for x:

$$\begin{aligned} 10000 &= 180(x - 1989) + 1827 \\ 8173 &= 180(x - 1989) \\ 2034.4 &= = x. \end{aligned}$$

Conclude the tuition is \$10,000 in the year 2035. □

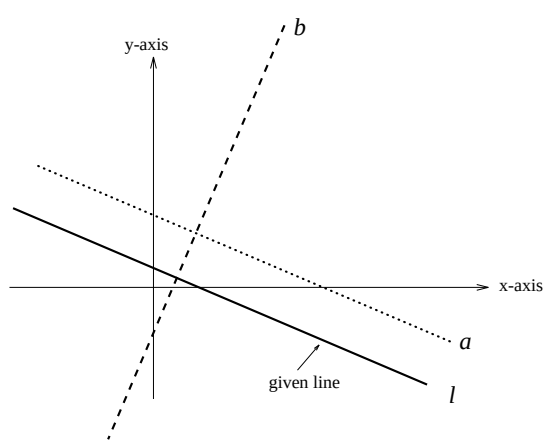
Perpendicular and Parallel Lines.

Here is a useful fact to keep in mind; an exercise at the end of the section will outline a justification.

Parallel and Perpendicular Lines 1.4.14: *Two non-vertical lines in the plane are parallel exactly when they both have the same slope. Two non-vertical lines are perpendicular exactly when their slopes are negative reciprocals of one another.*

Example 1.4.15:

Let ℓ be a line in the plane passing through the points $(1, 1)$ and $(6, -1)$. Find a linear equation whose graph is a line parallel to ℓ passing through 5 on the y-axis. Find a linear equation whose graph is perpendicular to ℓ and passes through $(4, 6)$.



Solution. Letting $P = (1,1)$ and $Q = (6,-1)$, apply the "two point formula":

$$y = \frac{-2}{5}(x-1)+1 = -\frac{2}{5}x + \frac{7}{5}.$$

The graph of this equation will be ℓ. This equation is in slope intercept form and we can read off that the slope is $m = \frac{-2}{5}$. The desired line a is parallel to ℓ; it must have slope $m = -\frac{2}{5}$ and y-intercept 5. Plugging into the "slope intercept form":

$$y = \frac{-2}{5}x + 5.$$

The desired line b is a line perpendicular to ℓ (so it's slope is $m' = \frac{-1}{\frac{-2}{5}} = \frac{5}{2}$) and passes through the point (4,6), so we can use the "point slope formula":

$$y = \frac{5}{2}(x-4)+6.$$

□

Intersecting Curves II (The Quadratic Formula).

We have already encountered problems which require that we investigate the intersection of two curves in the plane. Ultimately, this reduces to solving a system of two (or more) equations in the variables x and y. A useful tool, when working with equations involving squared terms (i.e. x^2 or y^2), is the *quadratic formula.*

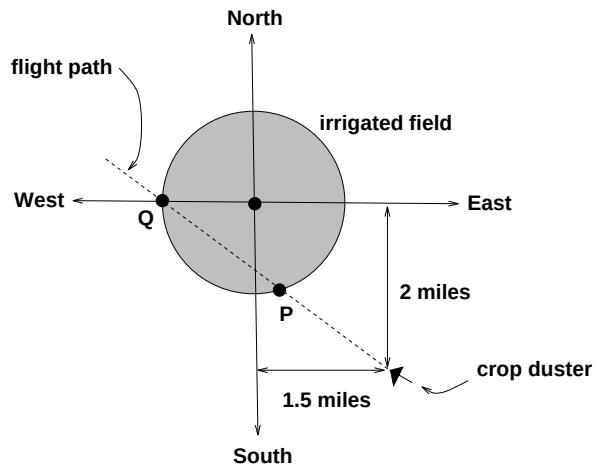
Quadratic Formula 1.4.16: *Consider the equation $az^2 + bz + c = 0$, where a, b, c are constants. The solutions for this equation are given by the formula*

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solutions are real numbers if and only if $b^2 - 4ac \geq 0$.

The next example illustrates a typical application of the quadratic formula. In addition, we describe a very useful technique for finding the shortest distance between a "line" and a "point".

Example 1.4.17: *A crop dusting airplane flying 120 mph is spotted 2 miles South and 1.5 miles East of the center of a circular irrigated field. The irrigated field has a radius of 1 mile. Impose a coordinate system as pictured. The flight path of the duster is a straight line passing over the labeled points P and Q. Assume that the point Q where the plane exits the airspace above the field is the Western-most location of the field. Answer these questions:*
(a) Find a linear equation whose graph is the line along which the crop duster travels.
(b) Find the location P where the crop duster enters airspace above the irrigated field.
(c) How much time does the duster spend flying over the irrigated field?
(d) Find the shortest distance from the flight path to the center of the irrigated field.



Solution. (a) Take $Q = (-1, 0)$ and $S = (1.5, -2)$= duster spotting point. Construct a line through Q and S. The slope is $-0.8 = m$ and the line equation becomes: $y = -0.8x - 0.8$.

(b) The equation of the boundary of the irrigated region is $x^2 + y^2 = 1$. We need to solve this equation AND the line equation $y = -0.8x - 0.8$ simultaneously. Plugging the line equation into the unit circle equation gives:

$$\begin{aligned} x^2 + (-0.8x - 0.8)^2 &= 1 \\ x^2 + 0.64x^2 + 1.28x + 0.64 &= 1 \\ 1.64x^2 + 1.28x - 0.36 &= 0 \end{aligned}$$

Apply quadratic formula and find $x = -1, 0.2195$. Conclude x coordinate of P is 0.2195. To find y coordinate, plug into line equation in (a) and get $y = -0.9756$. Conclude $P = (0.2195, -0.9756)$

(c) Find distance from P to Q by distance formula:

$$d = \sqrt{(-1 - 0.2195)^2 + (0 - (-0.9756))^2} = 1.562 \text{ miles}$$

Now, $\frac{1.562 \text{ miles}}{120 \text{ mph}} = 0.01302 \text{ hours} = 47 \text{ seconds}$

(d) The idea is to construct a line perpendicular to the flight path passing through the origin of the coordinate system. This line will have slope $m = -\frac{1}{-0.8} = 1.25$. So this perpendicular line has equation $y = 1.25x$. Intersecting this line with the flight path gives us the point closest to the center of the field. The x coordinate of this point is found by setting the two line equations equal and solving:

$$\begin{aligned} -0.8x - 0.8 &= 1.25x \\ x &= -0.3902 \end{aligned}$$

This means the closest point on the flight path is (-0.39,-0.49). Apply the distance formula and get the shortest distance to the flight path is $d = \sqrt{(-0.39)^2 + (-0.49)^2} = 0.6263$. □

Problems

1. This exercise emphasizes the "mechanical aspects" of working with linear equations:
   (a) Find the equation of a line passing thru the points $(1, -1)$ and $(-2, 4)$.
   (b) Find the equation of a line passing thru the point $(-1, -2)$ with slope $m = 40$.
   (c) Find the equation of a line with y-intercept $b = -2$ and slope $m = -2$.
   (d) Find the equation of a line passing thru the point $(4, 11)$ and having slope $m = 0$.
   (e) Find the equation of a line perpendicular to the line in a. and passing thru (1,1).
   (f) Find the equation of a line parallel to the line in b. and having y-intercept $b = -14$.
   (g) Find the slope of the line having equation $3x + 4y = 7$.
   (h) Find the equation of a line crossing the x-axis at $a = 1$ and having slope $m = 1$.

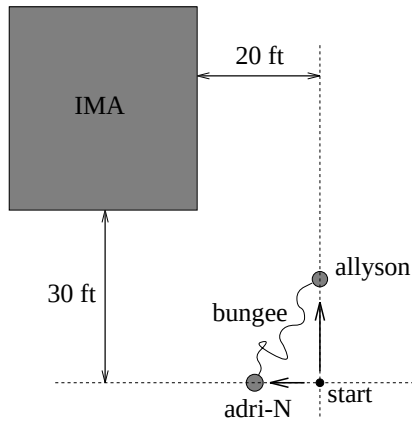
2. The (average) sale price for single family property in Seattle and Port Townsend is tabulated below:

YEAR	SEATTLE	PORT TOWNSEND
1970	\$38,000	\$8400
1990	\$175,000	\$168,400

   (a) Find a linear model relating the year x and the sales price y for a single family property in Seattle.
   (b) Find a linear model relating the year x and the sales price y for a single family property in Port Townsend.
   (c) Sketch the graph of both modeling equations in a common coordinate system; restrict your attention to $x \geq 1970$.
   (d) What is the sales price in Seattle and Port Townsend in 1983 and 1998?
   (e) When will the average sales price in Seattle and Port Townsend agree and what is this price?
   (f) When will the average sales price in Port Townsend be \$15,000 less than the Seattle sales price? What are the two sales prices at this time?
   (g) Is the Port Townsend sales price ever double the Seattle sales price?

3. Consider the equation: $\alpha x^2 + 2\alpha^2 x + 1 = 0$. Find the values of x that make this equation true (your answer will involve α). Find values of α that make this equation true (your answer will involve x).

4. Allyson and Adrienne have decided to connect their ankles with a *bungee cord*; one end is tied to each persons ankle. The cord is 30 feet long, but can stretch up to 90 feet. They both start from the same location. Allyson moves 10 ft/sec and Adrienne moves 8 ft/sec in the directions indicated. Adrienne stops moving at time $t = 5.5$ sec, but Allyson keeps on moving 10 ft/sec in the indicated direction.
   (a) Sketch an accurate picture of the situation at time $t = 7$ seconds. Make sure to label the locations of Allyson and Adriene; also, compute the length of the bungee cord at $t = 7$ seconds.
   (b) Where is Allyson when the bungee reaches its maximum length?

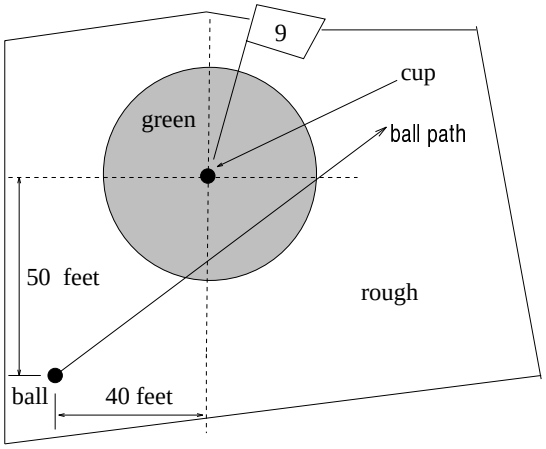


5. Complete the following table; in many cases there may be several possible correct answers:

Equation	slope	y-intercept	point on line	point on line
$y = 2x + 1$				
			(3,-4)	(-1,7)
	-2	1		
	$\frac{1}{2}$		(0,1)	

6. Draw the graphs of $x = -1$ and $y = 2$ in the xy coordinate system. An ant starts at the location $(6, 2)$ and moves to the left along the line $y = 2$. Assume the position of the ant after t seconds is the point $P(t) = (6 - 2t, 2)$. At the same instant, a spider starts at the location $(-1, -3)$ and moves upward along the line $x = -1$. Assume the position of the spider after t seconds is the point $Q(t) = (-1, -3 + t)$.
   (a) Find a formula for the distance between the ant and spider at time t seconds.
   (b) When is the distance between the two bugs exactly 4 units? Where are the bugs located?

7. The cup on the 9th hole of a golf course is located dead center in the middle of a circular green which is 70 feet in diameter. Your ball is located as in the picture below:



The ball follows a straight line path and exits the green at the right-most edge. Assume the ball travels 10 ft/sec.

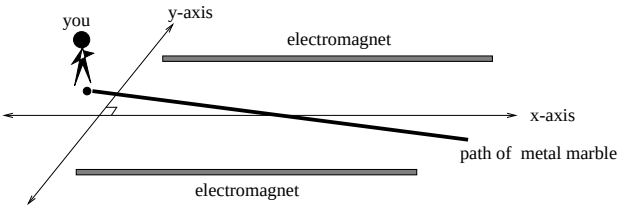
(a) Where does the ball enter the green?
(b) When does the ball enter the green?
(c) How long does the ball spend inside the green?
(d) Where is the ball located when it is closest to the cup and when does this occur.

Chapter 2

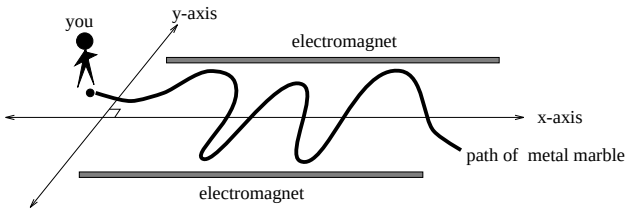
Functions

The power of mathematics when applied in the sciences is it's ability to precisely describe relationships between two (or more) changing quantities.

To set the stage, consider this contrived experiment: Two parallel electromagnets are lying on a level floor. An electromagnet is a device that can be turned "on" and "off" with a switching device and can have varying magnetic strength. If we have these magnets turned "off" and roll a metal marble as pictured at left (a 3D view), it will follow a simple straight line path; in this case, modeling the path of the marble involves linear equations. But, if instead we turn these magnets "on" and "off" with varying strengths, then the rolled marble is going to experience a varying attraction toward the two magnets and may follow a more exotic path as pictured at right (a 3D view):



MAGNETS OFF



MAGNETS USED

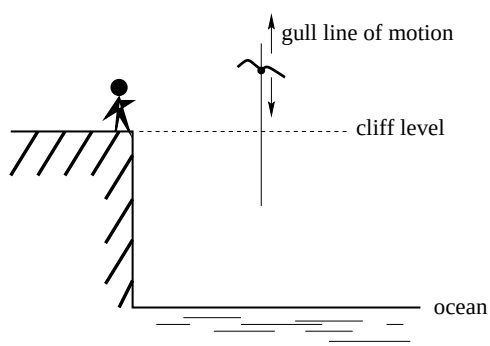
Now we are confronted with a natural question: *How can we "describe" the path in the right-hand picture?* Answering this question requires a certain amount of terminology and ultimately leads to the concept of a *function*. This is the most important concept in the book.

2.1 Functions and Graphs

Pictures are certainly important in the work of an *architect*, but it is perhaps less evident that visual aides can be powerful tools for solving mathematical problems. If we start with an equation and attach a picture, then the mathematics can come to life. This adds a new dimension to both interpreting and solving problems. One of the real triumphs of modern mathematics is a theory connecting pictures and equations via the concept of a *graph*. This transition from "equation" to "picture" (called graphing) and its usefulness (called graphical analysis) are the theme of the next two sections. The importance of these ideas is HUGE and cannot be overstated. Every moment spent studying these ideas will pay back dividends in this course and any future mathematics, science or engineering courses.

Relating Data, Plots and Equations.

Imagine you are standing high atop an oceanside cliff and spot a seagull hovering in the air-current. Assuming the gull moves up and down along a vertical line of motion, how can we best describe it's location at time t seconds?

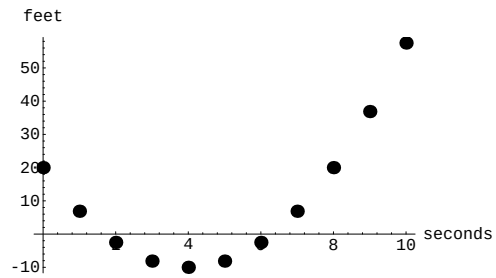


There are three different (but closely linked) ways to describe the location of the gull:

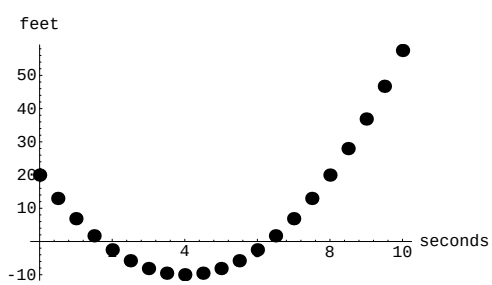
- a table of **data** of the gull's height above cliff level at various times t;
- a **plot** of the data in a "time" (seconds) vs. "height" (feet) coordinate system;
- an **equation** relating time t (seconds) and height s (feet).

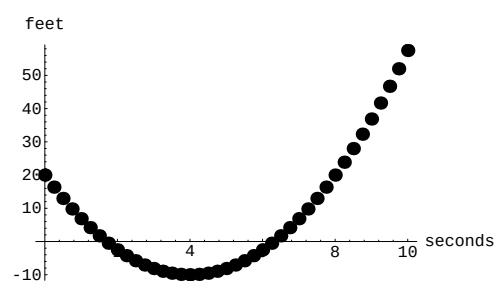
To make sure we really understand how to pass back and forth between these three descriptive modes, imagine we have tabulated (below) the height of the gull above cliff level at one-second time intervals for a 10 second time period. Here, a "negative height" means the gull is below cliff level. We can try to visualize the meaning of this data by plotting these 11 data points (t, s) in a time (sec.) vs. height (ft.) coordinate system.

gull height (feet)					
t (sec)	s (ft)	t (sec)	s (ft)	t (sec)	s (ft)
0	20	4	-10	8	20
1	6.88	5	-8.12	9	36.88
2	-2.5	6	-2.5	10	57.5
3	-8.12	7	6.88		



We can improve the quality of this description by increasing the number of data points. For example, if we tabulate the height of the gull above cliff level at 1/2 second or 1/4 second time intervals (over the same 10 second time period), we might get these two plots:





We have focused on how to go from data to a plot, but the reverse process is just as easy:

> *A point (t, s) in any of these three plots is interpreted to mean that the gull is s feet above cliff level at time t seconds.*

Furthermore, increasing the amount of data, we see how the plotted points are "filling in" a portion of a parabola. Of course, it is way too tedious to create longer and longer tables of data. What we really want is a "formula" (think of it as a prescription) that tells us how to produce a data point for the gull's height at any given time t. If we had such a formula, then we could completely dispense with the tables of data and just use the formula to crank out data points. For example, look at this equation involving the variables t and s:

$$s = \frac{15}{8}(t-4)^2 - 10.$$

If we plug in $t = 0, 1, 2, 9, 10$, then we get out $s = 20, 6.88, -2.5, 36.88, 57.5$, respectively; this was some of our initial tabulated data. This same equation produces ALL of the data points for the other two plots, using 1/2 second and 1/4 second time intervals. (Granted, we have swept under the rug the issue of *"...where the heck the equation comes from..."*; that is a consequence of mathematically modeling the motion of this gull. Right now, we are focusing on how the equation relates to the data and the plot, assuming the equation is in front of us to start with.) In addition, it is very important to notice that having this equation produces an infinite number of data points for our gull's location, since we can plug in any t value between 0 and 10 and get out a corresponding height s. In other words, the equation is A LOT more powerful than a finite (usually called *discrete*) collection of tabulated data.

What is a Function?.

Our lives are chocked full of examples where two changing quantities are related to one another:

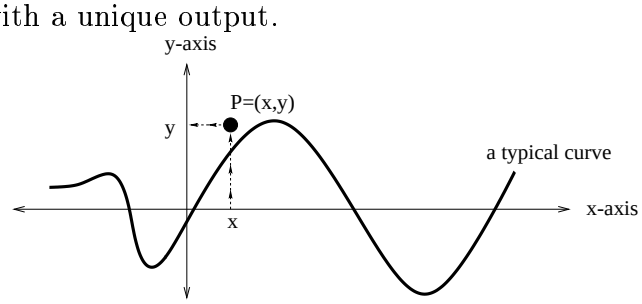
- The cost of *postage* is related to the *weight* of the item.
- The *value* of an investment will depend upon the *time* elapsed.
- The *population* of cells in a growth medium will be related to the amount of *time* elapsed.
- The *speed* of a chemical reaction will be related to the *temperature* of the reaction vessel.

In all such cases, it would be beneficial to have a "procedure" whereby we can assign a unique output value to any acceptable input value. For example, given the time elapsed (an input value), we would like to predict a unique future value of an investment (the output value). Informally, this leads to the broadest (and hence most applicable) definition of what we will call a *function*:

A function is a procedure for assigning a unique output to any allowable input.

The key word here is "procedure". Our discussion of the hovering seagull in §2.1.1 highlights three ways to produce such a "procedure" using data, plots of curves and equations. Why is this so?

- A table of data, by it's very nature, will relate two columns of data: The output and input values are listed as column entries of the table and reading across each row is the "procedure" which relates an input with a unique output.
- Given a curve as pictured, consider the "procedure" which associates to each x on the horizontal axis the y coordinate of the pictured point P on the curve.



- Given an equation relating two quantities x and y, plugging in a particular x value and going through the "procedure" of algebra often produces a unique output value y.

The definition of a function (equation viewpoint).

Now we focus on giving a precise definition of a function, in the situation when the "procedure" relating two quantities is actually given by an equation. Keep in mind, this is only one of three possible ways to describe a function; we could alternatively use tables of data or the plot of a curve. We focus on the equation viewpoint first, since it is no doubt the most familiar.

If we think of x and y as related physical quantities (e.g. *time* and *distance*), then it is sometimes possible (and often desirable) to express one of the variables in terms of the other. For example, by simple arithmetic, the equations

$$3x + 2y = 4 \qquad x^2 - x = \tfrac{1}{2}y - 4 \qquad y\sqrt{x^2+1} = 1,$$

can be rewritten as equivalent equations

$$y = \tfrac{1}{2}(4 - 3x) \qquad 2x^2 - 2x + 8 = y \qquad y = \tfrac{1}{\sqrt{x^2+1}}.$$

This leads to THE MOST IMPORTANT MATH DEFINITION IN THE WORLD:

Function Definition 2.1.1: *A function is a package, consisting of three parts:*

- *An equation of the form*

  $$y = \text{"a mathematical expression only involving the variable } x\text{"},$$

  which we usually indicate via the shorthand notation $y = f(x)$. This equation has the very special property that each time we plug in an x value, it produces exactly one (a unique) y value. We call the mathematical expression $f(x)$ the rule.
- *A set D of x-values we are allowed to plug into $f(x)$, called the domain of the function.*
- *The set R of output values $f(x)$, where x varies over the domain, called the range of the function.*

Anytime we have a function $y = f(x)$, we refer to x as the *independent variable* (the "input data") and y as the *dependent variable* (the "output data"). This terminology emphasizes the fact that we have freedom in the values of x we plug in, but once we specify an x value, the y value is uniquely determined by the rule $f(x)$.

In practice, we usually provide the rule and domain but do not specify the range upfront. This is ok, because given the domain and rule, the range is uniquely determined; i.e. the range will be the set of output values you get by plugging in domain values.

Examples 2.1.2: (i) The equation $y = -2x + 3$ is in the form $y = f(x)$, where the rule is $f(x) = -2x + 3$. Once we specify a domain of x values, we have a function. For example, we could let the domain be all real numbers. (ii) Take the same rule $f(x) = -2x + 3$ from (i) and the domain be all non-negative real numbers. This describes a function. However, the functions

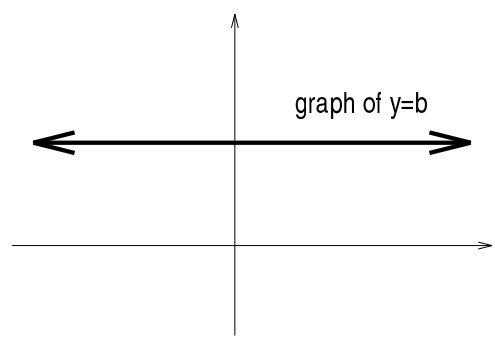
$$f(x) = -2x + 3 \text{ on the domain of } \underline{\text{all non-negative real numbers}}$$

and

$$f(x) = -2x + 3 \text{ on the domain of } \underline{\text{all real numbers}} \text{ (from (i))}$$

are *different*, even though they share the same rule; this is because their domains differ! This example illustrates the idea of what is called a *restricted domain*. In other words, we started with the function in (i) on the domain of all real numbers, then we "restricted" to the subset of non-negative real numbers.

(iii) The equation $y = b$, where b is a constant, defines a function on the domain of all real numbers, where the rule is $f(x) = b$; we call these the *constant functions*. Recall, in §1.3, we observed that the solutions of the equation $y = b$, plotted in the xy coordinate system, will give a horizontal line. For example, if $b = 0$, you get the horizontal axis.



(iv) Consider the equation $y = \frac{1}{x}$, then the rule $f(x) = \frac{1}{x}$ defines a function, as long as we do not plug in $x = 0$. For example, take the domain of to be the non-zero real numbers.

(v) Consider the equation $y = \sqrt{1 - x^2}$. Before we start plugging in x values, we want to know the expression under the radical symbol (square root symbol) is non-negative; this insures the square root is a real number. This amounts to solving an inequality equation: $0 \leq 1 - x^2$; i.e. $-1 \leq x \leq 1$. These remarks show that the rule $f(x) = \sqrt{1 - x^2}$ defines a function, where the domain of x values is $-1 \leq x \leq 1$.

Typically, the domain of a function $y = f(x)$ will either be the entire number line, an interval on the number line or a finite union of such intervals. We summarize the notation used to represent intervals in the table below.

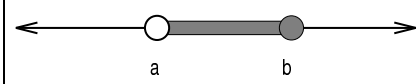
Common Intervals on the Number Line		
Description	Symbolic Notation	Picture
All numbers x between a and b, x possibly equal to either a or b	$a \leq x \leq b$	
All numbers x between a and b, $x \neq a$ and $x \neq b$	$a < x < b$	
All numbers x between a and b, $x \neq b$ and x possibly equal to a	$a \leq x < b$	
All numbers x between a and b, $x \neq a$ and x possibly equal to b	$a < x \leq b$	

Table 2.1.1: Interval Notations

We can interpret a function as a "prescription" that takes a given x value (in the domain) and produces a single unique y value (in the range). We need to be really careful and not fall into the trap of thinking that every equation in the world is a function. For example, if we look at this equation

$$x + y^2 = 1$$

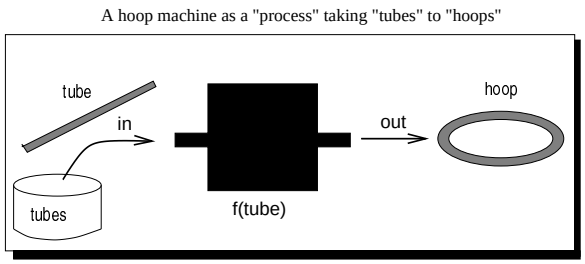
and plug in $x = 0$, the equation becomes

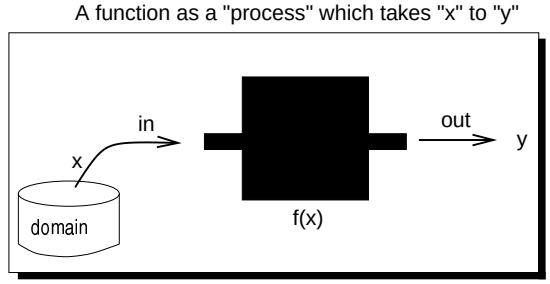
$$y^2 = 1.$$

This equation has two solutions, $y = \pm 1$, so the conclusion is that plugging in $x = 0$ does NOT produce a single output value. This violates one of the conditions of our function definition, so the equation $x + y^2 = 1$ is NOT a function in the independent variable x. Notice, if you were to try and solve this equation for y in terms of x, you'd first write $y^2 = 1 - x$ and then take a square root (to isolate y); but the square root introduces TWO roots, which is just another way of reflecting the fact there can be two y values attached to a single x value. Alternatively, you can solve the equation for x in terms of y, getting $x = 1 - y^2$; this shows the equation does define a function $x = g(y)$ in the independent variable y.

The definition of a function (conceptual viewpoint).

Conceptually, you can think of a function as a "process": An allowable input goes into a "black box" and out pops a unique new value denoted by the symbol $f(x)$; compare this with a machine that makes "hula-hoops":





This viewpoint is useful when problem solving, where a function is usually described in words.

Examples 2.1.3: *Here are four examples of relationships that are functions:*

(i) *The total amount of water used by a household since midnight on a particular day.* Let y be the total number of gallons of water used by a household between 12:00am and a particular time t; we will use time units of "hours". Given a time t, the household will have used a specific (unique) amount of water, call it $S(t)$. Then $y = S(t)$ defines a function in the independent variable t with dependent variable y. The domain would be $0 \leq t \leq 24$ and the largest possible value of $S(t)$ on this domain is $S(24)$. This tells us that the range would be the set of values $0 \leq y \leq S(24)$.

(ii) *The height of the center of a basketball as you dribble, depending on time.* Let s be the height of the basketball center at time t seconds after you start dribbling. Given a time t, if we freeze the action the center of the ball has a single unique height above the floor, call it $h(t)$. So, the height of the basketball center is given by a function $s = h(t)$. The domain would be a given interval of time you are dribbling the ball; for example, maybe $0 \leq t \leq 2$ (the first 2 seconds). In this case, the range would be all of the possible heights attained by the center of the basketball during this 2 seconds.

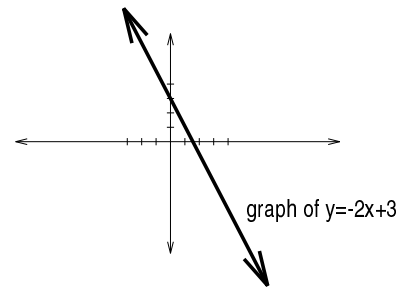
(iii) *The state sales tax due on a taxable item.* Let T be the state tax (in dollars) due on a taxable item that sells for z dollars. Given a taxable item that costs z dollars, the state tax due is a single unique amount, call it $W(z)$. So, $T = W(z)$ is a function, where the independent variable is z. The domain could be taken to be $0 \leq z \leq 1,000,000$, which would cover all items costing up to one-million dollars. The range of the function would be the set of all values $W(z)$, as z ranges over the domain.

(iv) *The speed of a chemical reaction depending on the temperature.* Let v be the speed of a particular chemical reaction and T the temperature in Celsius °C. Given a particular temperature T, one could experimentally measure the speed of the reaction; there will be a unique speed, call it $r(T)$. So, $v = r(T)$ is a function, where the independent variable is T. The domain could be taken to be $0 \leq T \leq 100$, which would cover the range of temperatures between the freezing and boiling points of water. The range of the function would be the set of all speeds $r(T)$, as T ranges over the domain.

The graph of a function.

Let's start with a concrete example; the function $f(x) = -2x + 3$ on the domain of all real numbers. We discussed this example in (2.1.2). Plug in the specific x values, where $x = -1, 0, 1, 2$ and tabulate the resulting y values of the function:

x	y	point (x, y)
-1	5	(-1,5)
0	3	(0,3)
1	1	(1,1)
2	-1	(2,-1)
$\vdots$	$\vdots$	$\vdots$
x	$-2x + 3$	$(x, -2x + 3)$



This tells us is that the points $(0, 3), (1, 1), (2, -1), (-1, 5)$ are solutions of the equation $y = -2x + 3$. For example,

$$\text{If } y = -2x + 3, x = 0, y = 3, \text{ then } 3 = -2 \cdot 0 + 3 \text{ (which is true)},$$

or

$$\text{If } y = -2x + 3, x = 2, y = -1, \text{ then } -1 = -2 \cdot 2 + 3 \text{ (which is true)},$$

etc. In general, if we plug in x we get out $-2x+3$, so the point $(x, -2x+3)$ is a solution to the function equation $y = f(x)$. We can plot all of these solutions in the xy-coordinate system. The set of points we obtain, as we vary over all x in the domain, is called the set of *solutions* of the equation $y = -2x + 3$:

$$\textit{Solutions} = \{(x, -2x + 3) | x \text{ any real number}\}.$$

Notice that plotting these points produces a line of slope $m = -2$ with y-intercept 3. In other words, the graph of the function $f(x) = -2x + 3$ is the same as the graph of the equation $y = -2x + 3$, as we discussed in §1.4.

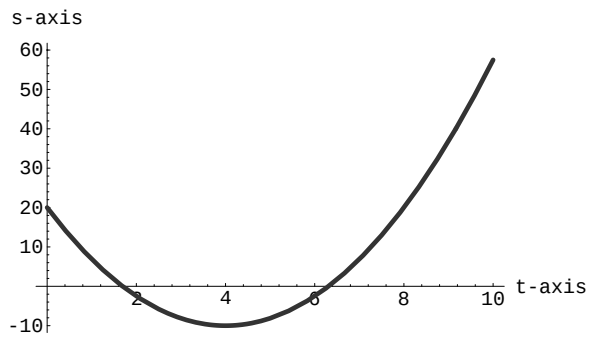
In general, by definition, we say that a point (x, y) is a *solution* to the function equation $y = f(x)$ if plugging x and y into the equation gives a true statement.

How can we find ALL the solutions of the equation $y = f(x)$?

In general, the definition of a function is "rigged" so it is easy to describe all solutions of the equation $y = f(x)$: *Each time we specify an x value (in the domain), there is only one y value, namely $f(x)$. This means the point $P = (x, f(x))$ is the ONLY solution to the equation $y = f(x)$ with first coordinate x.* We define the *graph* of the function $y = f(x)$ to be the plot of all solutions of this equation (in the xy coordinate system). It is common to refer to this as either the *graph of $f(x)$* or the *graph of f*.

$$Graph = \{(x, f(x)) \mid x \text{ in the domain}\}. \tag{2.1.4}$$

Example 2.1.5: *The function* $s = h(t) = \frac{15}{8}(t-4)^2 - 10$ *defines a function in the independent variable* t*. If we restrict to the domain* $0 \le t \le 10$*, then using a graphing device (or the discussion in §2.3), we see that the graph is a portion of a parabola, as pictured at right. Verify that the data points discussed in the seagull example (in §2.1.1) all lie on this parabola.*

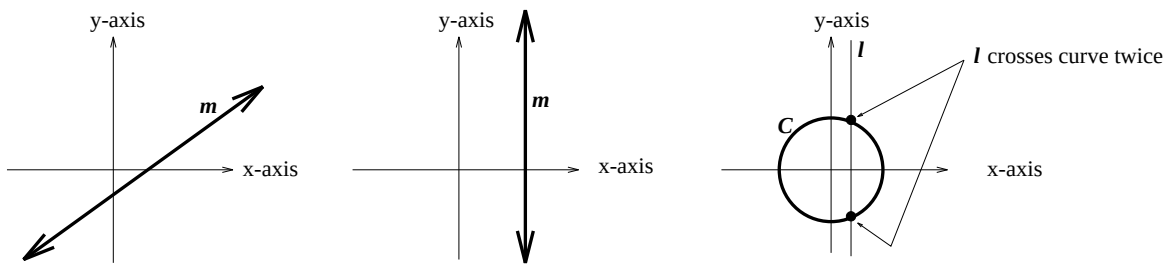


The vertical line test.

There is a pictorial aspect of the graph of a function which is very revealing: Since $(x, f(x))$ is the only point on the graph with first coordinate equal to x, a vertical line passing through x on the x-axis (with x in the domain) crosses the graph of $y = f(x)$ once and only once. This gives us a decisive way to test if a curve is the graph of a function.

Vertical Line Test 2.1.6: *Draw a curve in the* xy*-plane and specify a set* D *of* x*-values. Suppose every vertical line through a value in* D *intersects the curve exactly once. Then the curve is the graph of some function on the domain* D*. On the other hand, if we can find a single vertical line through some value in* D *that intersects the curve more than once, then the curve is not the graph of a function on the domain* D*.*

For example, draw any straight line m in the plane. By the vertical line test, if the line m is not vertical, m is the graph of a function. On the other hand, if the line m is vertical, then m is not the graph of a function. These two situations are illustrated below. As another example, consider the equation $x^2 + y^2 = 1$, whose graph is the unit circle and specify the domain D to be $-1 \le x \le 1$; recall (1.3.4). The vertical line passing through the point $(\frac{1}{2}, 0)$ will intersect the unit circle twice; by the vertical line test, the unit circle is **not** the graph of a function on $-1 \le x \le 1$.

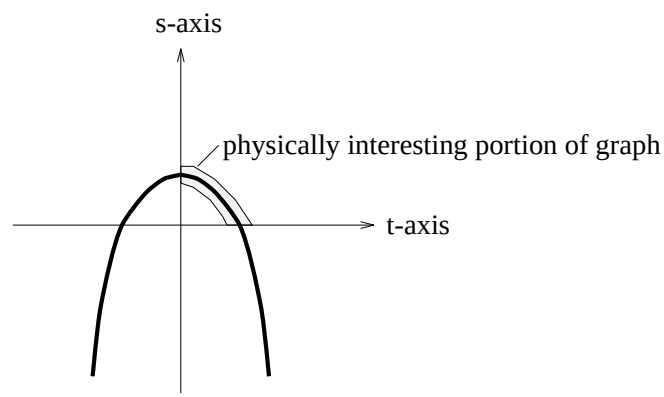


Imposed constraints.

In physical problems, it might be natural to *constrain* (meaning to "limit" or "restrict") the domain. As an example, suppose the height s (in feet) of a ball above the ground after t seconds is given by the function

$$s = h(t) = -16t^2 + 4.$$

We could look at the graph of the function in the ts-plane and we will review in §2.3 that the graph looks like a parabola. The physical context of this problem makes it natural to only consider the portion of the graph in the first quadrant; why? One way of specifying this quadrant would be to restrict the domain of possible t values to lie between 0 and $\frac{1}{2}$; notationally, we would write this constraint as $0 \leq t \leq \frac{1}{2}$.



<u>Linear Functions</u>.

A major goal of this course is to discuss several different kinds of functions. The work we did in §1.4 actually sets us up to describe one very useful type of function called a *linear function.* Back in section 1.4, we discussed how lines in the plane can be described using equations in the variables x and y. One of the key conclusions was:

A non-vertical line in the plane will be the graph of an equation $y = mx + b$, where m is the slope of the line and b is the y-intercept.

Notice that any non-vertical line will satisfy the conditions of the vertical line test, which means it must be the graph of a function. What is the function? The answer is to use the equation in x and y we already obtained in §1.4: The rule $f(x) = mx + b$ on some specified domain will have a line of slope m and y-intercept b as it's graph. We call a function of this form a *linear function.*

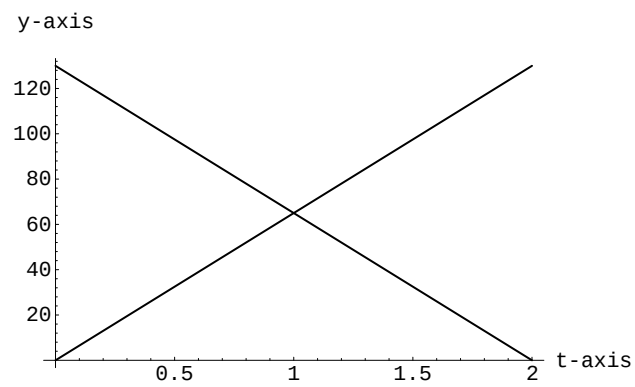
Examples 2.1.7: *You are driving 65 mph from the Kansas state line (mile marker 0) to Salina (mile marker 130) along I-35. Describe a linear function that calculates the mile marker you will be at after t hours. Describe another linear function that will calculate your distance from Salina after t hours.*

Solution. Define a function $d(t)$ to be the mile marker after t hours. Using "distance=rate×time", we conclude that $65t$ will be the distance traveled after t hours. Since we started at mile marker 0, $d(t) = 65t$ is the rule for the first function. A reasonable domain would be to take $0 \le t \le 2$, since it takes 2 hours to reach Salina.

For the second situation, we need to describe a different function, call it $s(t)$, that calculates your distance from Salina after t hours. To describe the rule of $s(t)$ we can use the previous work:

$$\begin{aligned} s(t) &= \text{(mile marker Salina)} - \\ &\quad \text{(your mile marker at } t \text{ hrs.)} \\ &= 130 - d(t) \\ &= 130 - 65t. \end{aligned}$$

For the rule $s(t)$, the best domain would again be $0 \le t \le 2$. We have graphed these two functions in the same coordinate system. (Which function goes with which graph?) □



Profit Analysis.

Let's give a first example of how to interpret the graph of a function in the context of an application.

> *A software company plans to bring a new product to market. The sales price per unit is \$15 and the expense to produce and market x units is* $\$100(1+\sqrt{x})$. *What is the profit potential?*

Two functions control the profit potential of the new software. The first tells us the *gross income*, in dollars, on the sale of x units. All of the costs involved in developing, supporting,

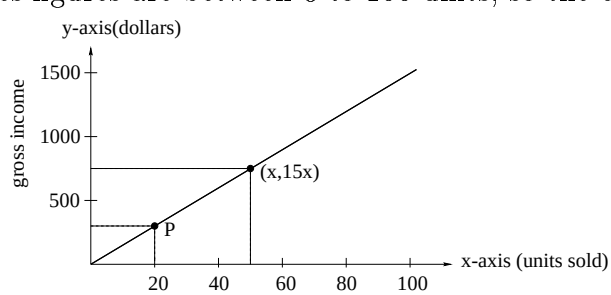
distributing and marketing x units are controlled by the *expense* equation (again in dollars):

$$g(x) = 15x \qquad \text{(gross income function)}$$

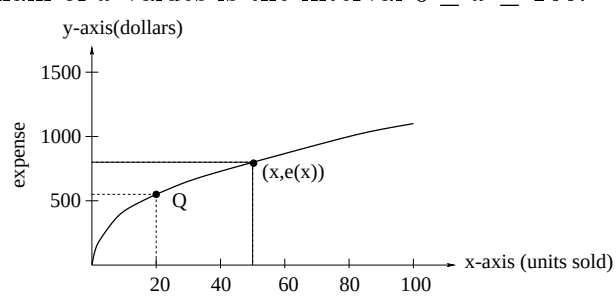
$$e(x) = 100(1+\sqrt{x}) \qquad \text{(expense function).}$$

A *profit* will be realized on the sale of x units whenever the gross income exceeds expenses; i.e. this occurs when $g(x) > e(x)$. A *loss* occurs on the sale of x units when expenses exceed gross income; i.e. this occurs when $e(x) > g(x)$. Whenever the sale of x units yields zero profit (and zero loss), we call x a *break-even point*; i.e. this occurs when $e(x) = g(x)$.

The above approach is "symbolic". Let's see how to study profit and loss visually, by studying the graphs of the two functions $g(x)$ and $e(x)$. To begin with, plot the graphs of the two individual functions in the xy-coordinate system. We will focus on the situation when the sales figures are between 0 to 100 units; so the domain of x values is the interval $0 \le x \le 100$.



Gross Income Graph



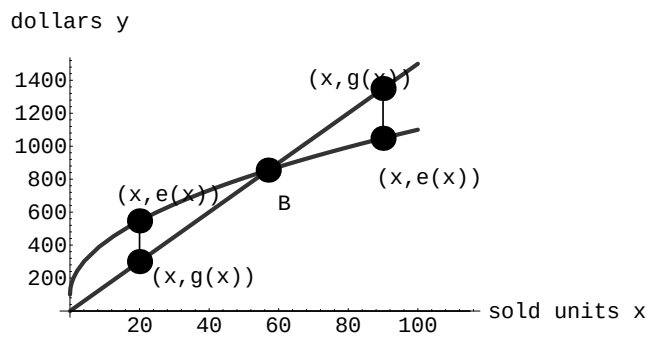
Expenses Graph

Given any sales figure x, we can graphically relate three things:

- x on the horizontal axis;
- a point on the graph of the gross income or expense function;
- y on the vertical axis.

If $x = 20$ units sold, there is a unique point $P = (20, g(20)) = (20, 300)$ on the gross income graph and a unique point $Q = (20, e(20)) = (20, 547)$ on the expenses graph. Since the y-coordinates of P and Q are the function values at $x = 20$, the height of the point above the horizontal axis is controlled by the function.

If we plot both graphs in the same coordinate system, we can visually study the distance between points on each graph, above x on the horizontal axis.



In the first part of this plot, the expense graph is above the income graph, showing a loss is realized; the exact amount of the loss will be $e(x) - g(x)$, which is the length of the pictured line segment. Further to the right, the two graphs cross at the point labeled "B"; this is the break-even point; i.e. expense and income agree, so there is zero profit (and zero loss). Finally, to the right of B the income graph is above the expense graph, so there is a profit; the exact amount of the profit will be $g(x) - e(x)$, which is the length of the right-most line segment. Our analysis will be complete, once we pin down the break-even point B. This amounts to solving the equation $g(x) = e(x)$.

$$\begin{aligned} 15x &= 100(1+\sqrt{x}) \\ 15x - 100 &= 100\sqrt{x} \\ 225x^2 - 3000x + 10000 &= 10000x \\ 225x^2 - 13000x + 10000 &= 0. \end{aligned}$$

Applying the quadratic formula, we get two answers: $x = 0.78, 57$.

Now, we face a problem: Which of these two solutions is the answer to the original problem? We are going to argue that only the second solution $x = 57$ gives us the break even point. What about the other "solution" at x=0.78? Try plugging $x = 0.78$ into the original equation: $15(0.78) \neq 100(1+\sqrt{0.78})$. What's happened? Well, when going from the second to the third line, both sides of the equation were squared. Whenever we do this, we run the risk of adding extraneous solutions. What should you do? After solving any equation, look back at your steps and ask yourself whether or not you may have added (or lost) solutions. In particular, be wary when squaring or taking the square root of both sides of an equation. Always check your final answer in the original equation.

We can now compute the coordinates of the break-even point using either function:

$$B = (57, g(57)) = (57, 855) = (57, e(57)).$$

Problems

1. In each of (a)-(h) decide which equations establish a function relationship between the independent variable x and the dependent variable y. Write out the rule for any functions.
   (a) $y = x + 1$
   (b) $2x - 3y = 5$
   (c) $x - 4 = 0$
   (d) $y + 2 = 7$
   e. $xy = 4$
   f. $x^2y = 4$
   g. $xy^2 = 4$
   h. $y - \sqrt{x^2 + 1} = 0$
2. This problem deals with the "mechanical aspects" of working with the rule of a function. For each of the functions listed in (a)-(c), calculate:
   (i) $f(0)=$
   (ii) $f(-2)=$
   (iii) $f(a)=$
   (iv) $f(x+3)=$
   (v) $f(\heartsuit)=$
   (vi) $f(\heartsuit + \triangle)=$
   (a) The function $f(x) = \frac{1}{2}(x - 3)$ on the domain of all real numbers.
   (b) The function $f(x) = x^2 + x + 1$ on the domain of all real numbers.
   (c) The function $f(x) = 4\pi^2$ on the domain of all real numbers.
3. For each function, complete this table with appropriate outputs.

Part	**(a)**	**(b)**	**(c)**
Function	$y = 4x - 7$	$y = x^2 + x + 1$	$y = \pi^3$
$f(0)$			
$f(1)$			
$f(a)$			
$f(a+1)$			
$f(x^2)$			

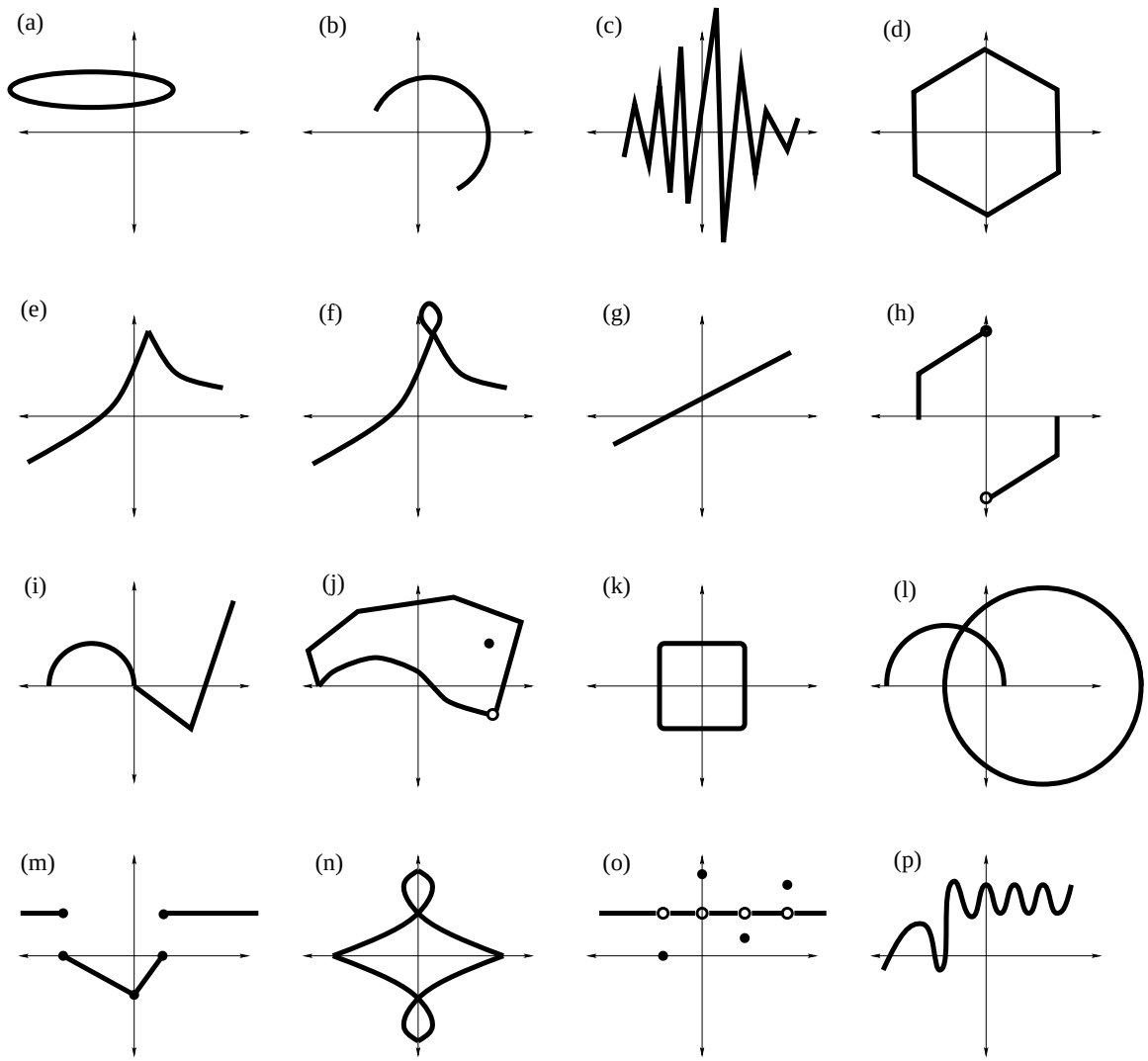
4. (a) Solve the equation
$$\frac{x^2}{9} + 4(y+1)^2 = 1$$
   for y in terms of x. You should get two different functions of x. Describe the largest possible domain of each function.
   (b) Let a and b be positive constants. Solve the equation
$$\left(\frac{x+1}{a}\right)^2 + \left(\frac{y-2}{b}\right)^2 = 1$$
   for y in terms of x. You should get two different functions of x. Describe the largest possible domain of each function.

(c) Let a and b be positive constants. Solve the equation

$$\left(\frac{x-1}{a}\right)^2 - \left(\frac{y+4}{b}\right)^2 = 1$$

for y in terms of x. You should get two different functions of x. Describe the largest possible domain of each function.

5. Which of these curves represent the graph of function? If the curve is not the graph of a function, describe what goes wrong and how you might "fix it". When you describe how to "fix" the graph, you are allowed to cut the curve into a finite number of pieces and study each piece individually. Many of these problems have more than one correct answer.

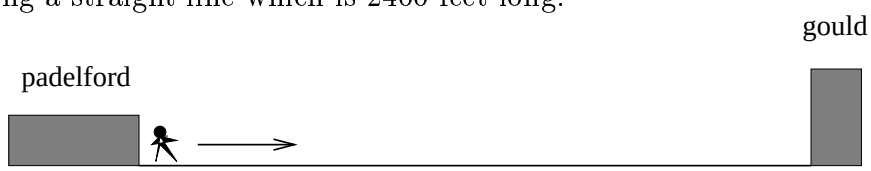


6. For each of the following functions, find the expression for

$$\frac{f(x+h)-f(x)}{h}.$$

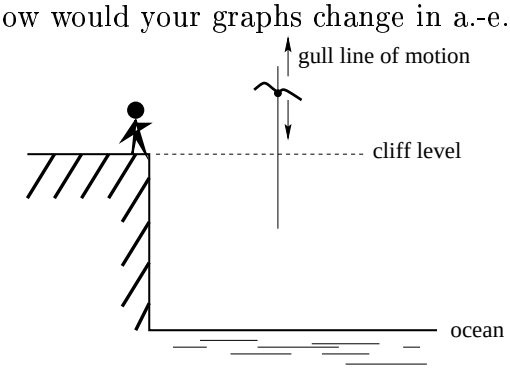
Simplify each of your expressions so that there is no h in the denominator.
(a) $f(x) = x^2 - 2x$.
(b) $f(x) = 2x + 3$

7. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s = d(t)$ which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes". Assume Dave's path to Gould Hall is along a straight line which is 2400 feet long.

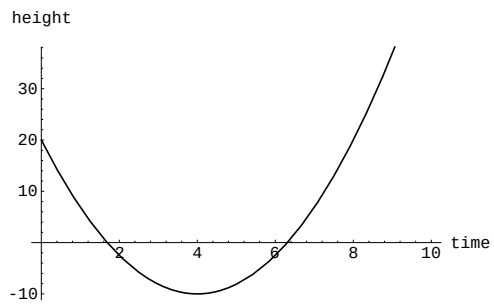


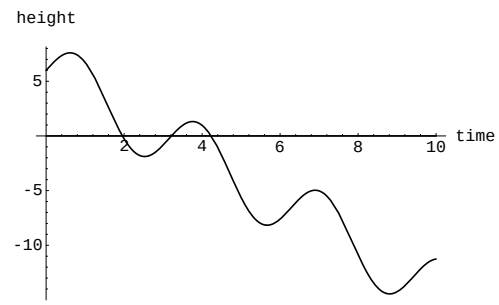
(a) Dave leaves Padelford Hall and walks at a constant speed until he reaches Gould Hall 10 minutes later.
(b) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.
(c) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.
(d) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Dave gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.
(e) Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of "distance from padelford" vs. "time" for both Angela and Dave.
(f) Suppose you want to sketch the graph of a *new* function $s = g(t)$ that keeps track of Dave's distance s from Gould Hall at time t. How would your graphs change in a.-e.?

8. Shannon is standing high atop an oceanside cliff and spots seagulls hovering in the air-current. Assume a gull moves up and down along a vertical line of motion, as indicated. The questions below deal with plots of functions giving the height of a gull above cliff level after t seconds.

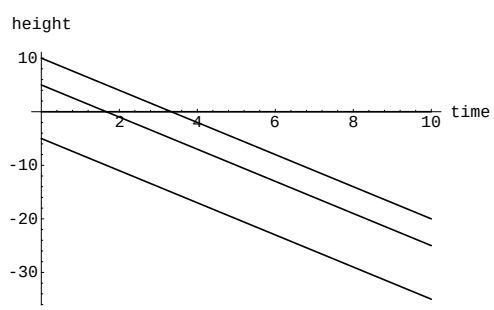


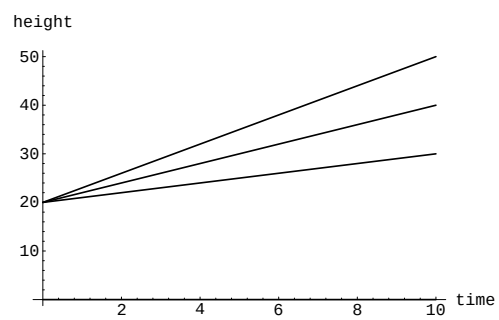
(a) Describe the motion of a gull given this two plots:



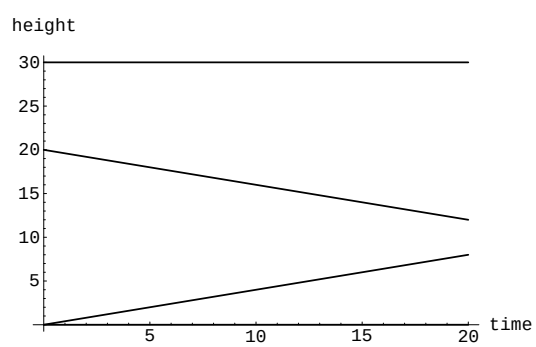


(b) Each plot simultaneously describes the motion of three different gulls. Describe what is happening in each scenario and what is the *vertical speed* of each gull?

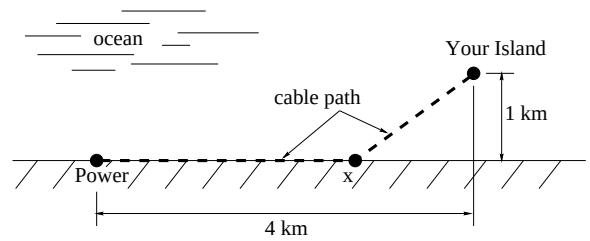




(c) Here you observe three different gulls simultaneously. What is happening in each case? Which gull has the largest and smallest vertical speed?



9. After winning the lottery, you decide to buy your own island. The island is located 1 km offshore from a straight portion of the mainland. You need to get power out to the island so you can listen to your extensive CD collection. A power sub-station is located 4 km from your island's nearest location to the shore. It costs \$50,000 per km to lay a cable in the water and \$30,000 per km to lay a cable over the land.



(a) Explain why we might as well assume that the cable follows the path indicated in the picture. In other words, explain why the path consists of two line segments, rather than a weird curved path. Also, as in the picture, explain why it is OK to assume the cable reaches shore to the right of the powerstation and the left of the island.

(b) Let x be the location downshore from the powerstation where the cable reaches the land. Find a function in the variable x that computes the cost to lay a cable out to your island.

2.2 Graphical Analysis

We ended the previous section with an in-depth look at a "Profit Analysis Problem". In that discussion, we looked at the graphs of the relevant functions and used these as visual aids to help us answer the questions posed. This was a concrete illustration of what is typically called "graphical analysis of a function". This is a fundamental technique we want to carry forward throughout the course. Let's highlight the key ideas for future reference.

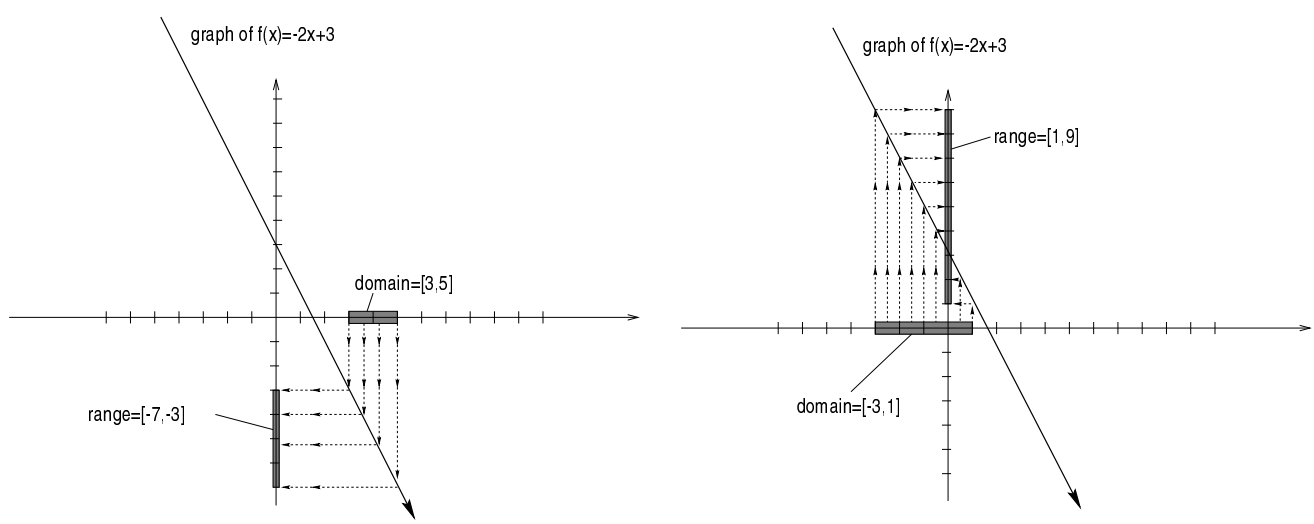
Visual Analysis of a Graph.

A variety of information can be visually read off of a function graph. To see this, we ask ourselves the following question: *What is the most basic qualitative feature of a graph?* To answer this, we need to return to the definition of the graph (see (2.1.4)) and the surrounding discussion. The key thing about the graph of a function $f(x)$ is that it keeps track of a particular set of points in the plane whose coordinates are related by the function rule. To be precise, a point $P = (x, y)$ will be on the graph of the function $f(x)$ exactly when $y = f(x)$.

Visualizing the domain and range.

A function is a package which consists of a rule $y = f(x)$, a domain of allowed x-values and a range of output y-values. The domain can be visualized as a subset of the x-axis and the range as a subset of the y-axis. If you are handed the domain, it is graphically easy to describe the range values obtained; here is the procedure:

> *Look at all points on the graph corresponding to domain values on the x-axis, then project these points to the y-axis. The collection of all values you obtain on the y-axis will be the range of the function. This idea of "projection" is illustrated in the two graphs below. We use arrows "$\rightarrow$" to indicate going from a domain x-value, up to the graph, then over to the y-axis:*



Interpreting Points on the Graph.

We can visually detect where a function has positive or negative values:

> *The function values $f(x)$ control the height of the point $P(x) = (x, f(x))$ on the graph above the x-axis; if the function value $f(x)$ is negative, the point $P(x)$ is below the x-axis.*

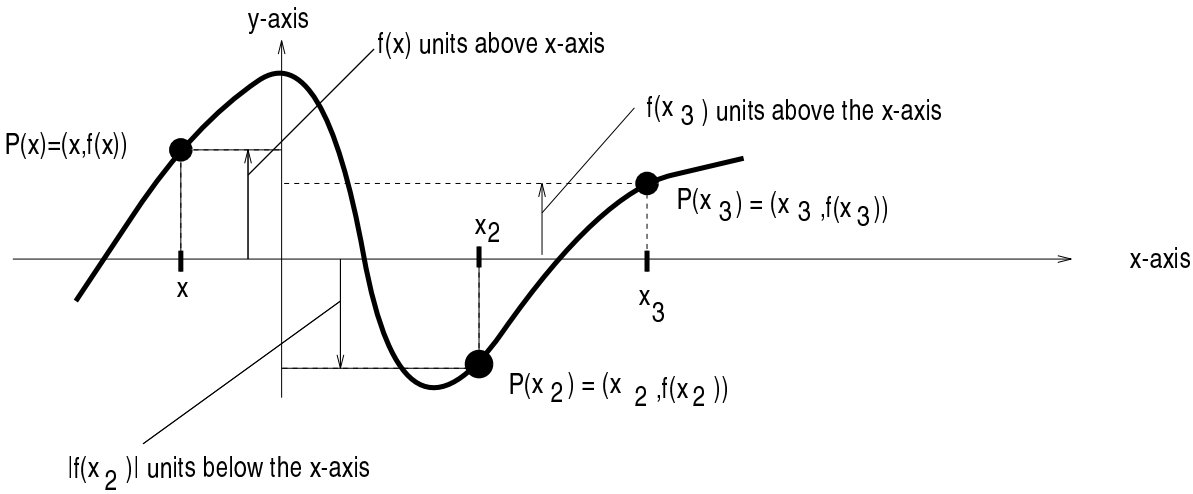
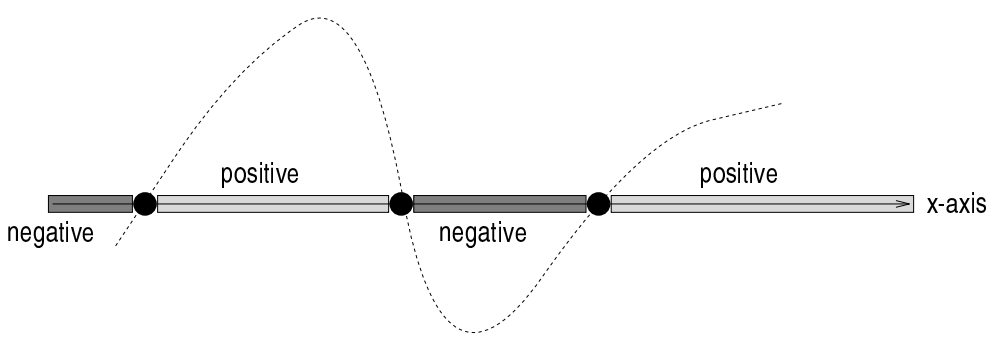


Figure 2.2.1: Interpreting Points on a Graph

In this pictured example, we can now divide the domain (in this case the whole number line) into segments where the function is above, below or crossing the axis. Keeping track of this information on a number line is called a *sign plot* for the function. We include a "shadow" of the graph in the picture below to emphasize how we arrived at our "positive" and "negative" labeling of the sign plot; in practice we would only provide a labeled number line.



By moving through a sequence of x values we can investigate how the corresponding points on the graph move "up and down"; this then gives us a dynamic visual sense of how the function values are changing. For example, in the picture below, suppose we let x move from

1 to 5, left to right; we have indicated how the corresponding points on the curve will move and how the function values will change.

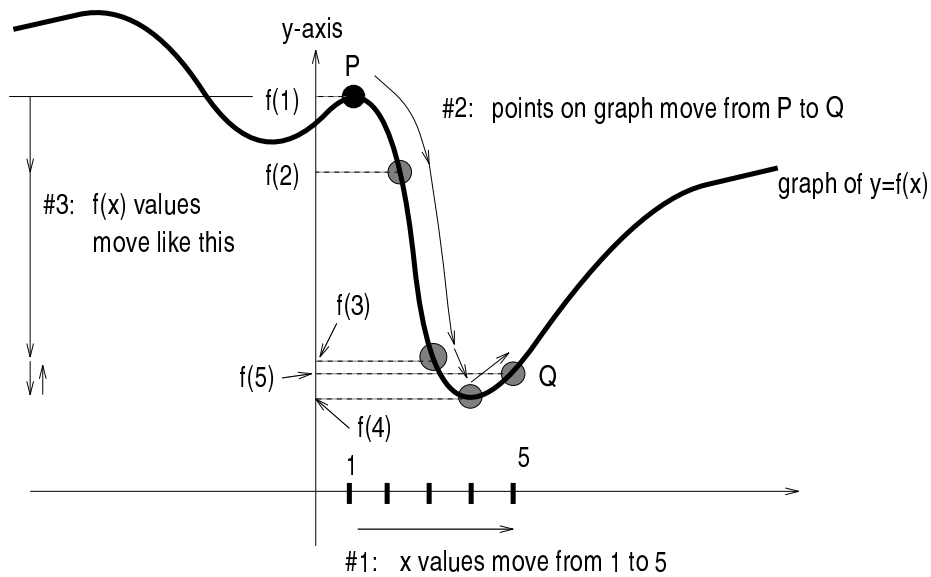


Figure 2.2.2: Dynamic Interpretation of a Graph

Interpreting Intercepts of a Graph.

The places where a graph crosses the axes are often significant. We isolate each as an important feature to look for when doing graphical analysis. The graph of the function $y = f(x)$ crosses the y-axis at the point $(0, f(0))$; so, the y-intercept of the graph is just $f(0)$. The graph of the function $y = f(x)$ crosses the x-axis at points of the form $(x_o, f(x_o))$, where $f(x_o) = 0$. The values x_o are called *roots* or *zeros* of the function $f(x)$. There can be at most one y-intercept, but there can be several x-intercepts or no x-intercept:

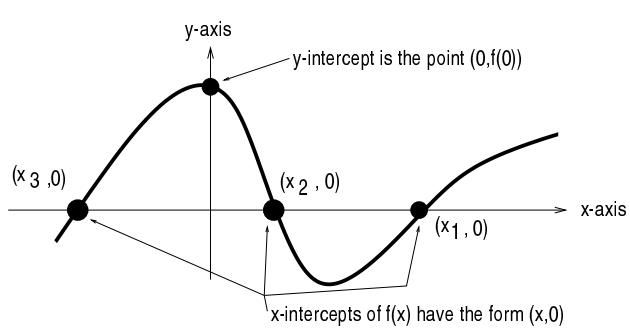
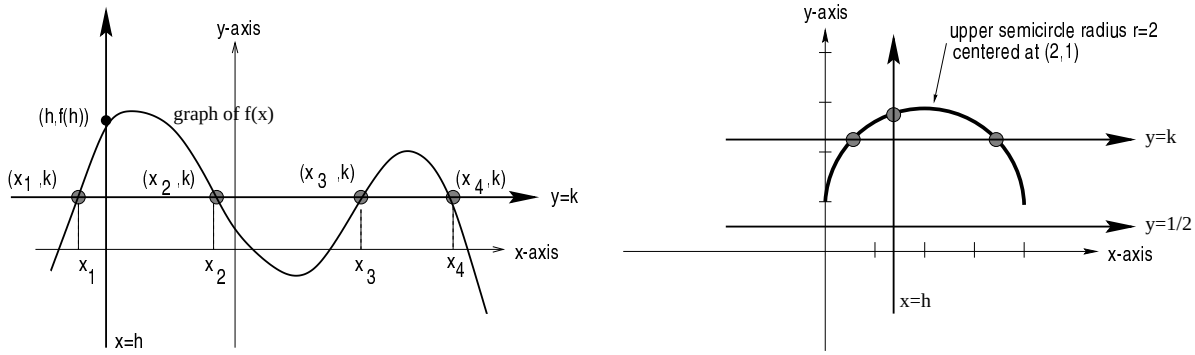


Figure 2.2.3: Intercepts of a Graph

The graph of a function $y = f(x)$ crosses the vertical line $x = h$ at the point $(h, f(h))$. To find where the graph of a function $y = f(x)$ crosses the horizontal line $y = k$, first solve the

equation $k = f(x)$ for x. If the equation $k = f(x)$ has solutions x_1, x_2, x_3, x_4, then the points of intersection would have the coordinates given.



As another example, the right-hand graph above will cross the horizontal line $y = k$ twice if and only if $1 \leq k < 3$; the graph will cross the horizontal line $y = k$ once if and only if $k = 3$. The graph will not intersect the line $y = \frac{1}{2}$ and the graph will cross the vertical line $x = h$ if and only if $0 \leq h \leq 4$.

Interpreting Increasing and Decreasing.

We use certain terms to describe how the function values are changing over some domain of x values. Typically, we want to study what is happening to the values $f(x)$ as x moves from "left to right" in some interval. This can be linked graphically with the study of "uphill" and "downhill" portions of the function graph: If you were "walking to the right" along the graph, the function values are *increasing* if you are walking uphill. Likewise, if you were "walking to the right" along the graph, the function values are *decreasing* if you are walking downhill.

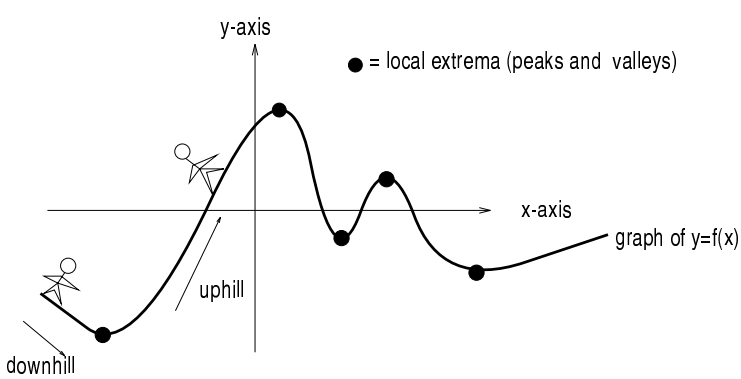
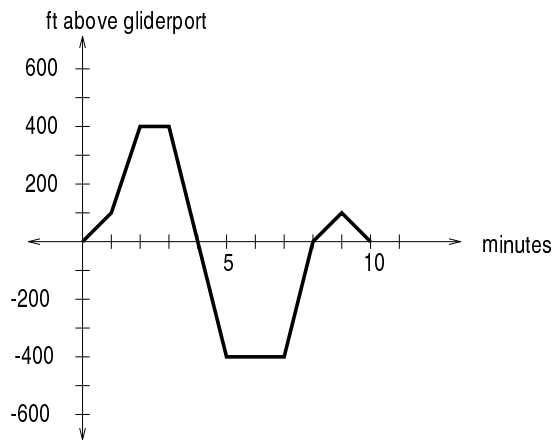


Figure 2.2.4: Graphically Interpreting Increasing and Decreasing

Once we understand where the graph is moving uphill and downhill, we can isolate the places where we change from moving uphill to downhill, or vice versa; these "peaks" and "valleys" are called *local maxima* and *local minima*. Some folks refer to either case as a *local extrema*. People have invested a lot of time (centuries!) and energy (lifetimes!) into the study of how to find local extrema for particular function graphs. We will see some basic examples

in this course and others will surface in future courses once you have the tools of calculus at your disposal. Examples range from business applications that involve optimizing profit to understanding the three-dimensional shape a of biological molecule.

Example 2.2.1: *A hang glider launches from a gliderport in La Jolla. The launch point is located at the edge of a 500 ft. high cliff over the Pacific Ocean. The elevation of the pilot above the gliderport after t minutes is given by the graph below:*



(a) When is the pilot climbing and descending?

(b) When is the pilot at the glider port elevation?

(c) How much time does the pilot spend flying level?

Solution. (a) Graphically, we need to determine the portions of the graph that are increasing or decreasing. In this example, it is increasing when $0 \leq t \leq 2$ and $7 \leq t \leq 9$. And, it is decreasing when $3 \leq t \leq 5$ and $9 \leq t \leq 10$.

(b) Graphically, this question amounts to asking when the elevation is 0, which is the same as finding when the graph crosses the horizontal axis. We can read off there are four such times: $t = 0, 4, 8, 10$.

(c) Graphically, we need to determine the portions of the graph which are made up of horizontal line segments. This happens when $2 \leq t \leq 3$ and $5 \leq t \leq 7$. So, our pilot flies level for a total of 3 minutes. □

<u>Circles and Semicircles</u>.

Back in section 1.3, we discussed equations whose graphs were circles: We found that the graph of the equation

$$(x-h)^2 + (y-k)^2 = r^2 \tag{2.2.2}$$

is a circle of radius r centered at the point (h, k). It is possible to manipulate this equation and become confused. We could rewrite this as $(y-k)^2 = r^2 - (x-h)^2$, then take the square root of each side. However, the resulting equivalent equation would be

$$y = k \pm \sqrt{r^2 - (x-h)^2}$$

and the presence of that $\pm$ sign is tricky; it means we have two equations:

$$y \;=\; k + \sqrt{r^2 - (x-h)^2} \quad \text{or} \quad y = k - \sqrt{r^2 - (x-h)^2}.$$

Each of these two equations defines a function:

$$f(x) \;=\; k + \sqrt{r^2 - (x-h)^2} \qquad \text{or} \qquad g(x) = k - \sqrt{r^2 - (x-h)^2}. \tag{2.2.3}$$

So, even though the equation in (2.2.2) is not a function, we were able to obtain *two* different functions $f(x)$ and $g(x)$ from the original equation. The relationship between the graph of the original equation and the graphs of the two functions in (2.2.3) is as follows: the upper semicircle is the graph of the function $f(x)$ and the lower semicircle is the graph of the function $g(x)$.

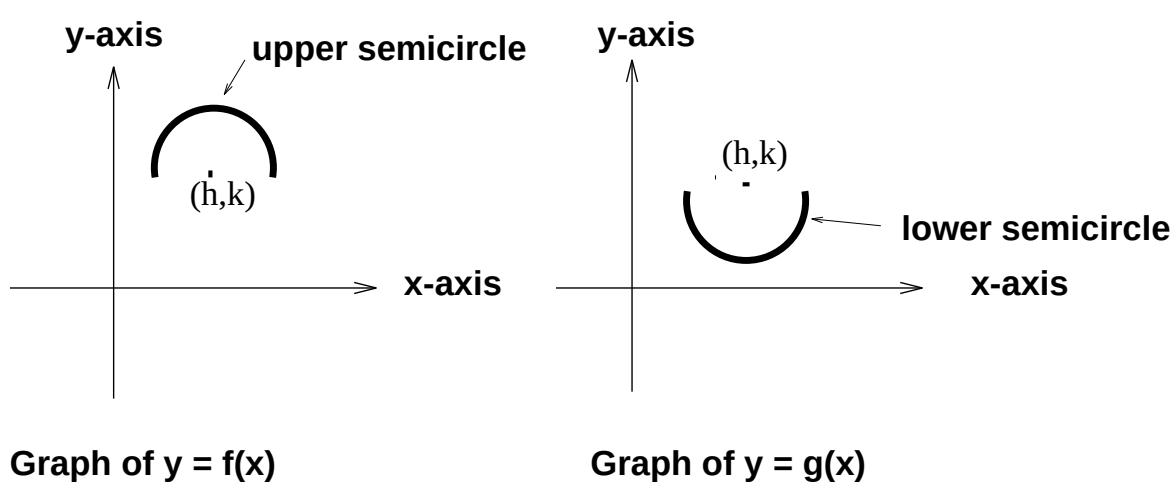
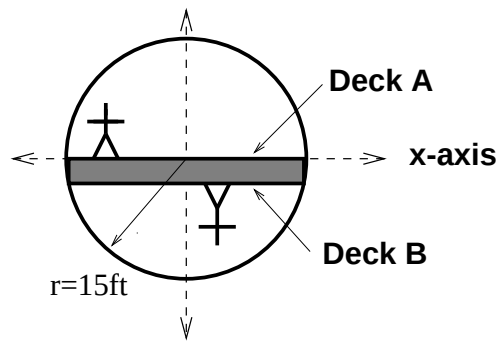


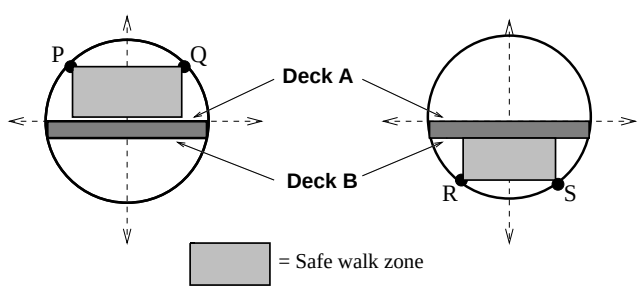
Figure 2.2.5: Upper and lower semicircles

Example 2.2.4: *A tunnel connecting two portions of a space station has a circular cross-section of radius 15 feet. Two walkway decks are constructed in the tunnel. Deck A is along a horizontal diameter and another parallel Deck B is 2 feet below Deck A. Because the space station is in a weightless environment, you can walk vertically upright along Deck A, or vertically upside down along Deck B. You have been assigned to paint "safety stripes" on each deck level, so that a 6 foot person can safely walk upright along either deck. Determine the width of the "safe walk zone" on each deck.*

Solution. Impose a coordinate system so that the origin is at the center of the circular crossection of the tunnel; by symmetry the walkway is centered about the origin. With this coordinate system, the graph of the equation $x^2 + y^2 = 15^2 = 225$ will be the circular cross-section of the tunnel. In the case of Deck A, we basically need to determine how close to each edge of the tunnel a 6 foot high person can stand without hitting his or her head on the tunnel; a similar remark applies to Deck B. This means we are really trying to fit two six-foot-high rectangular safe walk zones into the picture:



Crossection of tunnel



Our job is to find the coordinates of the four points P, Q, R, S. Let's denote by x_1, x_2, x_3, x_4 the x-coordinates of these four points, then $P = (x_1, 6)$, $Q = (x_2, 6)$, $R = (x_3, -8)$ and $S = (x_4, -8)$. To find x_1, x_2, x_3, x_4, we need to find the intersection of the pictured circle with two horizontal lines:

- Intersecting the the upper semicircle with the horizontal line having equation $y = 6$ will determine x_1 and x_2; the upper semicircle is the graph of $f(x) = \sqrt{225 - x^2}$.
- Intersecting the lower semicircle with the horizontal line having equation $y = -8$ will determine x_3 and x_4; the lower semicircle is the graph of $g(x) = -\sqrt{225 - x^2}$.

For Deck A, we simultaneously solve the system of equations

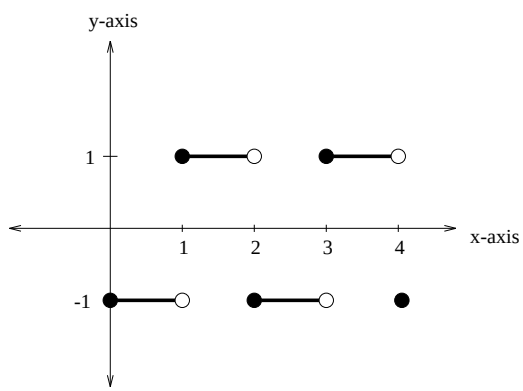
$$\left\{ \begin{array}{ll} y & = \sqrt{225 - x^2} \\ y & = 6 \end{array} \right\}.$$

Plugging in $y = 6$ into the first equation of the system gives $x^2 = 225 - 6^2 = 189$; i.e., $x = \pm\sqrt{189} = \pm 13.75$. This tells us that $P = (-13.75, 6)$ and $Q = (13.75, 6)$. In a similar way, for Deck B, we find $R = (-12.69, -8)$ and $S = (12.69, -8)$.

In the case of Deck A, we would paint a safety stripe 13.75 feet to the right and left of the centerline. In the case of Deck B, we would paint a safety stripe 12.69 feet to the right and left of the centerline. □

Multipart Functions.

So far, in all of our examples we have been able to write $f(x)$ as a nice compact expression in the variable x. Sometimes we have to work harder. As an example of what we have in mind, consider the left-hand picture below:



$$f(x) = \left\{ \begin{array}{rl} -1 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ -1 & \text{if } 2 \le x < 3 \\ 1 & \text{if } 3 \le x < 4 \\ -1 & \text{if } x = 4 \end{array} \right.$$

The curve we are trying to describe in this picture is made up of five pieces; four little line segments and a single point. The first thing to notice is that on the domain $0 \le x \le 4$, this curve will define the graph of some function $f(x)$. To see why this is true, imagine a vertical line moving from left to right within the domain $0 \le x \le 4$ on the x-axis; any one of these vertical lines will intersect the curve exactly once, so by the *vertical line test*, the curve must be the graph of a function. Mathematicians use the shorthand notation above to describe this function. Notice how the rule for $f(x)$ involves five cases; each of these cases corresponds to one of the five pieces that make up the curve. Finally, notice the care with the "open" and "closed" circles is really needed if we want to make sure the curve defines a function; in terms of the rule, these open and closed circles translate into strict inequalities like $<$ or weak inequalities like $\le$. This is an example of what we call a *multipart function*.

The symbolic appearance of multipart functions can be somewhat frightening. The key point is that the graph (and rule) of the function will be broken up into a number of separate cases. To study the graph or rule, we simply "home in" on the appropriate case. For example,

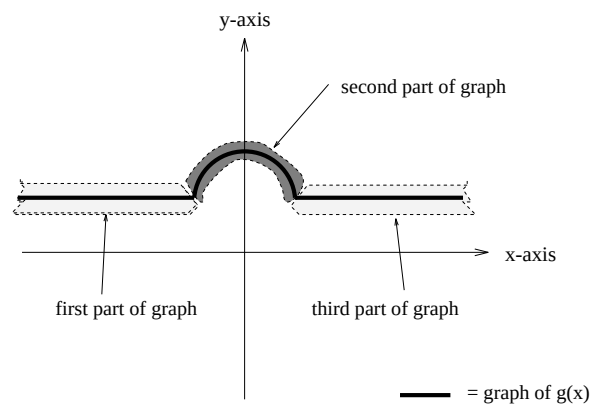
in the above illustration, suppose we wanted to compute $f(3.56)$. First, we would find which of the five cases covers $x = 3.56$, then apply that part of the rule to compute $f(3.56) = 1$.

Our first multipart function example illustrated how to go from a graph in the plane to a rule for $f(x)$; we can reverse this process and go from the rule to the graph.

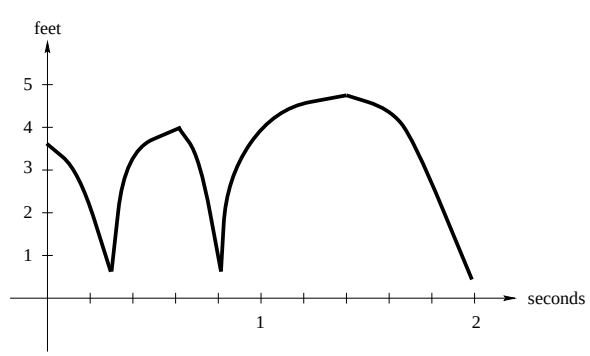
Example 2.2.5: *Sketch the graph of the multipart function*

$$g(x) = \begin{cases} 1 & \text{if } x \leq -1 \\ 1 + \sqrt{1 - x^2} & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Solution. The graph of $g(x)$ will consist of three pieces. The first case consists of the graph of the function $y = g(x) = 1$ on the domain $x \leq -1$, this consists of all points on the horizontal line $y = 1$ to the left of and including the point $(-1, 1)$. We have "lassoed" this portion of the graph in the picture below. Likewise, the third case in the definition yields the graph of the function $y = g(x) = 1$ on the domain $x \geq 1$; this is just all points on the horizontal line $y = 1$ to the right of and including the point $(1, 1)$. Finally, we need to analyze the middle case, which means we need to look at the graph of $1 + \sqrt{1 - x^2}$ on the domain $-1 \leq x \leq 1$. This is just the upper semicircle of the circle of radius 1 centered at $(0, 1)$. If we paste these three pieces together, we arrive at the graph of $g(x)$. □

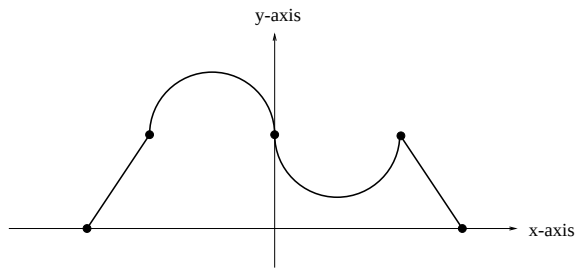


Example 2.2.6: *You are dribbling a basketball and the function $s = h(t)$ keeps track of the height of the ball's center above the floor after t seconds. If we take the domain to be $0 \leq t \leq 2$ (the first 2 seconds), here is a reasonable graph that could arise. This is a multipart function. Three portions of the graph are decreasing and two portions are increasing. Why doesn't the graph touch the t axis?*



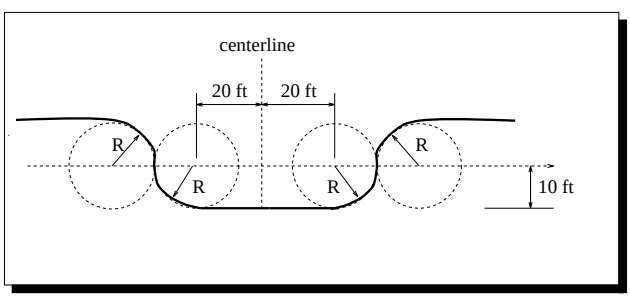
Problems

1. The graph of a function $y = g(x)$ on the domain $-6 \leq x \leq 6$ consistes of line segments and semicircles of radius 2 connecting the points $(-6, 0), (-4, 4), (0, 4), (4, 4), (6, 0)$.



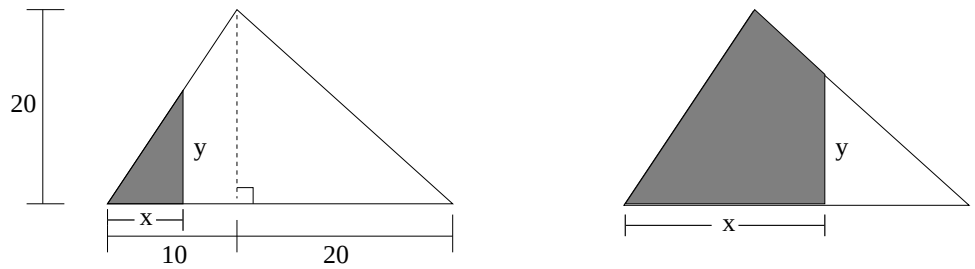
   (a) What is the range of g?
   (b) Where is the function increasing? Where is the function decreasing?
   (c) Find the multipart formula for $y = g(x)$.
   (d) If we restrict the function to the smaller domain $-1 \leq x \leq 5$, what is the range?

2. The vertical cross-section of a drainage ditch is pictured below:



   Here, R indicates a circle of radius 10 feet and all of the indicated circle centers lie along the common horizontal line 10 feet above and parallel to the ditch bottom. Assume that water is flowing into the ditch so that the level above the bottom is rising 2 inches per minute.
   (a) What is the width of the filled portion of the ditch after 1 hour and 18 minutes?
   (b) When will the filled portion of the ditch be 42 feet wide? 50 feet wide? 73 feet wide?
   (c) When will the ditch be completely full?
   (d) Find a multipart function that models the vertical cross-section of the ditch.

3. Pizzeria Buonapetito makes a triangular-shaped pizza with base width of 30 inches and height 20 inches as shown. Alice wants only a portion of the pizza and does so by making a vertical cut through the pizza and taking the shaded portion. Letting x be the bottom length of Alice's portion and y be the length of the cut as shown, answer the following questions:



shaded region is the piece Alice takes

(a) Find a formula for y as a multipart function of x, for $0 \leq x \leq 30$. Sketch the graph of this function and calculate the range.
(b) Find a formula for the area of Alice's portion as a multipart function of x, for $0 \leq x \leq 30$. Sketch the graph of this function and calculate the range.
(c) If Alice wants her portion to have half the area of the pizza, where should she make the cut?
(d) If Alice wants her portion to have $p\%$ of the area of the pizza, where should she make the cut?

4. This problem deals with cars traveling between Bellevue and Spokane, which are 280 miles apart. Let t be the time in hours, measured from 12:00 noon; so, for example, $t = -1$ is 11:00 am.
   (a) Joan drives from Bellevue to Spokane at a constant speed, departing from Bellevue at 11:00 am and arriving in Spokane at 3:30 pm. Find a function $j(t)$ that computes her distance from Bellevue at time t. Sketch the graph, specify the domain and determine the range.
   (b) Steve drives from Spokane to Bellevue at 70 mph, departing from Spokane at 12:00 noon. Find a function $s(t)$ for his distance from Bellevue at time t. Sketch the graph, specify the domain and determine the range.
   (c) Find a function $d(t)$ that computes the distance between Joan and Steve at time t. (Caution: This will be a multipart function.) Sketch the graph, specify the domain and determine the range.

2.3 Quadratic Modeling

If you kick a ball through the air enough times, you will find it's path tends to be parabolic. Before we can answer any detailed questions about this situation, we need to get our hands on a precise mathematical model for a parabolic shaped curve. This means we seek a function $y = f(x)$ whose graph reproduces the path of the ball.

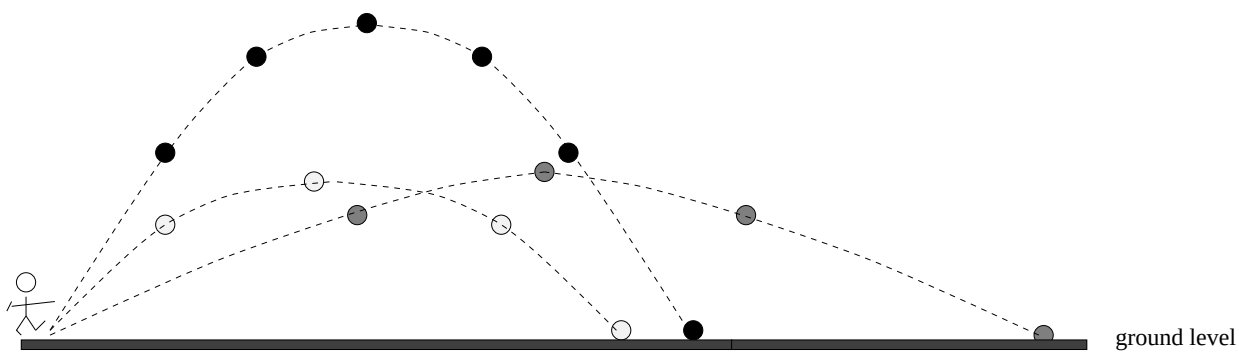
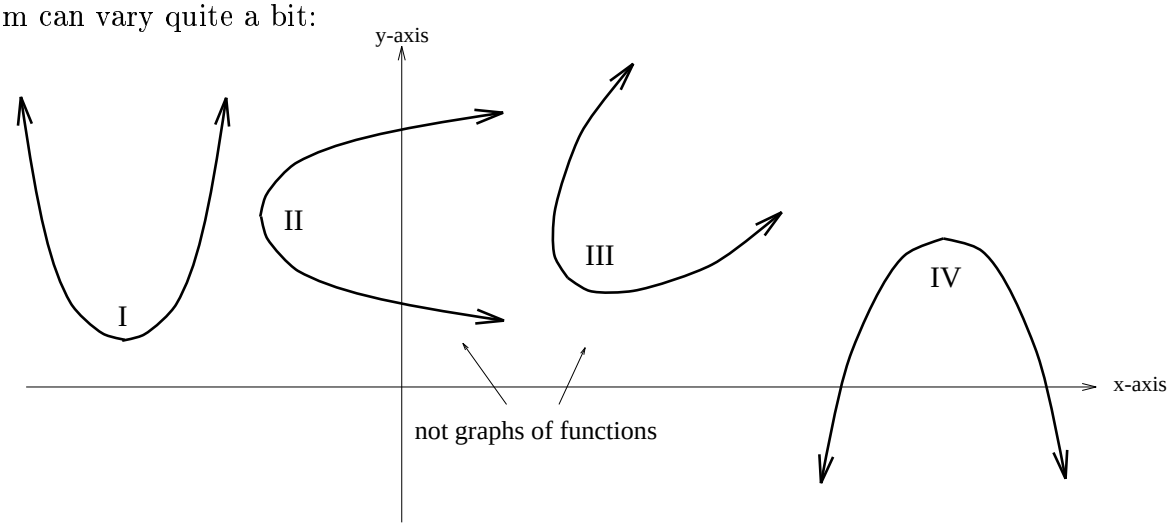


Figure 2.3.1: Possible paths for a kicked ball are parabolic

Parabolas and Vertex Form.

Ok, suppose we sit down with an xy-coordinate system and draw four random parabolas; let's label them I, II, III, IV. The relationship between these parabolas and the fixed coordinate system can vary quite a bit:



The key distinction between these four curves is that only I and IV are the graphs of functions; this follows from the *vertical line test*. A parabola which is the graph of a function is called a *standard parabola*. We can see that any standard parabola has three basic features:

- the parabola will either open "upward" or "downward";
- the graph will have either a "highest point" or "lowest point", called the *vertex*;

- the parabola will be symmetric about some vertical line called the *axis* of symmetry.

Our first task is to describe the mathematical model for any standard parabola. In other words, what kind of function equations $y = f(x)$ give us standard parabolas as their graphs? Our approach is geometric and visual:

- Begin with one specific example, then show every other standard parabola can be obtained from it via some specific geometric maneuvers.
- As we perform these geometric maneuvers, we keep track of how the function equation for the curve is changing.

This discussion will amount to a concrete application of a more general set of tools developed in the following section of this chapter.

Using a graphing device, it is an easy matter to plot the graph of $y = x^2$ and see we are getting the parabola pictured in Figure 2.3.2. (In fact, Exercise 2.1.12 describes a "pencil and paper" method for seeing this graph is a parabola.) The basic idea is to describe how we can manipulate this graph and obtain any standard parabola. In the end, we will see that standard parabolas are obtained as the graphs of functions having the form

$$y = ax^2 + bx + c,$$

for various constants a, b, c, with $a \neq 0$. A function of this type is called a *quadratic function* and these play a central role throughout the course. We will divide our task into two steps:

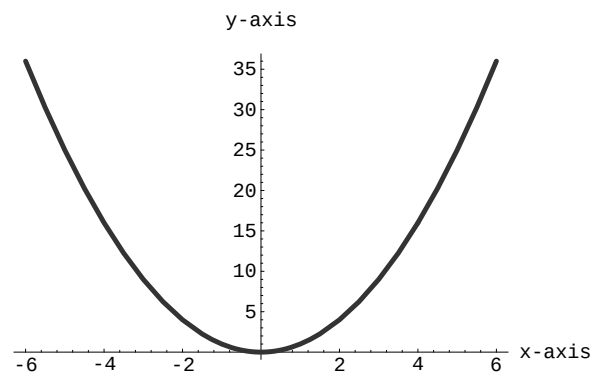


Figure 2.3.2

First we show every standard parabola arises as the graph of a function having the form

$$y = a(x-h)^2 + k,$$

for some constants a, h, k, with $a \neq 0$. This is called the *vertex form* of a quadratic function. Notice, if we were to algebracially expand out this equation, we could rewrite it in the $y = ax^2 + bx + c$ form. For example, suppose we start with the vertex form $y = 2(x-1)^2 + 3$, so that $a = 2, h = 1, k = 3$. Then we can rewrite the equation in the form $y = ax^2 + bx + c$ as follows:

$$2(x-1)^2 + 3 = 2(x^2 - 2x + 1) + 3 = 2x^2 - 4x + 5,$$

so $a = 2, b = -4, c = 5$. The second step is to show any quadratic function can be written in vertex form; the underlying algebraic technique used here is called *completing the square*. This is a bit more involved. For example, if you are simply handed the quadratic function

$y = -3x^2 + 6x - 1$, it not at all obvious why the vertex form is obtained by this equality:

$$-3x^2 + 6x - 1 \quad = \quad -3(x-1)^2 + 2.$$

The reason behind this equality is the technique of completing the square. In the end, we will almost always be interested in the vertex form of a quadratic. This is because a great deal of qualitative information about the parabolic graph can simply be "read off" from this form.

First Maneuver: Shifting.

Suppose we start with the graph in Figure 2.3.2 and horizontally shift it h units to the right. To be specific, consider the two cases $h = 2, 4$. To visualize this, imagine making a wire model of the graph, set in on top of the curve, then slide the wire model h units to the right. What you will obtain are the two "dashed curves" in Figure 2.3.3. We will call the process just described a *horizontal shift*. Since the "dashed curves" are no longer the original parabola in Figure 2.3.2, the corresponding function equations must have changed.

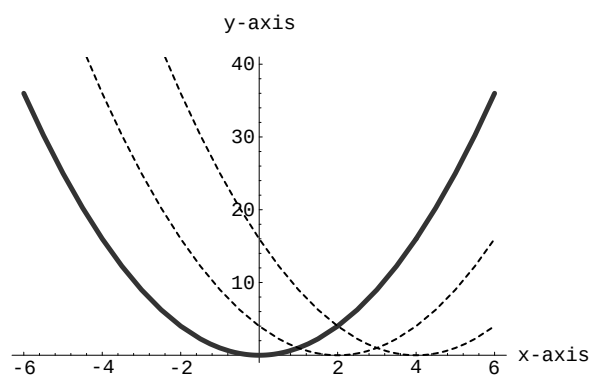


Figure 2.3.3

Using a graphing device, you can check that the corresponding equations for the dashed graphs would be

$$y \quad = \quad (x-2)^2 = x^2 - 4x + 4,$$

which is the plot with lowest point (2,0) and

$$y \quad = \quad (x-4)^2 = x^2 - 8x + 16,$$

which is the plot with lowest point (4,0). In general, if h is positive, the graph of the function $y = (x-h)^2$ is the parabola obtained by shifting the graph of $y = x^2$ by h units to the right.

Next, if h is negative, shifting h units to the right is the same as shifting $|h|$ units to the left!

On the domain $-6 \leq x \leq 6$, Figure 2.3.4 indicates this for the cases $h = -2, -4$, using "dashed curves" for the shifted graphs and a solid line for the graph of $y = x^2$. Using a graphing device, we can check that the corresponding equations for the dashed graphs would be

$$y = (x - (-2))^2 = (x + 2)^2 = x^2 + 4x + 4,$$

which is the plot with lowest point (-2,0) and

$$y = (x - (-4))^2 = (x + 4)^2 = x^2 + 8x + 16,$$

which is the plot with lowest point (-4,0).

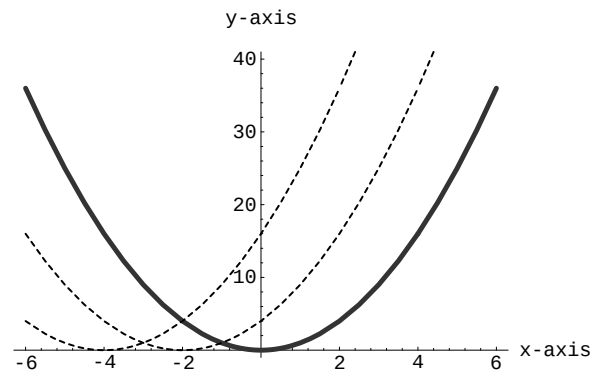


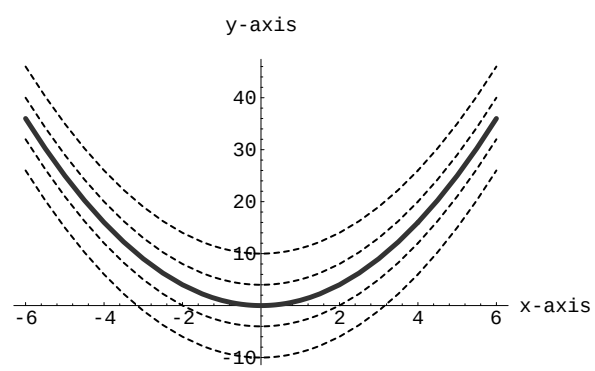
Figure 2.3.4

In general, if h is negative, the graph of the function $y = (x - h)^2$ gives the parabola obtained by shifting the graph of $y = x^2$ by $|h|$ units to the LEFT.

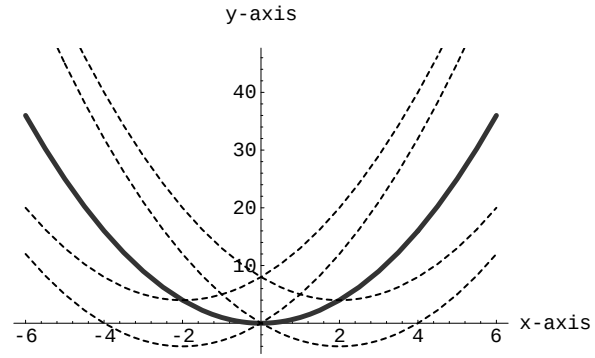
The conclusion thusfar is this: Begin with the graph of $y = x^2$ as pictured in Figure 2.3.2. Horizontally shifting this graph h units to the right gives a new (standard) parabola whose equation is $y = (x - h)^2$.

We can also imagine *vertically shifting* the graph in Figure 2.3.2. This amounts to moving the graph k units vertically upward. It turns out that this vertically shifted graph corresponds to the graph of the function $y = x^2 + k$. We can work out a few special cases and use a graphing device to illustrate what all this really means.

The figure at right illustrates the graphs of $y = x^2 + k$ in the cases when $k = 4, 10$ and $k = -4, -10$, leading to vertically shifted graphs. Positive (resp. negative) values of k lead to the upper two (resp. lower two) "dashed curves"; the plot of $y = x^2$ is again the solid line. The equations giving these graphs would be $y = x^2 - 10$, $y = x^2 - 4$, $y = x^2 + 4$ and $y = x^2 + 10$, from bottom to top dashed plot.

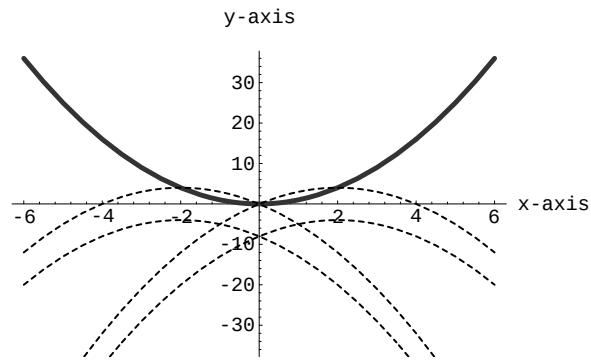


If we combine horizontal and vertical shifting, we end up with the graphs of functions of the form $y = (x - h)^2 + k$. The figure at right illustrates the four cases with corresponding equations $y = (x \pm 2)^2 \pm 4$; as an exercise, identify which equation goes with each curve.



Second Maneuver: Reflection.

Next, we can reflect any of the curves $y = p(x)$ obtained by horizontal or vertical shifting across the x-axis. This procedure will produce a new curve which is the graph of the new function $y = -p(x)$. For example, begin with the four dashed curves in the previous figure. Here are the reflected parabolas and their equations are $y = -(x \pm 2)^2 \pm 4$.

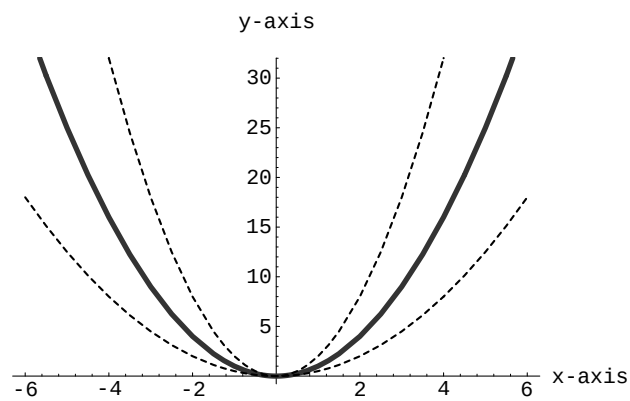


Third Maneuver: Vertical Dilation.

If a is a positive number, the graph of $y = ax^2$ is usually called a *vertical dilation* of the graph of $y = x^2$. There are two cases to distinguish here:

- If $a > 1$, we have a vertically expanded graph.
- If $0 < a < 1$, we have a vertically compressed graph.

This is illustrated for $a = 2$ (upper dashed plot) and $a = 1/2$ (lower dashed plot).



Conclusion.

Starting with $y = x^2$ in Figure 2.3.2, we can combine together all three of the operations: shifting, reflection and dilation. This will lead to the graphs of functions that have the form:

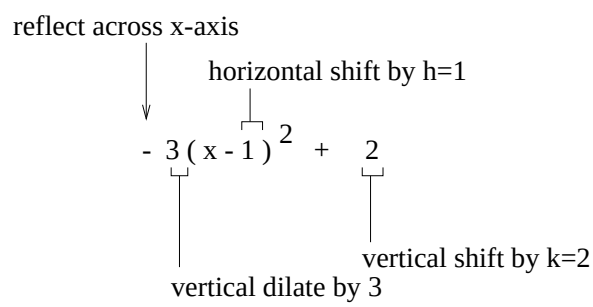
$$y = a(x - h)^2 + k,$$

for some a, h and k, $a \neq 0$. If you think about it for awhile, it seems pretty easy to believe that any standard parabola arises from the one in Figure 2.3.2 using our three geometric maneuvers. In other words, what we have shown is that any standard parabola is the graph of a quadratic equation in vertex form. Let's summarize this very important fact.

Parabolas and Vertex Form 2.3.1: *A standard parabola is the graph of a function $y = f(x) = a(x-h)^2 + k$, for some constants a, h, k and $a \neq 0$. The vertex of the parabola is (h, k) and the axis of symmetry is the line $x = h$. If $a > 0$, then the parabola opens upward; if $a < 0$, then the parabola opens downward.*

Example 2.3.2: *Describe a sequence of geometric operations leading from the graph of $y = x^2$ to the graph of $y = f(x) = -3(x-1)^2 + 2$.*

Solution. To begin with, we can make some initial conclusions about the specific shifts, reflections and dilations involved, based on looking at the vertex form of the equation. In addition, by (2.3.1), we know that the vertex of the graph of $y = f(x)$ is (1,2), the line $x = 1$ is a vertical axis of symmetry and the parabola opens downward.

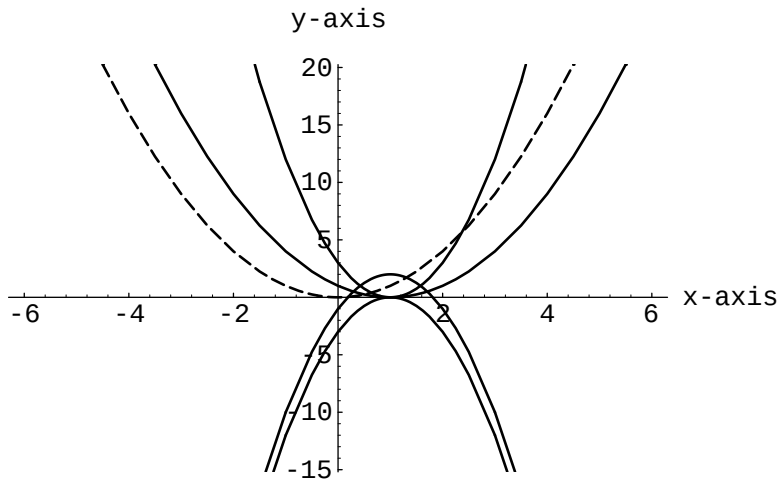


We need to be a little careful about the order in which we apply the four operations highlighted. We will illustrate a procedure that works. The full explaination for the success of our procedure involves *function compositions* and we will return to that at the end of §2.4. The order in which we will apply our geometric maneuvers is as follows:

horizontal shift $\Rightarrow$ vertical dilate $\Rightarrow$ reflect $\Rightarrow$vertical shift

The Figure below illustrates the four curves obtained by applying these successive steps, in this order. As a reference, we include the graph of $y = x^2$ as a "dashed curve":

- A horizontal shift by $h = 1$ yields the graph of $y = (x - 1)^2$; this is the fat parabola opening upward with vertex (1,0).
- A dilation by $a = 3$ yields the graph of $y = 3(x-1)^2$; this is the skinny parabola opening upward with vertex (1,0).
- A reflection yields the graph of $y = -3(x - 1)^2$; this is the downward opening parabola with vertex (1,0).
- A vertical shift by $k = 2$ yields the graph of $y = -3(x - 1)^2 + 2$; this is the downward opening parabola with vertex (1,2). □



Completing the Square.

By now it is pretty clear we can say a lot about the graph of a quadratic function which is in vertex form. We need a procedure for rewriting a given quadratic function in vertex form. Let's first look at an example.

Example 2.3.3: *Find the vertex form of the quadratic function* $y = -3x^2 + 6x - 1$.

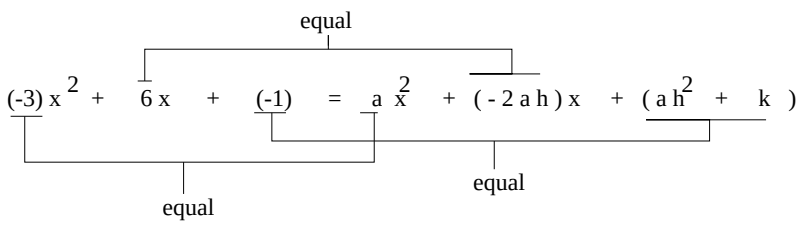
Solution. Since our goal is to put the function in vertex form, we can write down what this means, then try to solve for the unknown constants. Our first step would be to write

$$-3x^2 + 6x - 1 \quad = \quad a(x - h)^2 + k,$$

for some constants a, h, k. Now, expand the right hand side of this equation and factor out coefficients of x and x^2:

$$\begin{aligned} -3x^2 + 6x - 1 &= a(x - h)^2 + k = a(x^2 - 2xh + h^2) + k \\ -3x^2 + 6x - 1 &= ax^2 - 2xah + ah^2 + k \\ (-3)x^2 + 6x + (-1) &= ax^2 + (-2ah)x + (ah^2 + k). \end{aligned}$$

If this is an equation, then it must be the case that the coefficients of like powers of x match up on the two sides of the equation:



Now we have three equations and three unknowns (the a, h, k) and we can proceed to solve for these:

$$\begin{aligned} -3 &= a \\ 6 &= -2ah \\ -1 &= ah^2 + k \end{aligned}$$

The first equation just hands us the value of $a = -3$. Next, we can plug this value of a into the second equation, giving us

$$6 = -2ah = -2(-3)h = 6h,$$

so $h = 1$. Finally, plug the now known values of a and h into the third equation:

$$-1 = ah^2 + k = -3(1^2) + k = -3 + k,$$

so $k = 2$. Our conclusion is then

$$-3x^2 + 6x - 1 = -3(x-1)^2 + 2.$$

Notice, this is the quadratic we studied in Example 2.3.2. □

The procedure used in the preceeding example will always work to rewrite a quadratic function in vertex form. We refer to this as *completing the square*.

Example 2.3.4: *Describe the relationship between the graphs of* $y = x^2$ *and* $y = f(x) = -4x^2 + 5x + 2$.

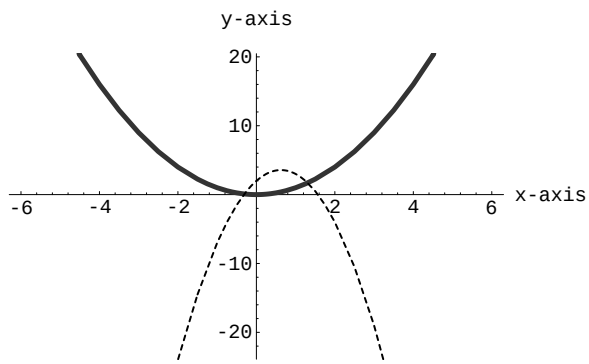
Solution. We will go through the algebra to complete the square, then interpret what this all means in terms of graphical maneuvers. We have

$$\begin{aligned} -4x^2+5x+2 &= a(x-h)^2+k \\ (-4)x^2+5x+2 &= ax^2+(-2ah)x+(ah^2+k). \end{aligned}$$

This gives us three equations:

$$\begin{aligned} -4 &= a \\ 5 &= -2ah \\ 2 &= ah^2+k. \end{aligned}$$

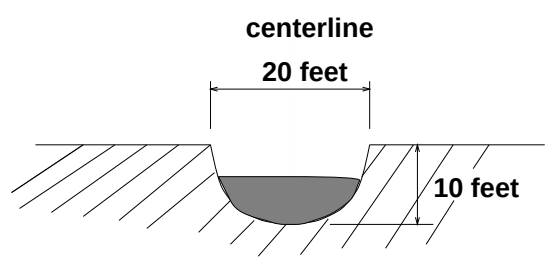
We conclude that $a=-4, h=\frac{5}{8}=0.625$ and $k=\frac{57}{16}=3.562$.



So, this tells us that we can obtain the graph of $y=f(x)$ from that of $y=x^2$ by these steps:

- Horizontally shifting by $h=0.625$ units gives $y=(x-0.625)^2$.
- Vertically dilate by the factor $a=4$ gives $y=4(x-0.625)^2$.
- Reflecting across the x-axis gives $y=-4(x-0.625)^2$.
- Vertically shifting by $k=3.562$ units gives $y=f(x)=-4(x-0.625)^2+3.562$. □

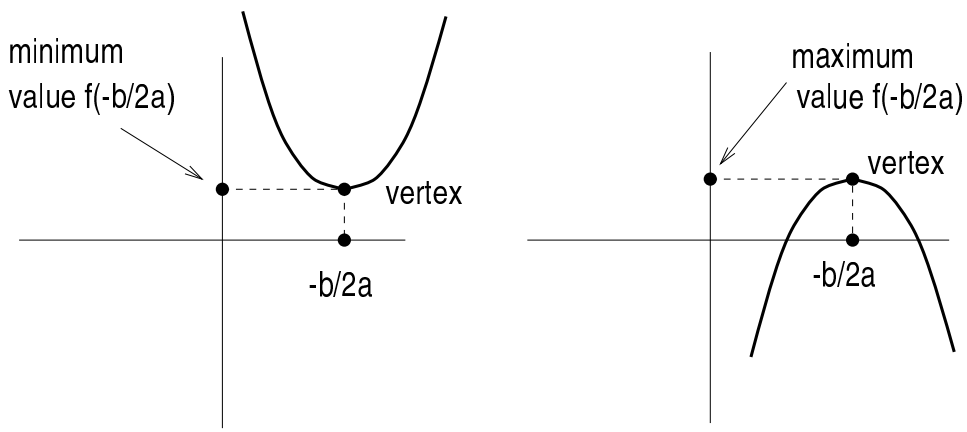
Example 2.3.5: *A drainage canal has a cross-section in the shape of a parabola. Suppose that the canal is 10 feet deep and 20 feet wide at the top. If the water depth in the ditch is 5 feet, how wide is the surface of the water in the ditch?*



Solution. Impose an xy-coordinate system so that the parabolic cross-section of the canal is symmetric about the y-axis and its vertex is the origin. The vertex form of any such parabola is $y=f(x)=ax^2$, for some $a>0$; this is because $(h,k)=(0,0)$ is the vertex and the parabola opens upward! The dimension information given tells us that the points $(10,10)$ and $(-10,10)$ are on the graph of $f(x)$. Plugging into the expression for f, we conclude that $10=100a$, so $a=0.1$ and $f(x)=(0.1)x^2$. Finally, if the water is 5 feet deep, we must solve the equation: $5=(0.1)x^2$, leading to $x=\pm\sqrt{50}=\pm 7.07$. Conclude the surface of the water is 14.14 feet wide when the water is 5 feet deep. □

Interpreting the Vertex.

If we begin with a quadratic function $y = f(x) = ax^2 + bx + c$, we know the graph will be a parabola. Graphically, the vertex will correspond to either the "highest point" or "lowest point" on the graph. If $a > 0$, the vertex is the lowest point on the graph; if $a < 0$, the vertex is the highest point on the graph. The maximum or minimum value of the function is the second coordinate of the vertex and the value of the variable x for which this extreme value is achieved is the first coordinate of the vertex. As we know, it is easy to read off the vertex coordinates when a quadratic function is written in vertex form. If instead we are given a quadratic function $y = ax^2 + bx + c$, we can use the technique of completing the square and arrive at a formula for the coordinates of the vertex in terms of a, b, c. We summarize this below and label the two situations (upward or downward opening parabola) in the Figure. Keep in mind, it is always possible to obtain this formula by simply completing the square:



Vertex Coordinates 2.3.6: *In applications involving a quadratic function*

$$f(x) = ax^2 + bx + c,$$

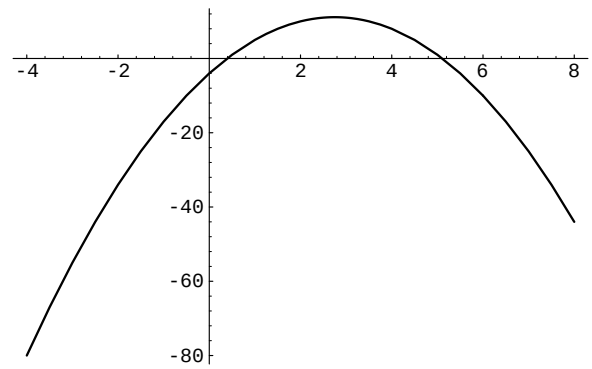
the vertex has coordinates $P = (\frac{-b}{2a}, f(\frac{-b}{2a}))$. *The second coordinate of the vertex will detect the maximum or minimum value of* $f(x)$*; this is often a key step in problem solving.*

Example 2.3.7: *Discuss the graph of the quadratic function* $y = f(x) = -2x^2 + 11x - 4$.

Solution. We need to place the equation $y = f(x)$ in vertex form. We can simply compute $a = -2$, $h = \frac{-b}{2a} = \frac{11}{4}$ and $k = f(\frac{11}{4}) = \frac{89}{8}$, using (2.3.6):

$$\begin{aligned} f(x) &= -2x^2 + 11x - 4 \\ &= -2(x - (\frac{11}{4}))^2 + \frac{89}{8}. \end{aligned}$$

This means that the graph of $f(x)$ is a parabola opening downward with vertex $(\frac{11}{4}, \frac{89}{8})$ and axis $x = \frac{11}{4}$; see the Figure at right.



□

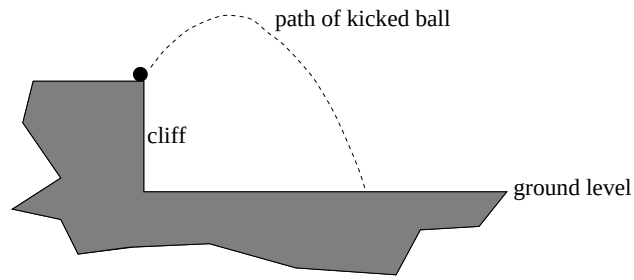
Quadratic Modeling Problems.

The real importance of quadratic functions stems from the connection with motion problems. Imagine one of the three kicked ball scenarios in Figure 2.3.1 and impose a coordinate system with the kicker located at the origin. We can study the motion of the ball in two ways:

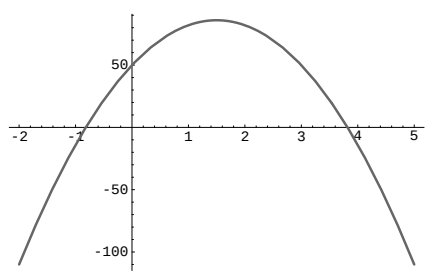
- Regard time t as the important variable and try to find a function $y(t)$ which describes the height of the ball t seconds after the ball is kicked; this would just be the y-coordinate of the ball at time t. We will see in Chapter 3 that $y(t)$ is a quadratic function. If we had this function in hand, we could determine **when** the ball hits the ground by solving the equation $0 = y(t)$, but we would not be able to determine **where** the ball hits the ground.
- A second approach is to forget about the time variable and simply try to find a function $y = f(x)$ whose graph models the exact path of the ball. In particular, we could find **where** the ball hits the ground by solving $0 = f(x)$, but we would not be able to determine **when** the ball hits the ground.

In Chapter 4, we will see these two viewpoints are very closely related. For now, it is most important to obtain some familiarity with the sort of motion problems we are interested in solving using quadratic functions.

Example 2.3.8: *A ball is located on the edge of a cliff as pictured below. The ball is kicked and its height (in feet) above the level ground is given by the function* $s = y(t) = -16t^2 + 48t + 50$*, where* t *represents seconds elapsed after kicking the ball. What is the maximum height of the ball and when is this height achieved? When does the ball hit the ground? How high is the cliff?*



Solution. The function $y(t)$ is a quadratic function with negative leading coefficient, so its graph in the ts-coordinate system will be a downward opening parabola. Using a graphing device, the picture looks like this:



The vertex is the highest point on the graph, which can be found by writing $y(t)$ in standard form using (2.3.6):

$$y(t) = -16t^2 + 48t + 50 = -16(t - \frac{3}{2})^2 + 86.$$

The vertex of the graph of $y(t)$ is $(\frac{3}{2}, 86)$, so the maximum height of the ball above the level ground is 86 feet, occuring at time $t = \frac{3}{2}$.

The ball hits the ground when its height above the ground is zero; using the quadratic formula:

$$\begin{aligned} y(t) &= -16t^2 + 48t + 50 = -16(t^2 - 3t - \frac{25}{8}) = 0 \\ t &= \frac{3 \pm \sqrt{(-3)^2 - 4(-\frac{25}{8})}}{2} = 3.818, -0.818\,\text{sec} \end{aligned}$$

Conclude the ball hits the ground after 3.818 seconds. Finally, the height of the cliff is the height of the ball zero seconds after release; i.e. $y(0) = 50$ feet is the height of the cliff. □

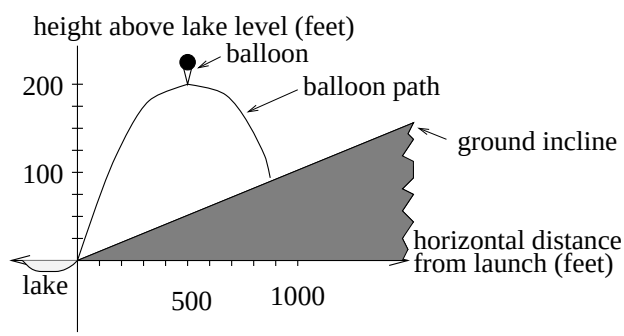
Two important cautions about 2.3.8:

1. The graph of $y(t)$ is NOT the path followed by the ball! Finding the actual path of the ball is not possible unless additional information is given. Can you see why?
2. The function $y(t)$ is defined for all t; however, in the context of the problem, there is no physical meaning when $t < 0$.

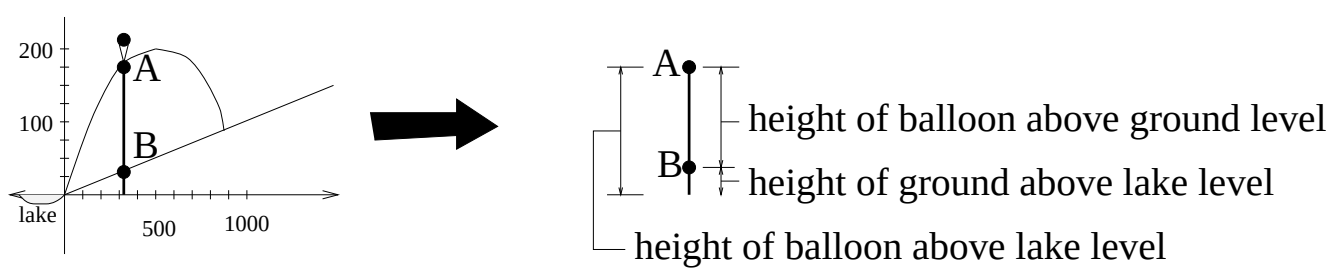
The next example illustrates how we must be *very* careful to link the question being asked with an appropriate function.

Example 2.3.9: *A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured below and assume that the path of the balloon follows the graph of* $y = f(x) = -\frac{2}{2500}x^2 + \frac{4}{5}x$. *The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. What is the maximum height of the balloon above lake level? What is the maximum height of the balloon above ground level? Where does the balloon land on the ground? Where is the balloon 50 feet above the ground?*

Solution. In the coordinate system indicated, the origin is the takeoff point and the graph of $y = f(x)$ is the path of the balloon. Since $f(x)$ is a quadratic function with negative leading coefficient, its graph will be parabola which opens downward:



The difficulty with this problem is that at any instant during the balloon's flight, the "height of the balloon above the ground" and the "height of the balloon above the lake level" are different! The picture below highlights this difference; consequently, two *different* functions will be needed to study these two different quantities.



The function $y = f(x)$ keeps track of the height of the balloon above lake level at a given x location on the horizontal axis. The line ℓ with slope $m = \frac{2}{20} = \frac{1}{10}$ passing through the origin

models the ground level. This says that the function

$$y = \frac{1}{10}x.$$

keeps track of the height of the ground above lake level at a given x location on the horizontal axis.

We can determine the maximum height of the balloon above lake level by analyzing the parabolic graph of $y = f(x)$. Putting $f(x)$ in vertex form, via (2.3.6),

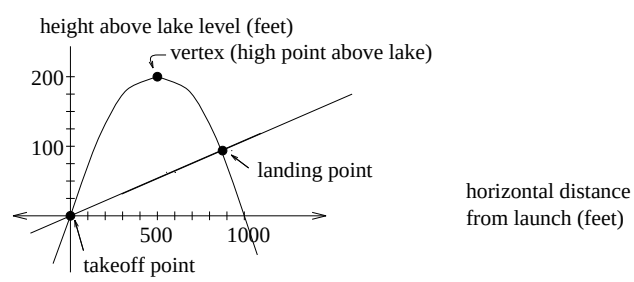
$$f(x) = -\frac{2}{2500}(x-500)^2 + 200.$$

The vertex of the graph of $y = f(x)$ is $(500, 200)$. This just tells us that the maximum height of the balloon above lake level is 200 feet. To find the landing point, we need to solve the system of equations

$$\left\{ \begin{array}{ll} y & = -\frac{2}{2500}x^2 + \frac{4}{5}x \\ y & = \frac{1}{10}x \end{array} \right\}.$$

As usual, plugging the second equation into the first and solving for x, we get

$$\begin{aligned} \frac{1}{10}x &= -\frac{2}{2500}x^2 + \frac{4}{5}x \\ x^2 &= 875x \\ x^2 - 875x &= 0 \\ x(x-875) &= 0 \end{aligned}$$



From the algebra, we see there are two solutions: $x = 0$ or $x = 875$; these correspond to the takeoff and landing points of the balloon, which are the two places the flight path and ground coincide. (Notice, if we had divided out x from the last equation, we would only get one solution; the tricky point is that we can't divide by zero!) The balloon lands at the position where $x = 875$ and to find the y coordinate of this landing point we plug $x = 875$ into our function for the balloon height above lake level: $y = f(875) = 87.5$ feet. So, the landing point has coordinates $(875, 87.5)$.

Next, we want to study the height of the balloon above the ground. Let $y = g(x)$ be the function which represents the height of the balloon above the ground when the horizontal

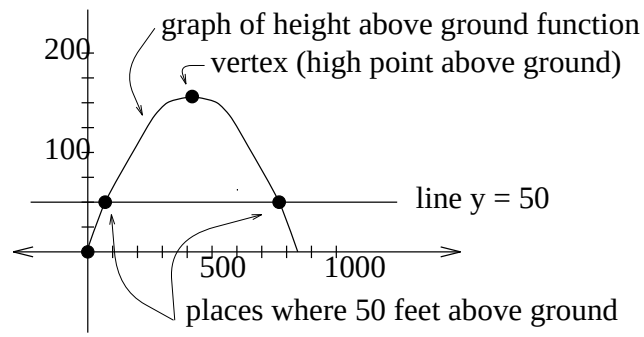
coordinate is x. We find

$$\begin{aligned} g(x) &= (\text{height of the balloon above lake level with horizontal coordinate } x) \\ &\qquad -(\text{elevation of ground above lake level with horizontal coordinate } x) \\ &= f(x) - g(x) \\ &= \left(-\frac{2}{2500}x^2 + \frac{4}{5}x\right)_{\text{balloon above lake level}} - \left(\frac{1}{10}x\right)_{\text{ground above lake level}} \\ &= -\frac{2}{2500}(x - 437.5)^2 + 153.12, \end{aligned}$$

Notice that $g(x)$ itself is a NEW quadratic function with negative leading coefficient, so the graph of $y = g(x)$ will be a downward opening parabola. The vertex of this parabola will be $(437.5, 153.12)$, so the highest elevation of the balloon above the ground is 153.12 feet.

We can now sketch the graph of $g(x)$ and the horizontal line determined by $y = 50$ in a common coordinate system, as below. Finding where the balloon is 50 feet above the ground amounts to finding where these two graphs intersect. We need to now solve the system of equations

$$\left\{ \begin{array}{ll} y & = -\frac{2}{2500}x^2 + \frac{7}{10}x \\ y & = 50 \end{array} \right\}.$$



Plug the second equation into the first and apply the quadratic formula to get $x = 796.54$ or 78.46. This tells us the two possible x coordinates when the balloon is 50 feet above the ground. In terms of the original coordinate system imposed, the two places where the balloon is 50 feet above the ground are $(78.46, 57.85)$ and $(796.54, 129.6)$. □

How many points determine a parabola?

We all recall from elementary geometry that two distinct points in the plane will uniquely determine a line; in fact, we used this to derive equations for lines in the plane. We could then ask if there is a similar characterization of parabolas.

Three Points Determine a Standard Parabola 2.3.10: *Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$ and $R = (x_3, y_3)$ be three distinct non-collinear points in the plane such that the x-coordinates are all different. Then there exists a unique standard parabola passing through these three points. This parabola is the graph of a quadratic function $y = f(x) = ax^2+bx+c$ and we can find these coefficients by simultaneously solving the system of three equations and three unknowns obtained by assuming P, Q and R are points on the graph of $y = f(x)$:*

$$\left\{ \begin{array}{ll} ax_1^2 + bx_1 + c & = y_1 \\ ax_2^2 + bx_2 + c & = y_2 \\ ax_3^2 + bx_3 + c & = y_3. \end{array} \right\}.$$

Example 2.3.11: *Assume the value of a particular house in Seattle has increased in value according to a quadratic function $y = v(x)$, where the units of y are in dollars and x represents the number of years the property has been owned. Suppose the house was purchased on January 1, 1970 and valued at \$50,000. In 1980, the value of the house on January 1 was \$80,000. Finally, on January 1, 1990 the value was \$200,000. Find the value function $v(x)$, determine the value on January 1, 1996 and find when the house will be valued at \$1,000,000.*

Solution. The goal is to explicitly find the value function $y = v(x)$. We are going to work in a xy-coordinate system in which the first coordinate of any point represents time and the second coordinate represents value. We need to decide what kind of units will be used. The x-variable, which represents time, will denote the number of years the house is owned. For the y-variable, which represents value, we could use dollars. But, instead, we will follow a typical practice in real estate and use the units of K, where K = \$1000. For example, a house valued at \$235,600 would be worth 235.6 K. These will be the units we use, which essentially saves us from drowning in a sea of zeros!

We are given three pieces of information about the value of a particular house. This leads to three points in our coordinate system: $P = (0, 50)$, $Q = (10, 80)$ and $R = (20, 200)$. If we plot these points, they do not lie on a common line, so we know there is a unique quadratic function $v(x) = ax^2 + bx + c$ whose graph (which will be a parabola) passes through these three points. In order to find the coefficients a, b, c, we need to solve the system of equations:

$$\left\{\begin{array}{ll} a0^2 + b0 + c & = 50 \\ a(10)^2 + b(10) + c & = 80 \\ a(20)^2 + b(20) + c & = 200, \end{array}\right\} \quad \text{which is equivalent to the system} \quad \left\{\begin{array}{ll} c & = 50 \\ 100a + 10b + c & = 80 \\ 400a + 20b + c & = 200. \end{array}\right\}.$$

Plugging $c = 50$ into the second two equations gives the system

$$\left\{\begin{array}{ll} 100a + 10b & = 30 \\ 400a + 20b & = 150. \end{array}\right\} \tag{2.3.12}$$

Solve the first equation for a, obtaining $a = \frac{30-10b}{100}$, then plug this into the second equation to get:

$$\begin{aligned} 400(\frac{30-10b}{100}) + 20b &= 150 \\ 120 - 40b + 20b &= 150 \\ b &= -\frac{3}{2}. \end{aligned}$$

Now, plug $b = -\frac{3}{2}$ into the first equation of (2.3.12) to get $100a + 10(-\frac{3}{2}) = 30$; i.e., $a = \frac{9}{20}$. We conclude that

$$y = v(x) = \frac{9}{20}x^2 - \frac{3}{2}x + 50,$$

keeping in mind the units here are K.

To find the value of the house on January 1, 1996, we simply note this is after $x = 26$ years of ownership. Plugging in, we get $y = v(26) = \frac{9}{20}(26)^2 - \frac{3}{2}26 + 50 = 315.2$; i.e. the value of the house is \$315,200. To find when the house will be worth \$1,000,000, we note that \$1,000,000=1000K and need to solve the equation

$$\begin{aligned} 1000 = v(x) &= \frac{9}{20}x^2 - \frac{3}{2}x + 50 \\ 0 &= \frac{9}{20}x^2 - \frac{3}{2}x - 950. \end{aligned}$$

By the quadratic formula,

$$x = \frac{\frac{3}{2} \pm \sqrt{(-\frac{3}{2})^2 - 4(\frac{9}{20})(-950)}}{2(\frac{9}{20})} = \frac{1.5 \pm \sqrt{1712.25}}{0.9} = 47.64 \text{ or } -44.31.$$

Because x represents time, we can ignore the negative solution and so the value of the house will be \$1,000,000 after approximately 47.64 years of ownership. □

What's Needed to Build a Quadratic Model?.

Back in (1.4.12), we highlighted the information required to determine a linear model; i.e. the data you need to come up with the linear equation. We do the analogous thing here for a quadratic model.

A quadratic model is completely determined by:

1. *Three distinct non-collinear points, or* *(2.3.12)*
2. *The vertex and one other point on the graph.*

The first approach is just (2.3.10). The second approach is based on the vertex form of a quadratic function. The idea is that we know any quadratic function $f(x)$ has the form

$$f(x) = a(x - h)^2 + k,$$

where (h, k) is the vertex. If we are given h and k, together with another point (x_o, y_o) on the graph, then plugging in gives this equation:

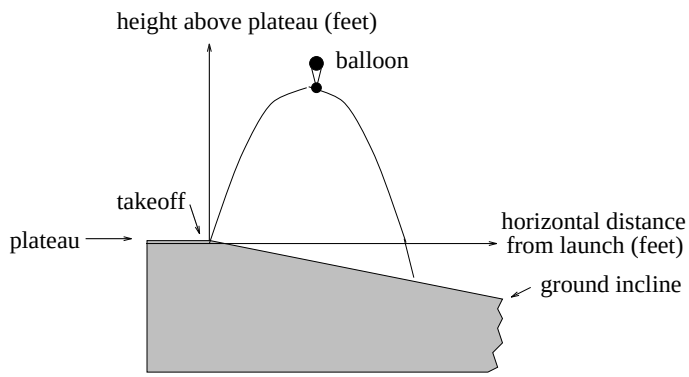
$$y_o = a(x_o - h)^2 + k.$$

The only unknown in this equation is a, which we can solve for using algebra. A couple of the exercises will depend upon these observations.

Problems

1. Write the following quadratic functions in vertex form, find the vertex and axis. Sketch a picture for each function.
   (a) $y = f(x) = 2x^2 - 16x + 41$.
   (b) $y = f(x) = x^2 - \frac{3}{7}x + 13$.
   (c) $y = f(x) = 2x^2$.

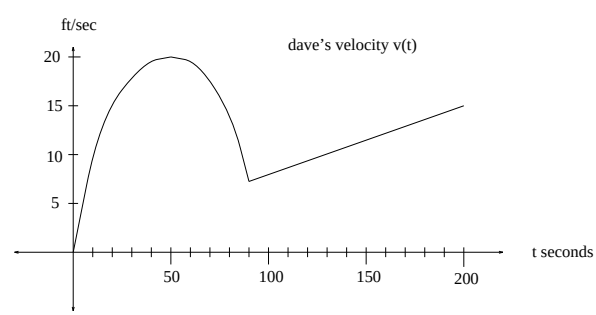
2. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows the graph of the quadratic function $y = f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. Answer the following questions:



   (a) What is the maximum height of the balloon above plateau level?
   (b) What is the maximum height of the balloon above ground level?
   (c) Where does the balloon land on the ground?
   (d) Where is the balloon 50 feet above the ground?

3. Steve likes to entertain friends at parties with "wire tricks". Suppose he takes a piece of wire 60 inches long and cuts it into two pieces. Steve takes the first piece of wire and bends it into the shape of a perfect circle. He then proceeds to bend the second piece of wire into the shape of a perfect square. Where should Steve cut the wire so that the total area of the circle and square combined is as small as possible? What is this minimal area?

4. Dave starts off on a run from the IMA. Assume he runs in a straight line for the first 200 seconds. Here is a picture of the graph of his velocity $v(t)$ at time t seconds; we will use distance units of FEET and time units of SECONDS. Here is the actual formula for Dave's velocity:



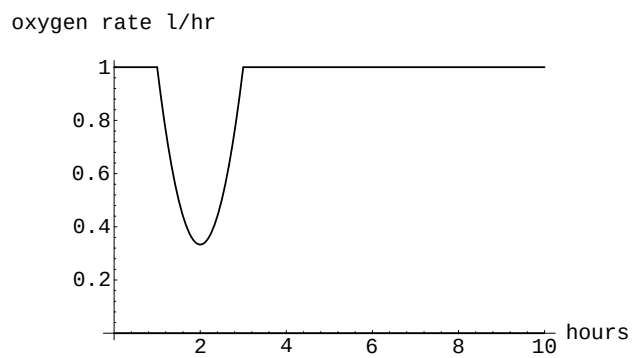
$$v(t) = \begin{cases} \frac{-1}{125}t^2 + \frac{8}{10}t, & 0 \le t \le 90 \\ 0.07091t - 0.818, & 90 \le t \le 200 \end{cases}$$

   (a) When does Dave have maximum velocity and what is his maximum velocity? What is his "pace" at this time in units of "minutes/mile"?
   (b) During the first 200 seconds, how much time does Dave spend running at a speed of at least 12 ft/sec?

2.4 Composition

A new home takes its shape from basic building materials and the skillful use of construction tools. Likewise, we can build *new* functions from known functions through the application of analogous mathematical tools. There are five tools we want to develop: composition, reflection, shifting, dilation, arithmetic. We will handle composition in this section, then discuss the others in the following two sections.

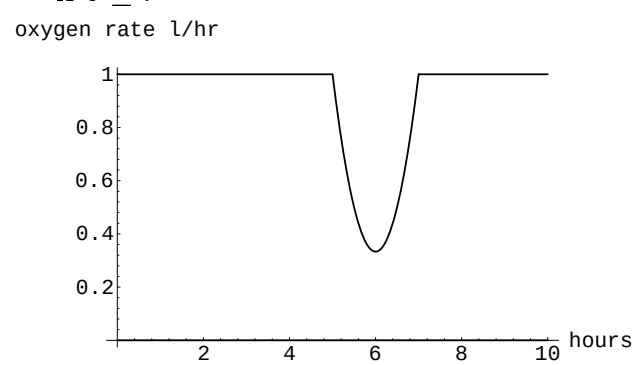
To set the stage, let's look at a simple botany experiment. Imagine a plant growing under a particular steady light source. Plants continually give off oxygen gas to the environment at some rate; common units would be liters/hour. If we leave this plant unbothered, we measure that the plant puts out 1 liter/hour; so, the oxygen output is a steady constant rate. However, if we apply a flash of high intensity green light at the time $t = 1$ and measure the oxygen output of the plant, we obtain the right-hand plot.



Using what we know from the previous section on quadratic functions, we can check that a reasonable model for the graph is this multipart function $f(t)$ (on the domain $0 \le t \le 10$):

$$f(t) = \begin{cases} 1 & \text{if } t \le 1 \\ \frac{2}{3}t^2 - \frac{8}{3}t + 3 & \text{if } 1 \le t \le 3 \\ 1 & \text{if } 3 \le t \end{cases}$$

Suppose we want to model the oxygen consumption when a green light pulse occurs at time $t = 5$ (instead of time $t = 1$), what is the mathematical model? For starters, it is pretty easy to believe that the graph for this new situation will look like the new graph pictured.

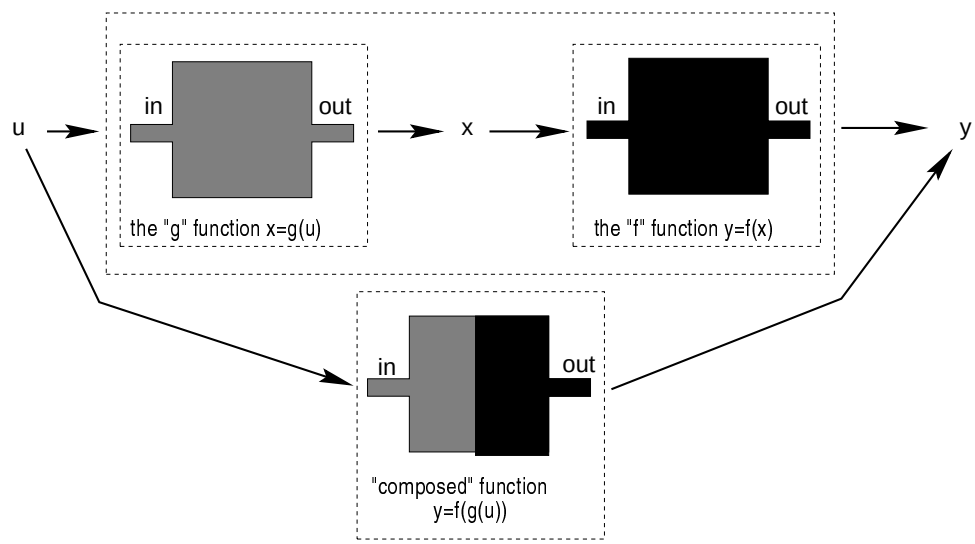


But, can we somehow use the model $f(t)$ in hand (the known function) to build the model we want (the new function)? We will return in Example 2.4.8 to see the answer is yes; first, we need to develop the tool of *function composition*.

The Formula for a Composition.

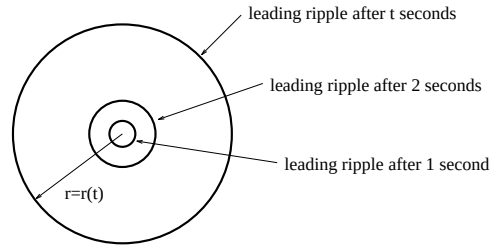
The basic idea is to start with two functions f and g and produce a new function called their *composition*. There are two basic steps in this process and we are going to focus on each seperately. The first step is fairly mechanical, though perhaps somewhat unnatural. It involves combining the formulas for the functions f and g together to get a new formula; we will focus on that step in this subsection. The next step is of varying complexity and involves analyzing how the domains and ranges of f and g affect those of the composition; we defer that to the next subsection once we have the mechanics down.

Here is a very common occurrence: We are handed a function $y = f(x)$, which means given an x value, the rule $f(x)$ produces a new y value. In addition, it my happen that the variable x is itself related to a third variable u by some *different* function equation $x = g(u)$. Given u, the rule $g(u)$ will produce a value of x; from this x we can use the rule $f(x)$ to produce a y value. In other words, we can regard y as a function depending on the *new* independent variable u. It is important to know the mechanics of working with this kind of setup. Abstractly, we have just described a situation where we take two functions and build a new function which "composes" the original ones together; schematically the situation looks like this:



Example 2.4.1: *A pebble is tossed into a pond. The radius of the first circular ripple is measured to increase at the constant rate of 2.3 ft/sec. What is the area enclosed by the leading ripple after 6 seconds have elapsed? How much time must elapse so that the area enclosed by the leading ripple is 300 square feet?*

Solution. We know that an object tossed into a pond will generate a series of concentric ripples, which grow steadily larger. We are asked questions which relate the area of the circular region bounded by the leading ripple and time elapsed.



Let r denote the radius of the leading ripple after t seconds; units of feet. The area A of a disc bounded by a leading ripple will be $A = A(r) = \pi r^2$. This exhibits A as a function in the variable r. However, the radius is changing with respect to time:

$$r = r(t) = \text{radius after } t \text{ seconds} = (2.3 \text{ feet/sec})t \text{ seconds} = 2.3t \text{ feet.}$$

So, $r = r(t)$ is a function of t. In the expression $A = A(r)$, replace "r" by "$r(t)$", then

$$A = \pi(2.3t)^2 = 5.29\pi t^2.$$

The new function $a(t) = 5.29\pi t^2$ gives a precise relationship between area and time.

To answer our first question, $a(6) = 598.3$ sq. ft. is the area of the region bounded by the leading ripple after 6 seconds. On the other hand, if $a(t) = 300$ sq. ft.: $300 = 5.29\pi t^2$, so $t = \pm\sqrt{300/(5.29\pi)} = \pm 4.25$. Since t represents time, only the positive solution $t = 4.25$ seconds makes sense. □

We can formalize the key idea used in solving this problem, which is familiar from previous courses. Suppose that

$$y = f(x)$$

and that additionally the independent variable x is itself a function of a different independent variable t; i.e.,

$$x = g(t).$$

Then we can replace every occurrence of "x" in $f(x)$ by the expression "$g(t)$", thereby obtaining y as a function in the independent variable t. We usually denote this new function of t:

$$y = f(g(t)).$$

We refer to $f(g(t))$ as the *composition* of f and g or the *composite function.*

The process of forming the composition of two functions is a mechanical procedure. If you are handed the actual formulas for $y = f(x)$ and $x = g(t)$, then :

To obtain the formula for $f(g(t))$, replace every occurrence of "x" in $f(x)$ by the expression "$g(t)$". (2.4.2)

Here are some examples of how to do this:

Examples 2.4.3: (i) If $y = f(x) = 2$ and $x = g(t) = 2t$, then $f(g(t)) = f(2t) = 2$.

(ii) If $y = 3x - 7$ and $x = g(t) = 4$, then $f(g(t)) = f(4) = 3 \cdot 4 - 7 = 5$.

(iii) If $y = f(x) = x^2 + 1$ and $x = g(t) = 2t - 1$, then

$$f(g(t)) = f(2t-1) = (2t-1)^2 + 1 = 4t^2 - 4t + 2.$$

(iv) If $y = f(x) = 2 + \sqrt{1 + (x-3)^2}$ and $x = g(t) = 2t^2 - 1$, then

$$f(g(t)) = f(2t^2 - 1) = 2 + \sqrt{1 + (2t^2 - 1 - 3)^2} = 2 + \sqrt{4t^4 - 16t^2 + 17}.$$

(v) If $y = f(x) = x^2$ and $x = g(t) = t + \heartsuit$, then

$$f(g(t)) = f(t + \heartsuit) = (t + \heartsuit)^2 = t^2 + 2t\heartsuit + \heartsuit^2.$$

It is natural to ask: What good is this whole business about compositions? One way to think of it is that we can use composite functions to break complicated functions into simpler parts. For example,

$$y = h(x) = \sqrt{x^2 + 1}$$

can be written as the composition $f(g(x))$, where $y = f(z) = \sqrt{z}$ and $z = g(x) = x^2 + 1$. Each of the functions f and g is "simpler" than the original h, which can help when studying h.

Examples 2.4.4: *(i) The function* $y = \frac{1}{(x-3)^2+2}$ *can be written as a composition* $y = f(g(x))$, *where* $y = f(z) = \frac{1}{z^2+2}$ *and* $z = g(x) = x - 3$.

(ii) The upper semicircle of radius 2 centered at (1,2) is the graph of the function $y = 2 + \sqrt{4 - (x-1)^2}$. *This function can be written as a composition* $y = f(g(x))$, *where* $y = f(z) = 2 + \sqrt{4 - z^2}$ *and* $z = g(x) = x - 1$.

Some notational confusion.

In our discussion above, we have used different letters to represent the domain variables of two functions we are composing. Typically, we have been writing: If $y = f(x)$ and $x = g(t)$, then $y = f(g(t))$ is the composition. This illustrates that the three variables t, x, y can all be of different types. For example, t might represent time, x could be speed and y could be distance.

If we are given two functions that involve the same independent variable, like $f(x) = x^2$ and $g(x) = 2x+1$, then we can still form a new function $f(g(x))$ by following the same prescription as in (2.4.2):

To obtain the formula for $f(g(x))$, replace every occurrence of "x" in $f(x)$ by the expression "$g(x)$". (2.4.5)

For our example, this gives us:

$$f(g(x)) \quad = \quad f(2x+1) = (2x+1)^2.$$

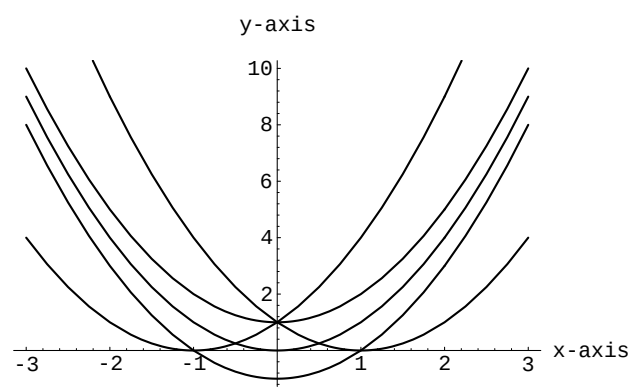
Here are three other examples:

If $f(x) = \sqrt{x}, g(x) = 2x^2 + 1$, then $f(g(x)) = \sqrt{2x^2+1}$.
If $f(x) = \dfrac{1}{x}, g(x) = 2x + 1$, then $f(g(x)) = \dfrac{1}{2x+1}$.
If $f(x) = x^2, g(x) = \triangle - x$, then $f(g(x)) = \triangle^2 - 2x\triangle + x^2$.

Example 2.4.6: *Let $f(x) = x^2$, $g(x) = x + 1$ and $h(x) = x - 1$. Find the formulas for $f(g(x)), g(f(x)), f(h(x))$ and $h(f(x))$. Discuss the a relationship between the graphs of these four functions?*

Solution. If we apply the procedure in (2.4.5), we obtain the composition formulas. The four graphs are given on the domain $-3 \le x \le 3$, together with the graph of $f(x) = x^2$.

$$\begin{aligned} f(g(x)) &= f(x+1) = (x+1)^2 \\ g(f(x)) &= g(x^2) = x^2 + 1 \\ f(h(x)) &= f(x-1) = (x-1)^2 \\ h(f(x)) &= h(x^2) = x^2 - 1. \end{aligned}$$



We can identify each graph by looking at it's vertex:

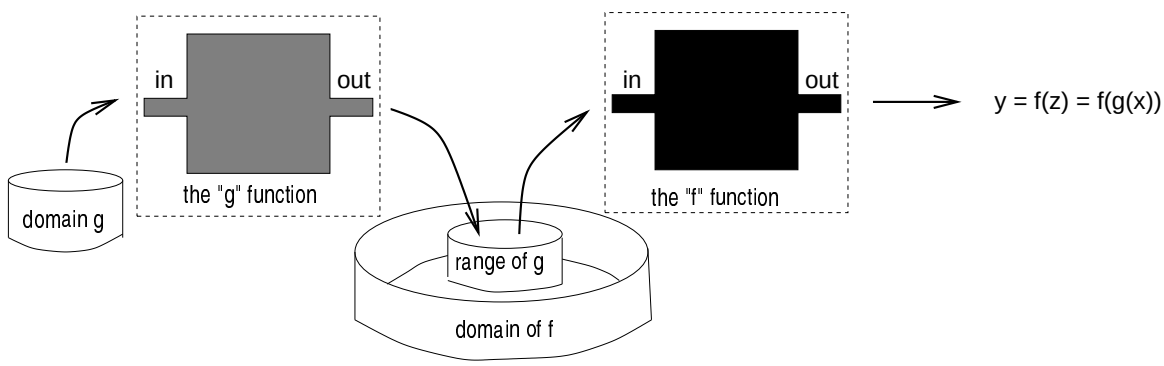
$f(x)$ has vertex (0,0)
$f(g(x))$ has vertex (-1,0)
$g(f(x))$ has vertex (0,1)

$f(h(x))$ has vertex (1,0)
$h(f(x))$ has vertex (0,-1)

Horizontal or vertical shifting of the graph of $f(x) = x^2$ gives the other four graphs. □

Domain, Range, etc. for a Composition.

A function is a "package" consisting of a rule, a domain of allowed values to plug in and a range of output values. When we start to compose functions we sometimes need to worry about how the domains and ranges of the composing functions affect the composed function. First off, when you form the composition $f(g(x))$ of $f(x)$ and $g(x)$, the range values for $g(x)$ must lie within the domain values for $f(x)$. This may require that you modify the range values of $g(x)$ by changing its domain. The domain values for $f(g(x))$ will be the domain values for $g(x)$:



In practical terms, here is how one deals with the domain issues for a composition. This is a refinement of (2.4.5).

To obtain the formula for $f(g(x))$, replace every occurrence of "x" in $f(x)$ by the expression "$g(x)$". In addition, if there is a condition on the domain of f that involves x, then replace every occurrence of "x" in that condition by the expression "$g(x)$". (2.4.7)

The next example illustrates how to use this principal.

Example 2.4.8: *Start with the function $y = f(x) = x^2$ on the domain $-1 \le x \le 1$. Find the rule and domain of $y = f(g(x))$, where $g(x) = x - 1$.*

Solution. We can apply the first statement in (2.4.7) to find the rule for $y = f(g(x))$:

$$\begin{aligned} y &= f(g(x)) \\ &= f(x-1) = (x-1)^2 \\ &= x^2 - 2x + 1. \end{aligned}$$

To find the domain of $y = f(g(x))$, we apply the second statement in (2.4.7); this will require that we solve an inequality equation:

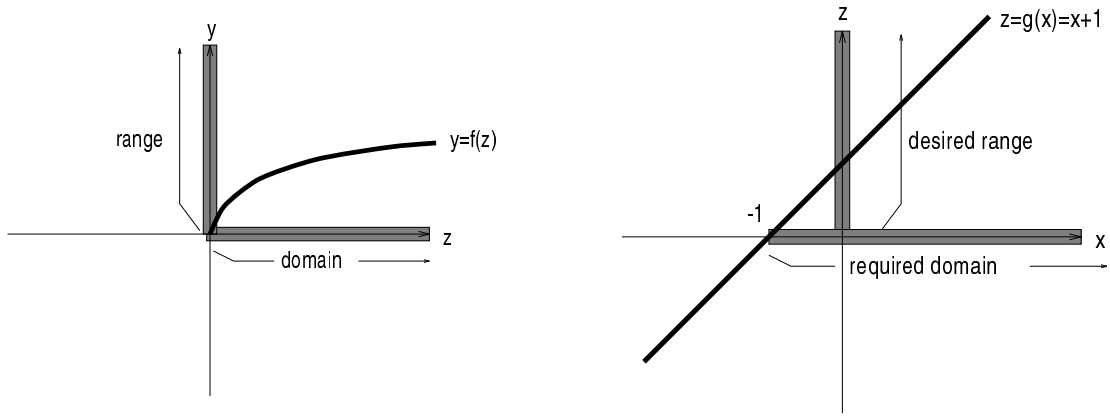
$$\begin{array}{rcl} -1 \leq & g(x) & \leq 1 \\ -1 \leq & x-1 & \leq 1 \\ 0 \leq & x & \leq 2 \end{array}$$

The conclusion is that $y = f(g(x)) = x^2 - 2x + 1$ on the domain $0 \leq x \leq 2$.

□

Example 2.4.9: *Let $y = f(z) = \sqrt{z}$, $z = g(x) = x + 1$. What is the largest possible domain so that the composition $f(g(x))$ makes sense?*

Solution. The largest possible domain for $y = f(z)$ will consist of all non-negative real numbers; this is also the range of the function $f(z)$ (see left-hand picture).



To find the largest domain for the composition, we try to find a domain of x-values so that the range of $z = g(x)$ is the domain of $y = f(z)$. So, in this case, we want the range of $g(x)$ to be all non-negative real numbers, denoted $0 \leq z$. We graph $z = g(x)$ in the xz-plane, mark the desired range $0 \leq z$ on the vertical z-axis, then determine which x-values would lead to points on the graph with second coordinates in this zone. We find that the domain of all

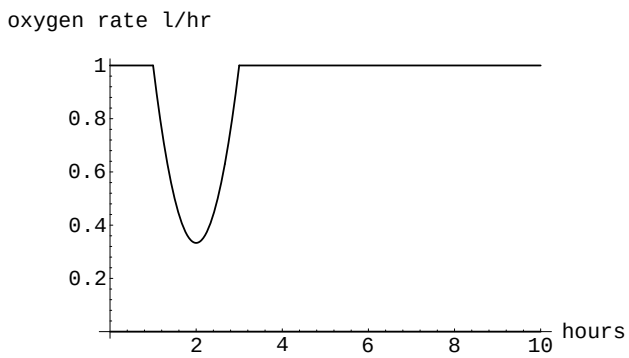
x-values greater or equal to -1 (denoted $-1 \leq x$) leads to the desired range. In summary, the composition $y = f(g(x)) = \sqrt{x+1}$ is defined on the domain of x-values $-1 \leq x$.

□

Let's return to the botany experiment that opened this section and see how composition of functions can be applied to the situation. Recall that plants continually give off oxygen gas to the environment at some rate; common units would be liters/hour.

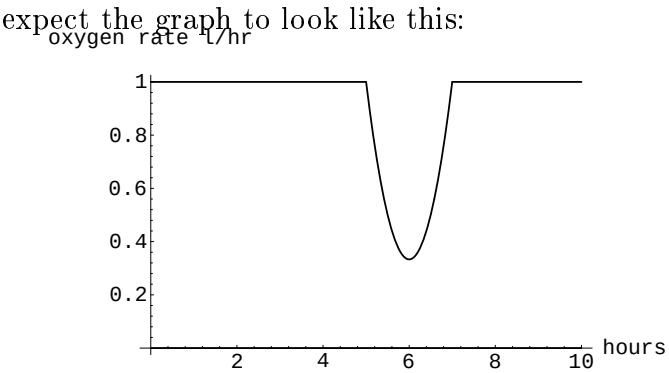
Example 2.4.10: *A plant is growing under a particular steady light source. If we apply a flash of high intensity green light at the time $t = 1$ and measure the oxygen output of the plant, we obtain the plot below and the mathematical model $f(t)$.*

$$f(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ \frac{2}{3}t^2 - \frac{8}{3}t + 3 & \text{if } 1 \leq t \leq 3 \\ 1 & \text{if } 3 \leq t \end{cases}$$



Now, suppose instead we apply the flash of high intensity green light at the time $t = 5$. Verify that the mathematical model for this experiment is given by f(g(t)), where $g(t) = t - 4$.

Solution. Our expectation is that the plot for this new experiment will have the "parabolic dip" shifted over to occur starting at time $t = 5$ instead of at time $t = 1$. In other words, we expect the graph to look like this:



Our job is to verify that this graph is obtained from the function $f(g(t))$, where $g(t) = t - 4$. This is a new terrain for us, since we need to look at a composition involving a multipart function. Here is how to proceed: When we are calculating a composition involving a multipart function, we need to look at each of the parts seperately; so there will be three cases to consider:

First part: $f(t) = 1$ when $t \leq 1$. To get the formula for $f(g(t))$, we now appeal to (2.4.7) and just replace every occurance of t in $f(t)$ by $g(t)$. That gives us this NEW domain condition and function equation:

$$\begin{aligned} f(g(t)) &= f(t-4) = 1 \quad \text{when } t - 4 \leq 1 \\ &= 1 \quad \text{when } t \leq 5. \end{aligned}$$

Second part: $f(t) = \frac{2}{3}t^2 - \frac{8}{3}t + 3$ when $1 \leq t \leq 3$. We now appeal to (2.4.7) and just replace every occurance of t in this function by $g(t)$. That gives us this NEW domain condition and function equation:

$$\begin{aligned} f(g(t)) &= f(t-4) = \frac{2}{3}(t-4)^2 - \frac{8}{3}(t-4) + 3 \quad \text{when } 1 \leq t - 4 \leq 3 \\ &= \frac{2}{3}t^2 - 8t + \frac{73}{3} \quad \text{when } 5 \leq t \leq 7. \end{aligned}$$

Third part: $f(t) = 1$ when $3 \leq t$. We now appeal to (2.4.7) and just replace every occurance of t in this function by $g(t)$. That gives us this NEW domain condition and function equation:

$$\begin{aligned} f(g(t)) &= f(t-4) = 1 \quad \text{when } 3 \leq t - 4 \\ &= 1 \quad \text{when } 7 \leq t. \end{aligned}$$

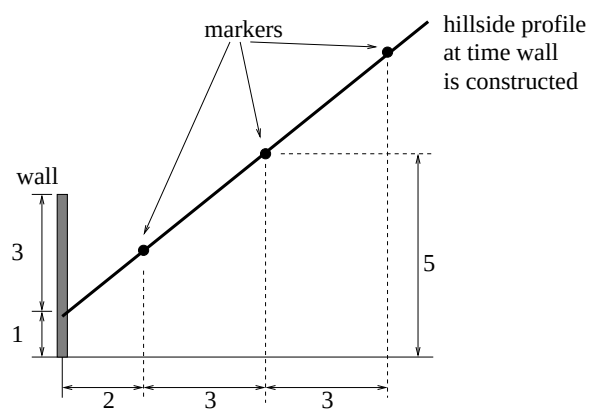
The multipart rule for this composition can now be written down and using a graphing device you can verify it's graph is the model for our experiment.

$$f(g(t)) = \begin{cases} 1 & \text{if } t \leq 5 \\ \frac{2}{3}t^2 - 8t + \frac{73}{3} & \text{if } 5 \leq t \leq 7 \\ 1 & \text{if } 7 \leq t \end{cases}$$

□

Problems

1. Compute the compositions $f(g(x))$, $f(f(x))$ and $g(f(x))$ in each case:
   (a) $f(x) = 1/x, g(x) = \sqrt{x}$.
   (b) $f(x) = 9x + 2, g(x) = \frac{1}{9}(x - 2)$.
   (c) $f(x) = 6x^2 + 5, g(x) = x - 4$.
   (d) $f(x) = 3, g(x) = 4x^2 + 2x + 1$.
2. A car leaves Seattle heading east. The speed of the car in mph after m minutes is given by the function
$$C(m) = \frac{70m^2}{10 + m^2}.$$
   (a) Find a function $m = f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C(f(s))$; what does this function calculate?
   (b) Find a function $m = g(h)$ that converts hours h into minutes m. Write our the formula for the new function $C(g(h))$; what does this function calculate?
3. A contractor has just built a retaining wall to hold back a sloping hillside. To monitor the movement of the slope the contractor places marker posts at the positions indicated in the picture; all dimensions are taken in units of meters.



   (a) Find a function $y = f(x)$ that models the profile of the hillside.
   (b) Assume that the hillside moves as time goes by and the profile is modeled by a function $g_n(x)$ after n years. If $n = 0$, then $g_0(x) = f(x)$. After one year, the profile is modeled by the function $g_1(x) = f(f(x))$. After two years, the profile is modeled by the function $g_2(x) = f(f(f(x)))$. After n years, it is modeled by the function $g_n(x) = f(f(\ldots f(x)\ldots))$, where we have composed the original function $n + 1$ times. Find a formula for $g_n(x)$ that does not involve compositions. (Hint: To do this, start by writing out the formulas for $n = 1, 2, 3, 4$. You will see a pattern developing. To get the general formula, the following fact will be useful: Given a real number $0 < r < 1$

and a positive integer k,

$$1 + r + r^2 + r^3 + r^4 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}.$$

Your final formula for $g_n(x)$ will involve both x and n.)

(c) Sketch the graphs of $g_n(x)$ for $n = 0, 1, 2, 3, 4, 5$ in the same coordinate system.

(d) What is happening to the marker posts?

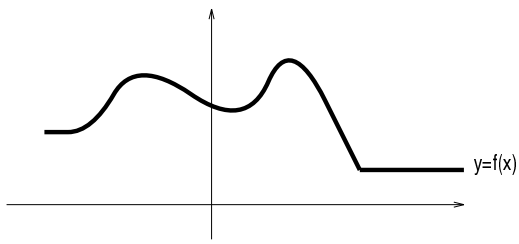
(e) Estimate when the hillside will start to spill over the retaining wall.

2.5 Three Construction Tools

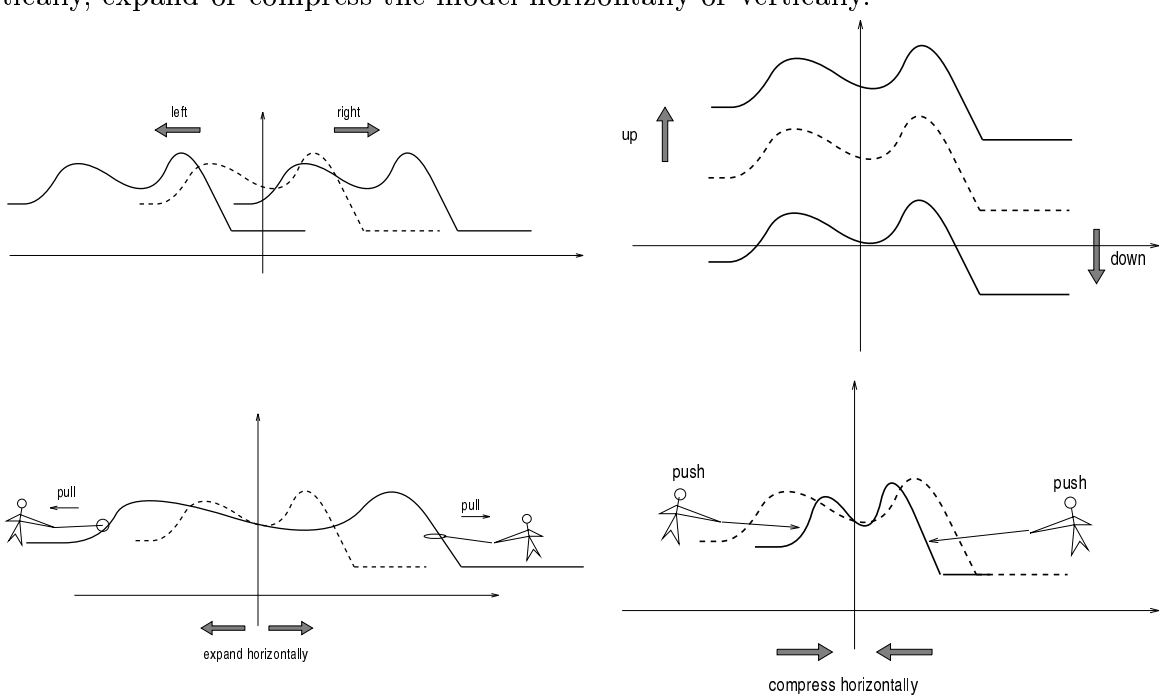
Sometimes the composition of two functions can be understood by graphical manipulation. When we discussed quadratic functions and parabolas in the previous section, certain key graphical manuevers were laid out. In this section, we extend those graphical techniques to general function graphs.

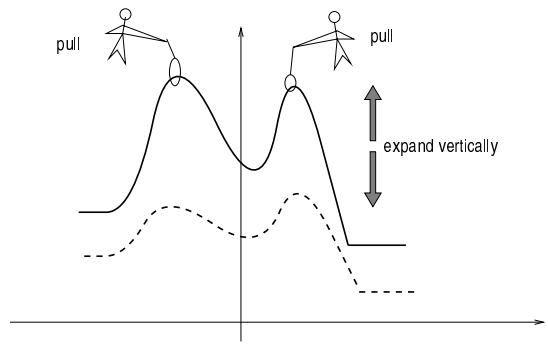
A Low-tech Exercise.

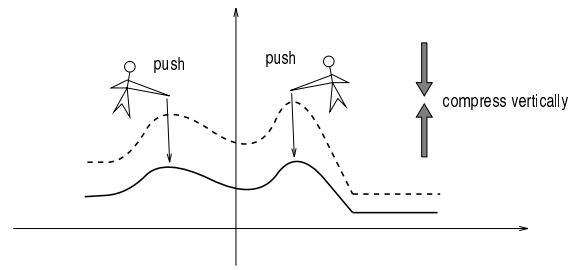
This section is all about building new functions from ones we already have in hand. This can be approached symbolically or graphically. Let's begin with a simple hands-on exercise involving this curve:



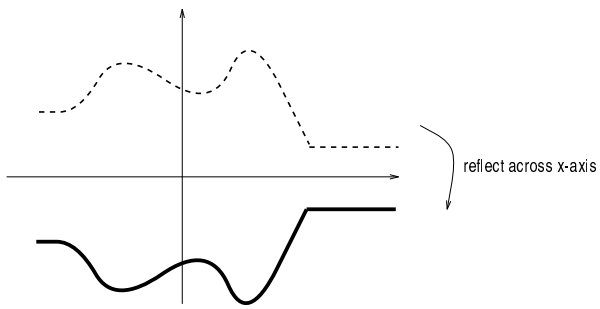
By the vertical line test, we know this represents the graph of a function $y = f(x)$. With this picture and a piece of bendable wire we can build an INFINITE number of new functions from the original function. Begin by making a "model" of this graph by bending a piece of wire to the exact shape of the graph and place it right on top of the curve. The wire model can be manipulated in a variety of ways: slide the model back and forth horizontally, up and down vertically, expand or compress the model horizontally or vertically.

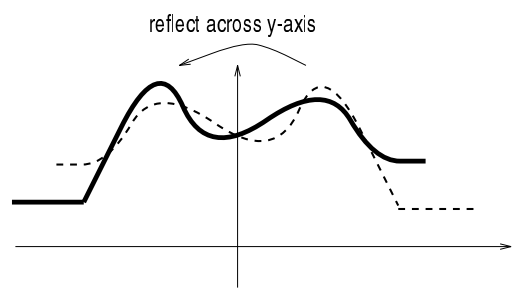




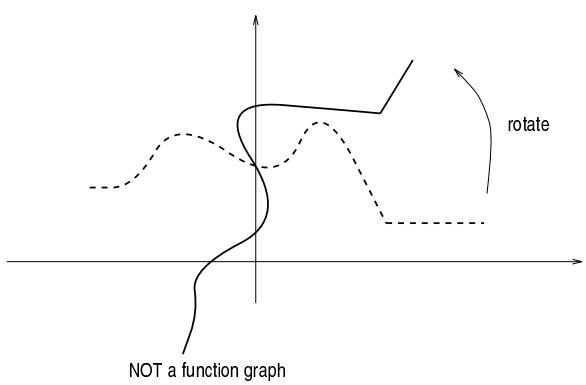


Another way to build new curves from old ones is to exploit the built in symmetry of the xy-coordinate system. For example, imagine reflecting the graph of $y = f(x)$ across the x-axis or the y-axis:





In all of the above cases, we moved from the original wire model of our function graph to a *new curve* which (by the vertical line test) is the graph of a *new function*. The big caution in all this is that we are NOT ALLOWED to rotate or twist the curve; this kind of maneuver does lead to a new curve, but it may not be the graph of a function:



These pictures highlight most of what we have to say in this section; the hard work remaining is a symbolic reinterpretation of these graphical operations.

Reflection.

In order to illustrate the technique of *reflection*, we will use a concrete example:

$$y = p(x) = 2x + 2, \text{domain: } -2 \leq x \leq 2, \text{range: } -2 \leq y \leq 6.$$

As we know, the graph of $y = p(x)$ on the domain $-2 \leq x \leq 2$ is a line of slope 2 with y-intercept 2, as pictured at right. Now, start with the function equation $y = p(x) = 2x + 2$ and replace every occurrence of "y" by "$-y$". This produces the new equation $-y = 2x + 2$; or, equivalently,

$$y = q(x) = -2x - 2.$$

The domain is still $-2 \leq x \leq 2$, but the range will change; we obtain the new range by replacing "y" by "$-y$" in the original range: $-2 \leq -y \leq 6$; so $2 \geq y \geq -6$. The graph of this function is a DIFFERENT line; this one has slope -2 and y-intercept -2. We contrast these two curves in the left-hand picture below, where $q(x)$ is graphed as the "dashed line" in the same picture with the original $p(x)$. Once we do this, it is easy to see how the graph of $q(x)$ is really just the original curve reflected across the x-axis.

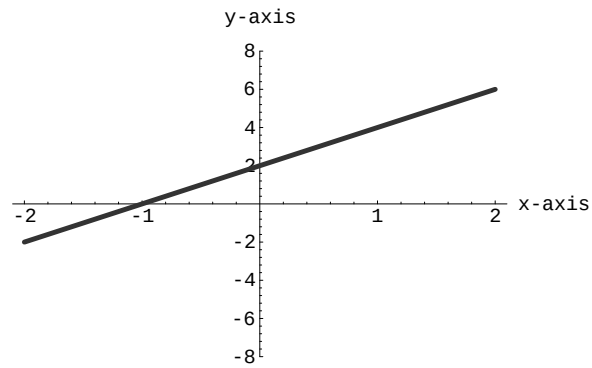
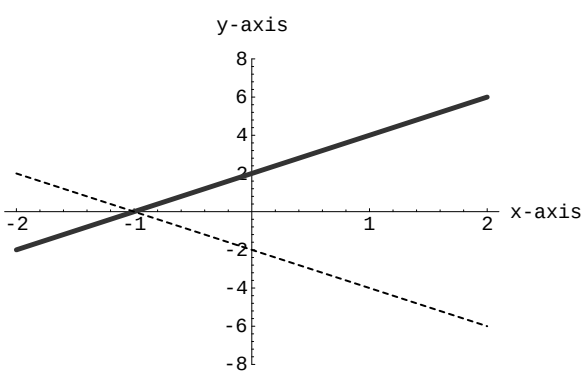
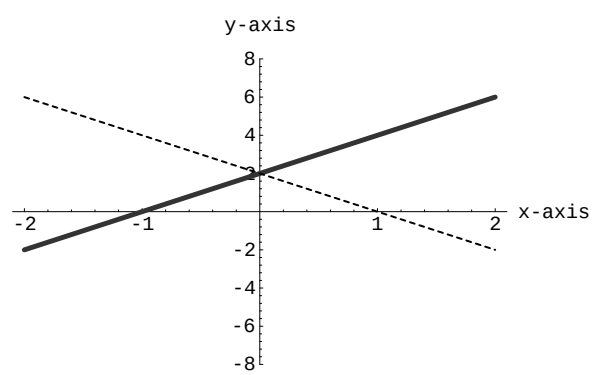


Figure 2.5.1: Graph of $y = 2x + 2$





Next, take the original function equation $y = p(x) = 2x + 2$ and replace every occurrence of "x" by "$-x$". This produces a new equation $y = 2(-x) + 2$; or, equivalently,

$$y \;=\; r(x) = -2x + 2.$$

The domain must also be checked by replacing "x" by "$-x$" in the original domain condition: $-2 \leq -x \leq 2$, so $2 \geq x \geq -2$. It just so happens in this case, the domain is unchanged. This is yet another DIFFERENT line; this one has slope -2 and y-intercept 2. We contrast these two curves in the right-hand picture above, where $r(x)$ is graphed as the "dashed line" in the same picture with the original $p(x)$. Once we do this, it is easy to see how the graph of $r(x)$ is really just the original curve reflected across the y-axis.

This example illustrates a general principle referred to as the *reflection principle*.

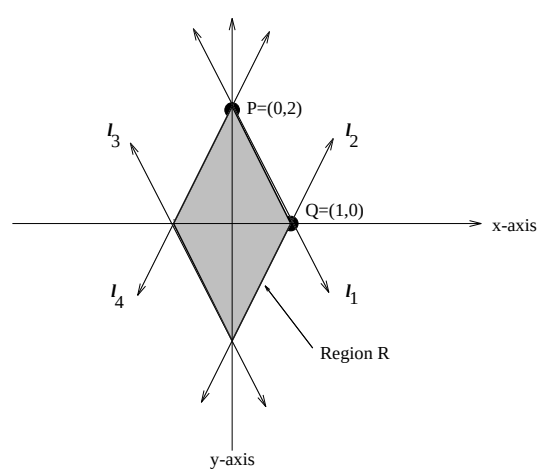
Reflection Principle 2.5.1: *Let $y = f(x)$ be a function equation.*

(i) We can reflect the graph across the x-axis and the resulting curve is the graph of the new function obtained by replacing "y" by "$-y$" in the original equation. The domain is the same as the domain for $y = f(x)$. If the range for $y = f(x)$ is $c \leq y \leq d$, then the range of $-y = f(x)$ is $c \leq -y \leq d$. In other words, the reflection across the x-axis is the graph of $y = -f(x)$.

(ii) We can reflect the graph across the y-axis and the resulting curve is the graph of the new function obtained by replacing "x" by "$-x$" in the original equation. The range is the same as the range for $y = f(x)$. If the domain for $y = f(x)$ is $a \leq x \leq b$, then the domain of $y = f(-x)$ is $a \leq -x \leq b$. Using composition notation, the reflection across the y-axis is the graph of $y = f(-x)$.

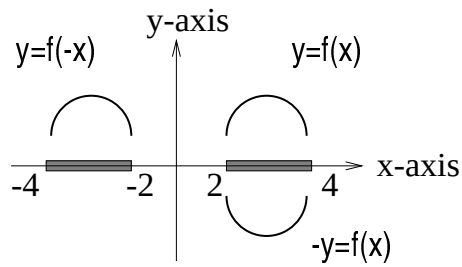
Example 2.5.2: *Consider the parallelogram-shaped region $\mathcal{R}$ with vertices (0,2), (0,-2), (1,0) and (-1,0). Use the reflection principle to find functions whose graphs bound $\mathcal{R}$.*

Solution. Here is a picture of the region $\mathcal{R}$: First off, using the two point formula for the equation of a line, we find that the line ℓ_1 passing through the points $P = (0, 2)$ and $Q = (1, 0)$ is the graph of the function $y = f_1(x) = -2x+2$. By (2.5.1)(i), ℓ_2 is the graph of the equation $-y = -2x + 2$, which we can write as the function $y = f_2(x) = 2x - 2$. By (1.7.1)(ii) applied to ℓ_2, the line ℓ_3 is the graph of the function $y = f_3(x) = -2x - 2$. Finally, by (2.5.1)(i) applied to ℓ_3, the line ℓ_4 is the graph of the equation $-y = -2x-2$, which we can write as the function $y = f_4(x) = 2x + 2$. $\square$



Here is an illustration of the fact that we need to be careful about the domain of the original function when using the reflection principle. For example, consider $y = f(x) = 1 + \sqrt{1-(x-3)^2}$. The largest possible domain of x-values is $2 \le x \le 4$ and the graph is an upper semicircle of radius 1 centered at the point (3,1).

Reflection across the x-axis gives the graph of $y = -1 - \sqrt{1-(x-3)^2}$ on the same domain; reflection across the y-axis gives the graph of $y = 1 + \sqrt{1-(x+3)^2}$ on the new domain $-4 \le x \le -2$.



Shifting.

Let's start out with the function

$$y = f(x) = \sqrt{4-x^2},$$

which has largest possible domain $-2 \le x \le 2$. From §2.2, the graph of this equation is an upper semicircle of radius 2 centered at the origin (0,0). Sliding the graph back and forth horizontally or vertically (or both), never rotating or twisting, we are led to the "dashed curves" below (contrasted with the original graph which is plotted with a solid curve). This describes some shifted curves on a pictorial level, but what are the underlying equations? For this example, we can use the fact all of the shifted curves are still semicircles and §2.2 tells us how to find their equations.

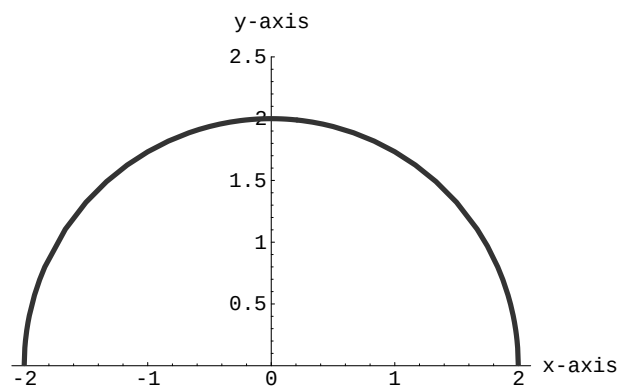
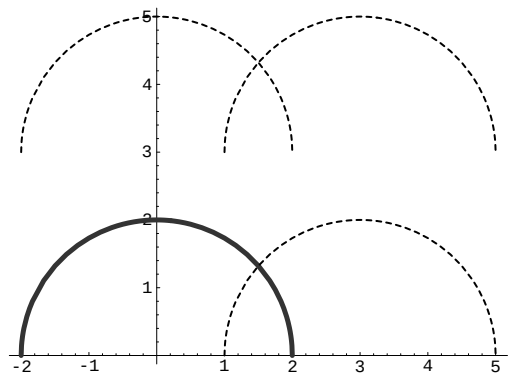


Figure 2.5.2: Graph of $y = \sqrt{4-x^2}$

The lower right-hand dashed semicircle is of radius 2 and centered at (3,0), so the corresponding equation must be $y = \sqrt{4-(x-3)^2}$. The upper left-hand dashed semicircle is of radius 2 and centered at (0,3), so the corresponding equation must be $y = 3 + \sqrt{4-x^2}$. The upper right-hand dashed semicircle is of radius 2 and centered at (3,3), so the corresponding equation must be $y = 3 + \sqrt{4-(x-3)^2}$.



Keeping this same example, we can continue this kind of shifting more generally by thinking about the effect of making the following three replacements in the equation $y = \sqrt{4 - x^2}$:

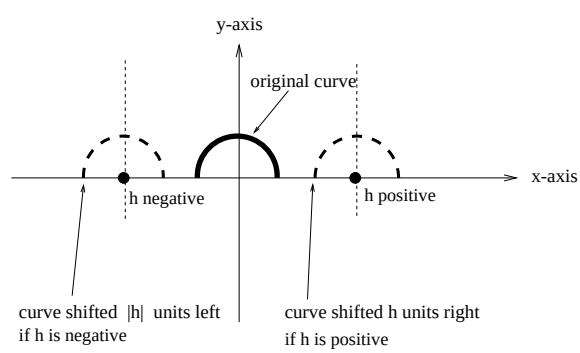
$$x \rightsquigarrow x - h; \qquad y \rightsquigarrow y - k; \qquad (x \text{ and } y) \rightsquigarrow (x - h \text{ and } y - k).$$

These substitutions lead to three new equations, each the equation of a semicircle:

$$\begin{aligned} y &= \sqrt{4 - (x-h)^2}; && \Leftarrow \text{upper semicircle with radius 2 and center } (h, 0) \\ y - k &= \sqrt{4 - x^2}; && \Leftarrow \text{upper semicircle with radius 2 and center } (0, k) \\ y - k &= \sqrt{4 - (x-h)^2} && \Leftarrow \text{upper semicircle with radius 2 and center } (h, k). \end{aligned}$$

There are three potentially confusing points with this example:

- Be careful with the sign (i.e.$\pm$) of h and k. In the above picture, if $h = 1$, we horizontally shift the semicircle 1 unit to the right; whereas, if $h = -1$, we horizontally shift the semicircle -1 units to the right. But, shifting -1 unit to the right is the same as shifting 1 unit to the left! In other words, if h is positive (resp. negative), then a horizontal shift by h will move the graph $|h|$ units to the right (resp. left).



- If k is positive (resp. negative), then a vertical shift by k will move the graph $|k|$ units up (resp. down). These conventions insure "positivity" of h and k match up with "rightward" and "upward" movement of the graph.
- When shifting, the domain of allowed x-values may change.

This example illustrates the an important general principle referred to as the *shifting principle*.

Shifting Principle 2.5.3: *Let $y = f(x)$ be a function equation.*

(i) If we replace "x" by "$x - h$" in the original function equation, then the graph of the resulting new function $y = f(x - h)$ is obtained by horizontally shifting the graph of $f(x)$ by h. If h is positive, the picture shifts to the right h units; if h is negative, the picture shifts to the left h units. If the domain of $f(x)$ is an interval $a \leq x \leq b$, then the domain of $f(x - h)$ is $a \leq x - h \leq b$. The range remains unchanged under horizontal shifting.

(ii) If we replace "y" by "$y - k$" in the original function equation, then the graph of the resulting new function $y = f(x) + k$ is obtained by vertically shifting the graph of $f(x)$ by k. If k is positive, the picture shifts upward k units; if k is negative, the picture shifts downward k units. If the range of $f(x)$ is an interval $c \leq y \leq d$, then the range of $f(x) + k$ is $c \leq y - k \leq d$. The domain remains unchanged under vertical shifting.

<u>Dilation</u>.

To introduce the next graphical principle we will look at the function

$$y \;=\; f(x) = \frac{x}{x^2+1}.$$

Using a graphing device, we have produced a plot of the graph on the domain $-6 \leq x \leq 6$.

From the picture, we see that the curve has a high point H (like a "mountain peak") and a low point L (like a "valley"). Using a graphing device, we can determine that the high point is $H = (1, \frac{1}{2})$ (it lies on the line with equation $y = \frac{1}{2}$) and the low point is $L = (-1, -\frac{1}{2})$ (it lies on the line with equation line $y = -\frac{1}{2}$). So, this all tells us that the range is $-\frac{1}{2} \leq y \leq \frac{1}{2}$. Draw two new horizontal lines with equations $y = 2 \cdot \pm\frac{1}{2} = \pm 1$. Grab the high point H on the curve and uniformly pull straight up, so that the high point now lies on the horizontal line $y = 1$ at $(1, 1)$. Repeat this process by pulling L straight downward, so that the low point is now on the line $y = -1$ at $(-1, -1)$. We end up with the "stretched dashed curve" illustrated in the left-hand figure below. In terms of the original function equation $y = \frac{x}{x^2+1}$, we are simply describing the graphical effect of multiplying the y-coordinate of every point on the curve by the positive number 2. In other words, the left-hand dashed curve is the graph of $y = \frac{2x}{x^2+1} = 2\left(\frac{x}{x^2+1}\right)$.

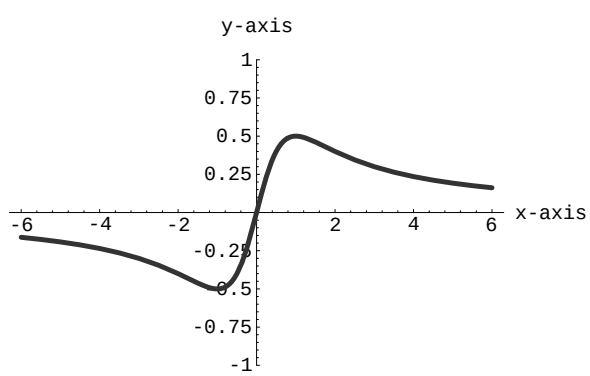
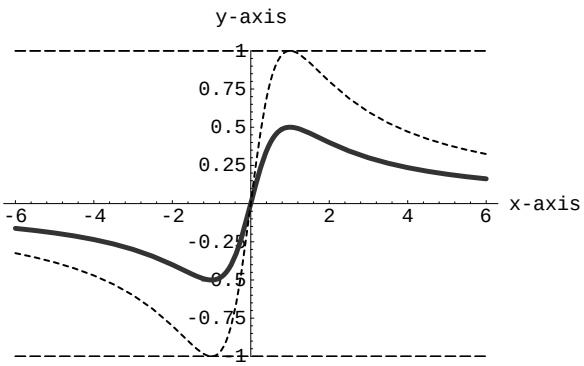
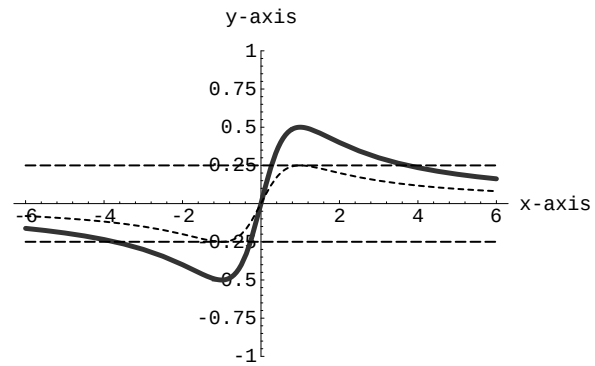


Figure 2.5.3: Graph of $y = \frac{x}{x^2+1}$

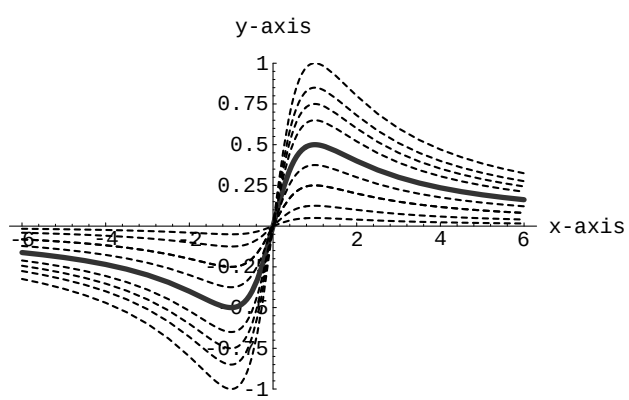




Next, draw the two horizontal lines with equations $y = \frac{1}{2} \cdot \pm\frac{1}{2} = \pm\frac{1}{4}$. Grab the high point H on the curve (in Figure 2.5.3) and uniformly push straight down, so that the high point now lies on the horizontal line $y = \frac{1}{4}$ at $(1, \frac{1}{4})$. Repeat this process at the low point L by pushing the curve straight upward, so that the low point is now on the line $y = -\frac{1}{4}$ at $(-1, -\frac{1}{4})$. We end up with the new "dashed curve" illustrated in the right-hand figure above. In terms of the original function equation $y = \frac{x}{x^2+1}$, we are simply describing the graphical effect of multiplying the y-coordinate of every point on the curve by the positive number $\frac{1}{2}$. In other words, the dashed curve is the graph of $y = \frac{x}{2(x^2+1)}$.

We could repeat this process systematically:

- Pick a positive number c.
- Draw the two horizontal lines $y = c \cdot \pm\frac{1}{2}$. If $c > 1$, then the graph of $y = \frac{c}{2}$ is parallel and above the graph of $y = \frac{1}{2}$. On the other hand, if $0 < c < 1$, then this new line is parallel and below $y = \frac{1}{2}$.
- Uniformly deform the original graph (in Figure 2.5.3) so that the new curve has it's high and low points just touching $y = \pm\frac{c}{2}$. This will involve vertically stretching or compressing, depending on whether $1 < c$ or $0 < c < 1$, respectively. A number of possibilities are pictured.



We refer to each new dashed curve as a *vertical dilation* of the original (solid) curve. This example illustrates an important principle.

Vertical Dilation Principle 2.5.4: *Let $c > 0$ be a positive number and $y = f(x)$ a function equation.*

(i) If we replace "y" by "$\frac{y}{c}$" in the original equation, then the graph of the resulting new equation is obtained by vertical dilation of the graph of $y = f(x)$. The domain of x-values is not affected.

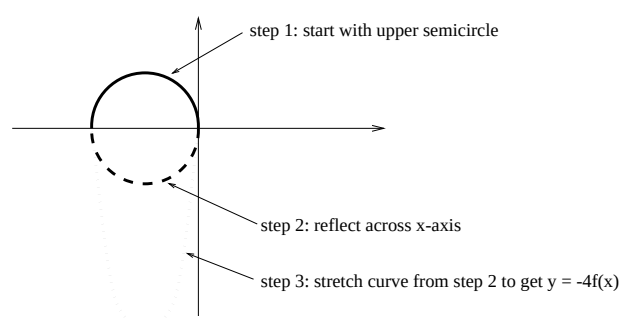
(ii) If $c > 1$, then the graph of $\frac{y}{c} = f(x)$ (i.e. $y = cf(x)$) is a vertically stretched version of the original graph.

(iii) If $0 < c < 1$, then the graph of $\frac{y}{c} = f(x)$ (i.e. $y = cf(x)$) is a vertically compressed version of the original graph.

If we combine dilation with reflection across the x-axis, we can determine the graphical relationship between $y = f(x)$ and $y = cf(x)$, for *any* constant c. The key observation is that reflection across the x-axis corresponds to the case $c = -1$.

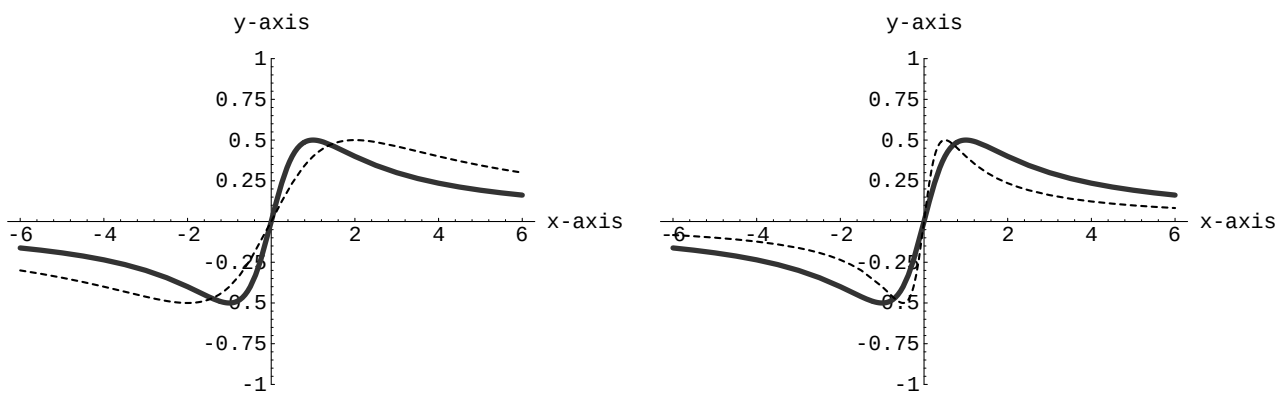
Example 2.5.5: *Describe the relationship between the graphs of $y = f(x) = \sqrt{1-(x+1)^2}$, $y = -f(x) = -\sqrt{1-(x+1)^2}$ and $y = -4f(x) = -4\sqrt{1-(x+1)^2}$.*

Solution. The graph of $y = f(x)$ is an upper semicircle of radius 1 centered at the point $(-1, 0)$. To obtain the picture of the graph of $y = -4f(x)$, we first reflect $y = f(x)$ across the x-axis; this gives us the graph of $y = -f(x)$. Then, we vertically dilate this picture by a factor of $c = 4$ to get the graph of $\frac{y}{4} = -f(x)$; which is the same as the graph of the equation $y = -4f(x)$. □



Let's return to the original example $y = \frac{x}{x^2+1}$ and investigate a different type of dilation where the action is taking place in the horizontal direction (whereas it was in the vertical direction before). Grab the right-hand end of the graph (in Figure 2.5.3) and pull to the right, while at the same time pulling the left-hand end to the left. We can quantify this by asking that the high point $H = (1, \frac{1}{2})$ of the original curve moves to the new location $(2, \frac{1}{2})$ and the low point $L = (-1, -\frac{1}{2})$ moves to the new location $(-2, -\frac{1}{2})$. The result will be somewhat analogous to stretching a spring. By the same token, we could push the right-hand end to the

left and push the left-hand end to the right, like compressing a spring. We can quantify this by asking that the high point $H = (1, \frac{1}{2})$ of the original curve moves to the new location $(\frac{1}{2}, \frac{1}{2})$ and the low point $L = (-1, -\frac{1}{2})$ moves to the new location $(-\frac{1}{2}, -\frac{1}{2})$. These two situations are indicated below:



We refer to each of the dashed curves as a *horizontal dilation* of the original (solid) curve.

The tricky point is to understand what happens on the level of the original equation. In the case of the left-hand stretched graph, you can use a graphing device to check this looks like the graph of $y = \frac{x/2}{(x/2)^2+1}$; in other words, we replaced "x" by "$x/2$" in the original equation. In the case of the right-hand compressed graph, you can use a graphing device to check this looks like the graph of $y = \frac{2x}{(2x)^2+1}$; in other words, we replaced "x" by "$\frac{x}{1/2} = 2x$" in the original equation.

The process just described leads to a general principle.

Horizontal Dilation Principle 2.5.6: *Let $c > 0$ be a positive number and $y = f(x)$ a function equation.*

(i) If we replace "x" by "$\frac{x}{c}$" in the original function equation, then the graph of the resulting new function $y = f(\frac{x}{c})$ is obtained by a horizontal dilation of the graph of $y = f(x)$. If the domain of $f(x)$ is $a \le x \le b$, then the domain of $y = f(\frac{x}{c})$ is $a \le \frac{x}{c} \le b$.

(ii) If $c > 1$, then the graph of $y = f(\frac{x}{c})$ is a horizontal stretch.

(iii) If $0 < c < 1$, then the graph of $y = f(\frac{x}{c})$ is a horizontal compression.

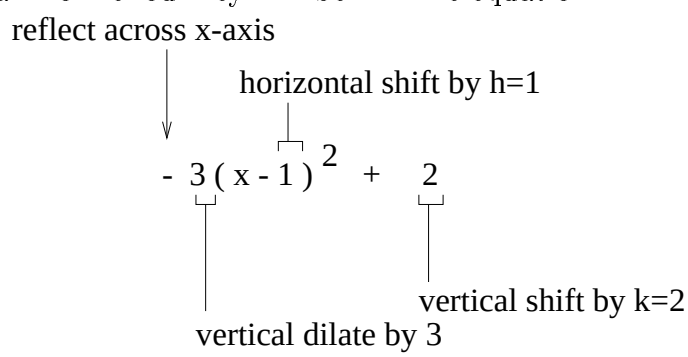
Vertex Form and Order of Operations.

Using the language of function compositions we can clarify our discussion in Example 2.3.2. Let's revisit that example:

Example 2.5.7: *The problem is to describe a sequence of geometric maneuvers that transform the graph of $y = x^2$ into the graph of $y = -3(x-1)^2 + 2$.*

Solution. The idea is to rewrite $y = -3(x-1)^2 + 2$ as a composition of $y = x^2$ with four other functions, each of which corresponds to a horizontal shift, vertical shift, reflection or dilation. Once we have done this, we can read off the order of geometric operations using the order of composition. Along the way, it is good to pay special attention to the exact order in which we will be composing our functions; this will make a big difference and is usually very confusing the first time through.

To begin with, we can isolate four key numbers in the equation:



We want to use each number to define a *new* function, then compose these in the correct order. We will also give our starting function $y = x^2$ a specific name to make things definite:

$$\begin{aligned} f(x) &= x^2 & \qquad h(x) &= x - 1 \\ v(x) &= x + 2 & \qquad r(x) &= -x \\ d(x) &= 3x. \end{aligned}$$

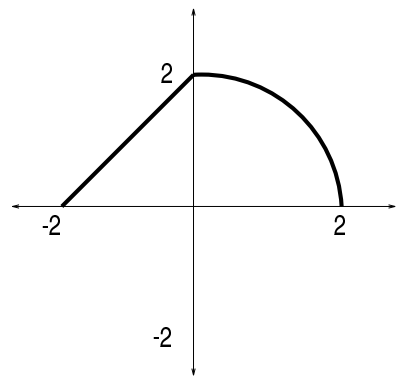
Now, verify that

$$\begin{aligned} v(r(d(f(h(x))))) &= v(r(d(f(x-1)))) = v(r(d((x-1)^2))) \\ &= v(r(3(x-1)^2)) = v(-3(x-1)^2) \\ &= -3(x-1)^2 + 2 = -3x^2 + 6x - 1. \end{aligned}$$

□

Summary of Rules.

For quick reference, we summarize the consequence of shifting and expanding symbolically and pictorially. The running example for the table will be a multipart function $y = f(x)$ whose graph consists of a line segment and a quarter circle on the domain $-2 \leq x \leq 2$:



$$f(x) = \begin{cases} x+2 & \text{if } -2 \leq x \leq 0 \\ \sqrt{4-x^2} & \text{if } 0 \leq x \leq 2 \end{cases}$$

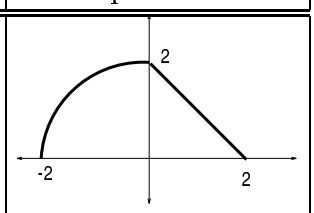
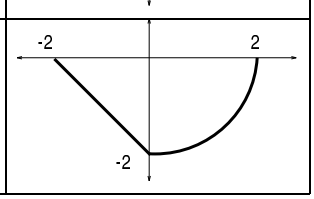
REFLECTION			
symbolic change	new equation	graphical consequence	picture
x by $-x$	$y = f(-x)$	flip across y-axis	
$f(x)$ by $-f(x)$	$y = -f(x)$	flip across x-axis	

Table 2.5.1: Reflection

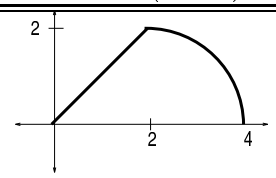
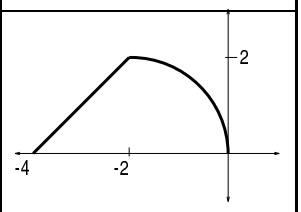
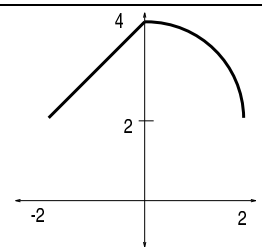
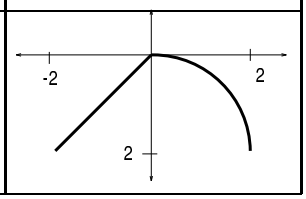
SHIFTING (Assume $c > 0$)			
symbolic change	new equation	graphical consequence	picture ($c = 2$)
x by $x - c$	$y = f(x - c)$	shifts right c units	
x by $x + c$	$y = f(x + c)$	shifts left c units	
$f(x)$ by $f(x) + c$	$y = f(x) + c$	shifts up c units	
$f(x)$ by $f(x) - c$	$y = f(x) - c$	shifts down c units	

Table 2.5.2: Shifting

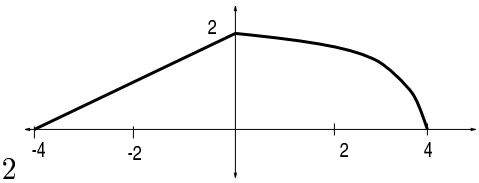
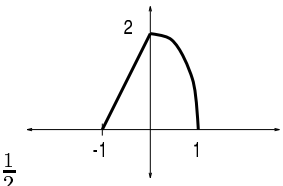
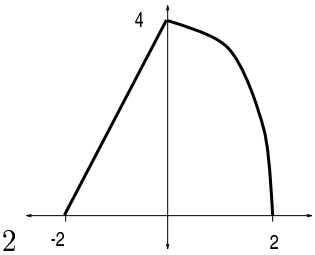
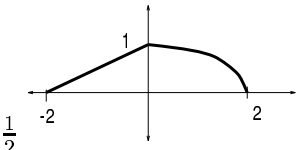
DILATION			
symbolic change	new equation	graphical consequence	picture
If $c > 1$, x by $\frac{x}{c}$	$y = f(\frac{x}{c})$	expand horizontally	$c = 2$ 
If $0 < c < 1$, x by $\frac{x}{c}$	$y = f(\frac{x}{c})$	compress horizontally	$c = \frac{1}{2}$ 
If $c > 1$, $f(x)$ by $cf(x)$	$y = cf(x)$	expand vertically	$c = 2$ 
If $0 < c < 1$, $f(x)$ by $cf(x)$	$y = cf(x)$	compress vertically	$c = \frac{1}{2}$ 

Table 2.5.3: Dilation

Problems

1. This problem is related to Example 2.5.7. Recall, we have these five functions:

$$\begin{aligned} f(x) &= x^2 & \qquad h(x) &= x-1 \\ v(x) &= x+2 & \qquad r(x) &= -x \\ d(x) &= 3x. \end{aligned}$$

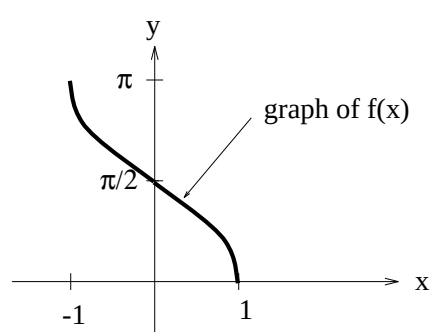
Calculate these compositions:
(a) $v(r(d(h(f(x)))))$.
(b) $h(f(d(r(v(x)))))$.
(c) $h(f(v(r(d(x)))))$.

2. Consider the function $y = f(x)$ with multipart definition

$$f(x) = \begin{cases} 0 & \text{if } x \le -1 \\ 2x+2 & \text{if } -1 \le x \le 0 \\ -x+2 & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x \ge 2 \end{cases}$$

(a) Sketch the graph of $y = f(x)$.
(b) Is $y = f(x)$ an even function; the definition of an even function is given in the previous problem.
(c) Sketch the reflection of the graph across the x-axis and y-axis. Obtain the resulting equations for these reflected curves.
(d) Sketch the vertical dilations $y = 2f(x)$ and $y = \frac{1}{2}f(x)$.
(e) Sketch the horizontal dilations $y = f(2x)$ and $y = f(\frac{1}{2}x)$.
(f) Find a number $c > 0$ so that the highest point on the graph of the vertical dilation $y = cf(x)$ has y-coordinate 11.
(g) Using horizontal dilation, find a number $c > 0$ so that the function values $f(\frac{x}{c})$ are non-zero for all $-\frac{5}{2} < x < 5$; sketch a picture.
(h) Using horizontal dilation, find a number $c > 0$ so that the function values $f(\frac{x}{c})$ are non-zero for all $-\frac{1}{6} < x < \frac{1}{3}$; sketch a picture.

3. The graph of a function $y = f(x)$ is pictured. It's domain is the interval $-1 \le x \le 1$. Sketch the graph of $y = \frac{1}{\pi}f(3x) - 0.5$. Find the largest possible domain of the function $y = g(x) = \sqrt{\frac{1}{\pi}f(3x) - 0.5}$.



4. A typical home gas furnace contains a *heat exchanger* which consists of a bunch of cylindrical metal tubes through which air passes as the flames heat the tube. As a furnace ages, these tubes can deform and crack, allowing deadly carbon monoxide gas to leak into the house. Suppose the cross-section of a heat exchanger tube is a circle of radius 5 cm when it is brand new. As the furnace ages, assume the bottom semicircular cross-section deforms according to the vertical dilation principle 2.5.4 with a deformation constant of

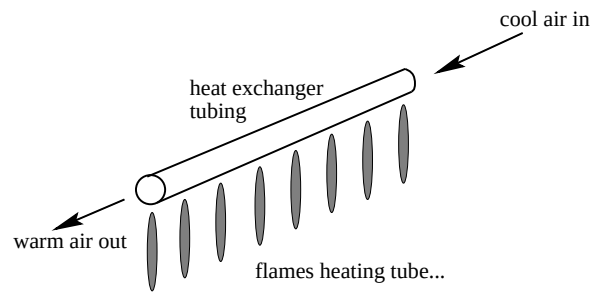
$$c = c(t) = 1 + \frac{1}{500}(t^2 + t)$$

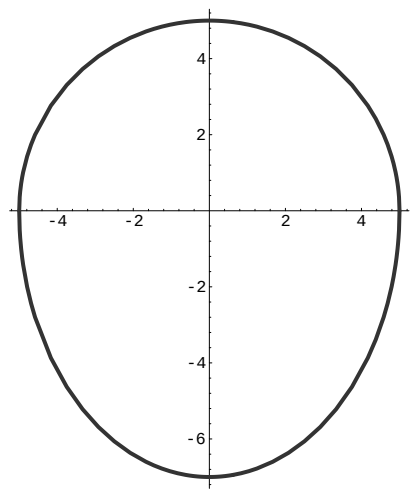
after t years. The top semicircular cross-section does not deform.

(a) Sketch an accurate picture of the cross-section of the heat exchanger tubing after 10 and 15 years.

(b) Suppose the metal in the tubing will crack if it deforms more than 3 cm out of the original shape. When will the exchanger crack?

(c) When will the cross-section of the heat exchanger tubing look like the picture at right?

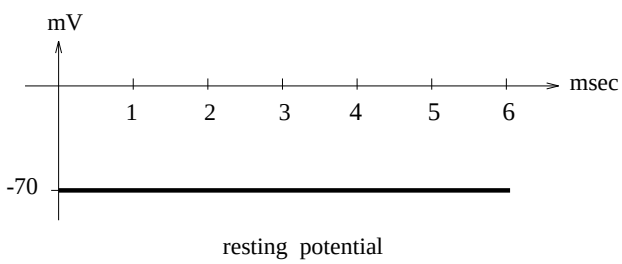


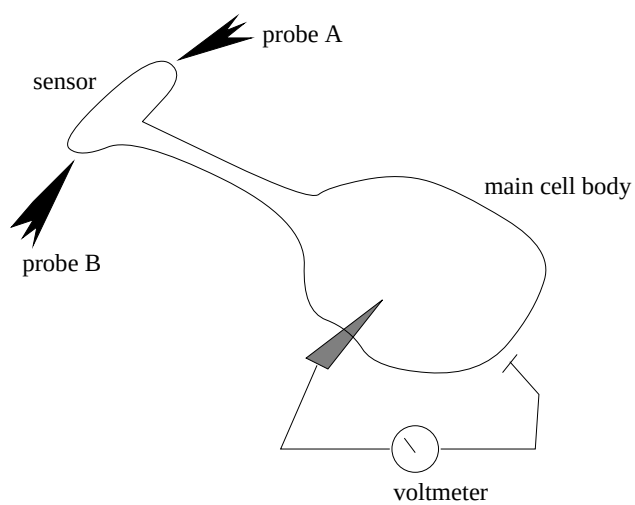


2.6 Arithmetic

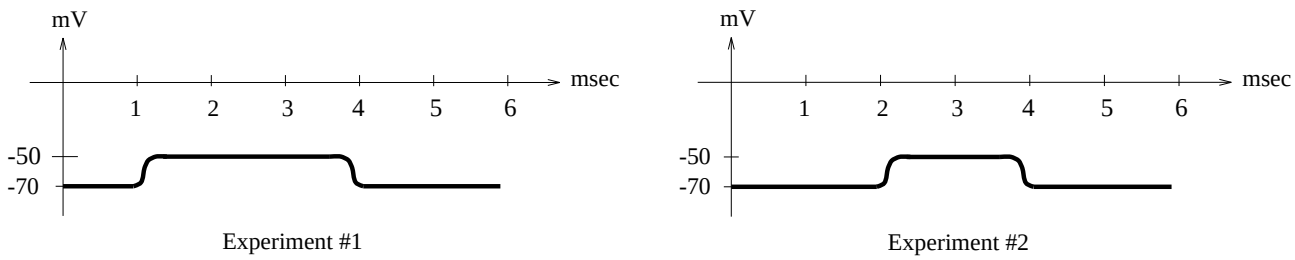
There is yet another way to build new functions from known functions, this time relying on what we know about the arithmetic operations for numbers. We will begin by indicating how this naturally arises in the context of a simple cellular biology experiment.

A biologist has isolated a single nerve cell (a *neuron*) and plans to make some experimental measurements. It turns out that there is always a voltage difference between the inside and outside of a living cell. We won't concern ourselves with why this is so, but focus on an experiment involving the measurement of this voltage, as a function of time. To begin with, the measurement of voltage in this experiment is in units of *volts*; the number tells us the tendency of a single unit of positive charge to move from the inside to the outside of the cell. In any event, if we do nothing to this nerve cell, the voltage measured turns out to be -70 mV; "mV" stands for "millivolts", or -70×10^{-3} volts. The fact we have a minus sign means a positive charge will move inward rather than outward. The plot below indicates a steady constant voltage (called the *resting potential*), where we will use msec=milliseconds for time units. Our nerve cell is attached to a touch sensor near the surface of an animal and the biologist has set up an appartus which will activate two different probes which will push on the sensor with equal force upon command:





In experiment #1, the biologist has probe A touch the sensor at time $t = 1$ and the voltage recordings for the cell are given in the lefthand plot below. In experiment #2, the biologist has probe B touch the sensor at time $t = 2$ and the voltage recordings for the cell are given in the righthand plot below.



What we want to indicate is how each of the plots in experiments #1 and #2 could have been obtained by an "addition process". For example, the plot in experiment #1 can be obtained by addition together the two plots, as pictured below:

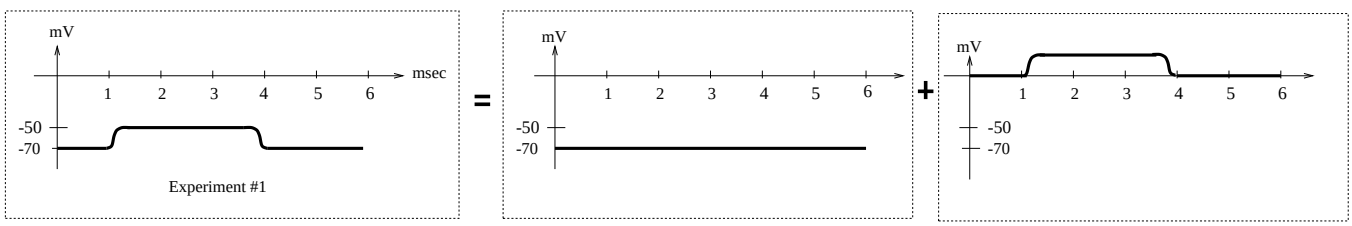


Figure 2.6.1

<u>Function Arithmetic</u>.

We want to use the arithmetic operations for numbers: $+ \quad - \quad \times \quad \div$, to combine together two functions. On a symbolic equations level, this is easy to do. If we begin with two functions $y = f(x)$ and $y = g(x)$, we can form the new functions:

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x); \\
(f-g)(x) &= f(x) - g(x); \\
(f \cdot g)(x) &= f(x)g(x); \\
(\frac{f}{g})(x) &= \frac{f(x)}{g(x)} \text{ as long as the denominator } g(x) \neq 0.
\end{aligned} \tag{2.6.1}$$

Examples 2.6.2: (i) Let $y = f(x) = 4x^3 - 6x^2 + x - 7$ and $y = g(x) = x^2 + 4$ with the domain all real numbers. Then a direct calculation shows that $(f+g)(x) = 4x^3 - 5x^2 + x - 3$, $(f-g)(x) = 4x^3 - 7x^2 + x - 11$, $(f \cdot g)(x) = 4x^5 - 6x^4 + 17x^3 - 31x^2 + 4x - 28$, $(\frac{f}{g})(x) = \frac{4x^3 - 6x^2 + x - 7}{x^2 + 4}$. Notice that $\frac{f}{g}$ is defined for all x, since the denominator is never going to be zero.

(ii) Let $y = f(x) = 3$ and $y = g(x) = \sqrt{1 - x^2}$. The largest possible common domain is $-1 \leq x \leq 1$; why? We get $(f+g)(x) = 3 + \sqrt{1 - x^2}$, $(f-g)(x) = 3 - \sqrt{1 - x^2}$, $(f \cdot g)(x) = 3\sqrt{1 - x^2}$, $(\frac{f}{g})(x) = \frac{3}{\sqrt{1 - x^2}}$. Notice that the domain of allowed values is $-1 \leq x \leq 1$ for each of the first three functions and $-1 < x < 1$ for the last one; why?

Sometimes it is useful to recognize a function can be written as an arithmetic combination of two "simpler" functions. Here are some examples of this sort of thing:

Examples 2.6.3: *(i) Suppose $y = x^3 + x - 2$. Find simpler functions $f(x)$ and $g(x)$ so that $y = f(x) + g(x)$. To do this, let $f(x) = x^3$ and $g(x) = x - 2$. Another correct answer would be $f(x) = x^3 - 2$ and $g(x) = x$.*

(ii) Suppose $y = 3x\sqrt{x^2+1}$. Find simpler functions $f(x)$ and $g(x)$ so that $y = f(x)g(x)$. To do this, let $f(x) = 3x$ and $g(x) = \sqrt{x^2+1}$.

(iii) Suppose $y = \frac{x-1}{x^4+1}$. Find simpler functions $f(x)$ and $g(x)$ so that $y = \frac{f(x)}{g(x)+h(x)}$. To do this, let $f(x) = x - 1$, $g(x) = x^4$ and $h(x) = 1$.

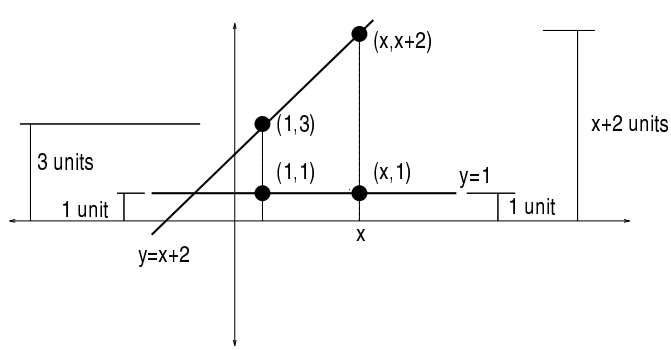
(iv) Suppose $y = 4(x^2 + x) + 2x(x^2 + x)$. Find simpler functions $f(x)$ and $g(x)$ so that $y = f(x)g(x)$. To do this, first factor $y = (4 + 2x)(x^2 + x)$, then let $f(x) = 4 + 2x$ and $g(x) = x^2 + x$.

What about the domain?

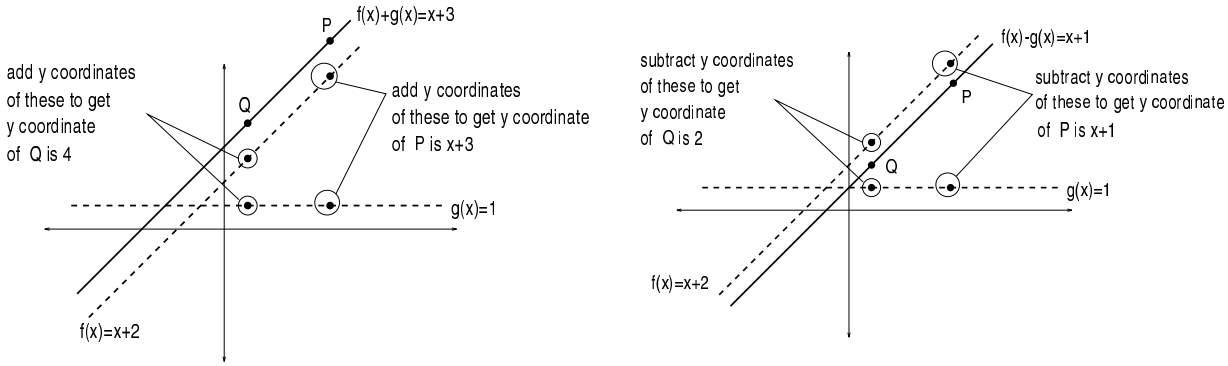
When you start combining functions as in (2.6.1), you need to assume you are using the same domain for each function. For example, suppose $f(x) = x^2$ and $g(x) = \frac{1}{x}$. We can take the domain of $f(x)$ to be all real numbers, but the largest possible domain of $g(x)$ would be the non-zero numbers. When we form the new functions $f(x) + g(x) = x^2 + \frac{1}{x}$ and $f(x) - g(x) = x^2 - \frac{1}{x}$, the largest possible domain would be the non-zero numbers.

Graphical Interpretation.

What do these function arithmetic operations mean on the level of their graphs? In the case of the addition and subtraction operations, this is easy to illustrate by way of example. If $f(x) = x + 2$ and $g(x) = 1$, here is a sketch of the graph of each function, interpreting graphically what is happening at $x = 1$ and general x.



In the left-hand graph, we sketch $f(x) + g(x) = x + 3$, together with graphical interpretation at $x = 1$ and general x; the right-hand graph deals with $f(x) - g(x) = x + 1$.

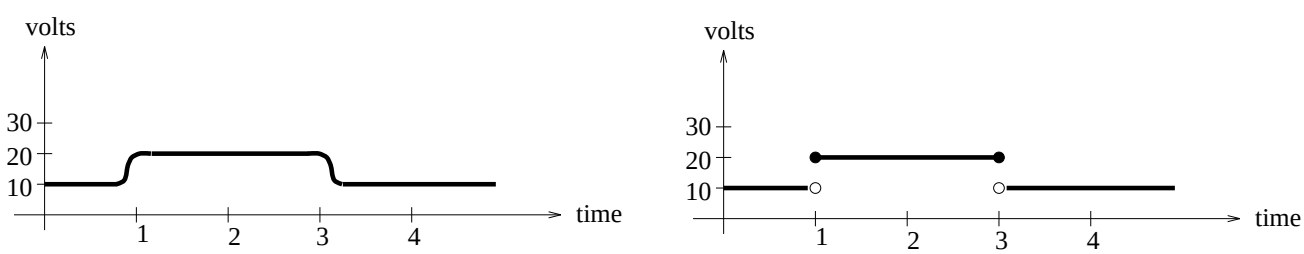


This simple example illustrates how you can "add" or "subtract" two graphs: Note that $(x, f(x))$ is the point on the graph of $f(x)$ above x (on the x-axis) and $(x, g(x))$ is on the point on the graph of $g(x)$ above x (on the x-axis). The point on the graph of $f(x) \pm g(x)$ above x on the x-axis is just obtained by combining the y-coordinates:

$$(x, f(x) \pm g(x)).$$

Step Functions.

Now we are in a position to describe somewhat complicated functions in terms of simpler ones; as suggested in Figure 2.6.1. For example, the measurement of voltage as a function of time often leads to plots which involve jumps in voltage over a short time period. This was illustrated in the experimental graphs #1,2 at the start of this section. In practice, it is common to model these sorts of plots with what are called *step functions* as sketched on the right below:



the actual voltage plot

a step function model

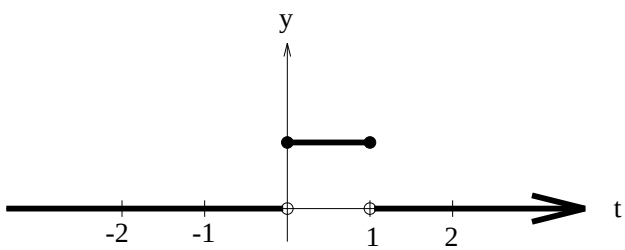
Roughly speaking, if we were to interpret the lefthand plot, it is telling us that in the time interval between $t = 1$ and $t = 3$, the voltage is 20 volts; otherwise it is 10 volts. Of course, this is not quite accurate really close to $t = 1$ or $t = 3$, but those portions of the graph are nearly vertical and rather insignificant when you look at the scale of the entire graph. We could, to a first approximation, replace the lefthand graph with the righthand graph above. Notice, this is the graph of the multipart function:

$$y(t) = \begin{cases} 10 & \text{if } 0 \le t < 1 \\ 20 & \text{if } 1 \le t \le 3 \\ 10 & \text{if } 3 < t \end{cases}$$

This is an example of what is usually called a *step function*.

Building Step Functions.

We can begin with a *basic step function* $u(t)$ and show how to build all others using our function building tools:



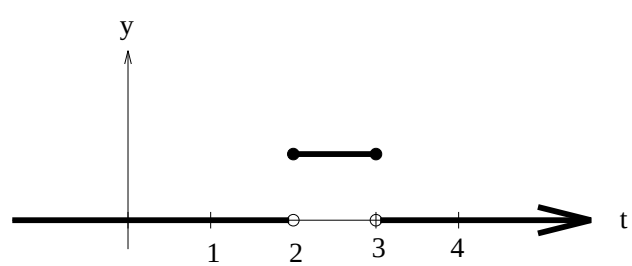
$$y = u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } 1 < t \end{cases} \qquad (2.6.4)$$

Example 2.6.5: *Let $u(t)$ be the basic step function in (2.6.4). Compute the multipart formula for $u(t-2)$ and sketch the graph.*

Solution. To start, notice this is really the calculation of a composition of two functions. Moreover, by (2.5.3), this is a horizontal shift of the basic step function in (2.6.4). The complication is that one of these functions is a multipart function. The key is to study how the multipart rule is affected for each of the multipart cases of $u(t)$ seperately. Here is the procedure and the resulting graph:

$$y = u(t-2) = \begin{cases} 0 & \text{if } t-2 < 0 \\ 1 & \text{if } 0 \le t-2 \le 1 \\ 0 & \text{if } 1 < t-2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } 2 \le t \le 3 \\ 0 & \text{if } 3 < t \end{cases}$$



□

Example 2.6.6: *Let $u(t)$ be the basic step function in (2.6.4). Compute the multipart rule for*

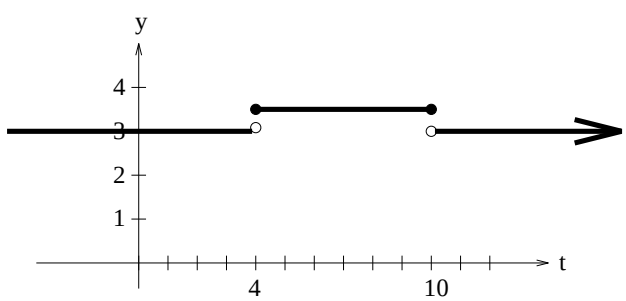
$$y = \frac{1}{2}u(\frac{1}{6}(t-4)) + 3$$

and sketch the graph. If this was modeling the voltage in an experiment, interpret what the graph is telling you.

Solution. As in the preceeding example, we study how the multipart rule is affected for each of the multipart cases of $u(t)$ seperately. We will start by finding the affect of the horizontal shift and dilation:

$$y = u(\frac{1}{6}(t-4)) = \begin{cases} 0 & \text{if } \frac{1}{6}(t-4) < 0 \\ 1 & \text{if } 0 \le \frac{1}{6}(t-4) \le 1 \\ 0 & \text{if } 1 < \frac{1}{6}(t-4) \end{cases} = \begin{cases} 0 & \text{if } t < 4 \\ 1 & \text{if } 4 \le t \le 10 \\ 0 & \text{if } 10 < t \end{cases}$$

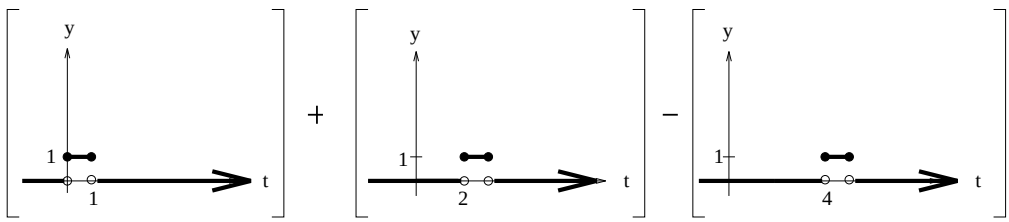
$$y = \left(\frac{1}{2}u(\frac{1}{6}(t-4))\right) + 3 = \left\{ \begin{array}{ll} (1/2) \times 0 & \text{if } t < 4 \\ 1/2 & \text{if } 4 \le t \le 10 \\ (1/2) \times 0 & \text{if } 10 < t \end{array} \right\} + 3 = \begin{cases} 3 & \text{if } t < 4 \\ \frac{7}{2} & \text{if } 4 \le t \le 10 \\ 3 & \text{if } 10 < t \end{cases}$$



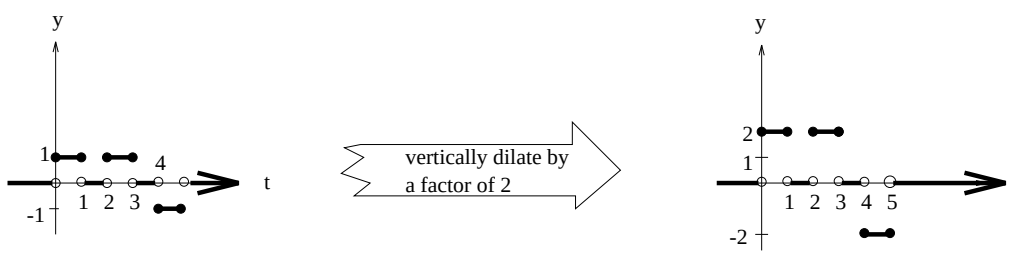
If this graph represents a voltage plot, the interpretation is as follows: The voltage is a constant 3 volts except during the time interval $4 \le t \le 10$, during which time it jumps up to a constant 3.5 volts. □

Example 2.6.7: *Let $u(t)$ be the basic step function in (2.6.4). Sketch the graph of $y = 2u(t) + 2u(t-2) - 2u(t-4)$.*

Solution. This is the same as $y = 2(u(t) + u(t-2) - u(t-4))$. We first graph $y = u(t) + u(t-2) - u(t-4)$:



which gives the lefthand plot below. Then we vertically dilate by a factor of 2 to get the righthand plot of $y = 2u(t) + 2u(t-2) - 2u(t-4)$.



□

Problems

1. In each of the following cases, find a formula for $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$ and $(\frac{f}{g})(x)$. In the division case, comment on any conditions required for the domain values.
   (a) $f(x) = 2x - 4$ and $g(x) = 18x^2 + 3x$.
   (b) $f(x) = 2 - \sqrt{5 - (x-3)^2}$ and $g(x) = x + 1$.
   (c) $f(x) = 21x^2 + 2x - 1$ and $g(x) = 8x^2 - 3x - 2$.
   (d) $f(x) = 0$ and $g(x) = x^3 + x$.
2. Let $u(t)$ be as in (2.6.4). Compute the multipart rules and graphs for $u(\frac{t}{2} - 1)$ and $u(\frac{1}{2}(t-1))$. Explain how to obtain each function from $u(t)$ by applying the operations of §2.5; be very specific about the order in which you apply the operations.
3. Let $u(t)$ be the basic step function in (2.6.4). Suppose we start with a step function of the form $y = Au(\frac{1}{B}(t - C)) + D$, where A, B, C, D are all non-negative constants with both A and B positive. (For example, $A = B = 1$, $C = D = 0$ is just $u(t)$; if $A = 1/2, B = 6, C = 4, D = 3$, we get the function in Example (2.6.6).) The graph of this function determines a rectangular region on the domain $C \le t \le C + B$. Sketch this region and compute it's area in terms of A, B, C, D. *Note:* This problem is related to a topic in cellular biology called *neuronal arithmetic*.

2.7 Polynomial Modeling

We have looked at modeling problems that involve linear and quadratic functions; here, the key players were equations of the form $y = mx + b$ or $y = ax^2 + bx + c$, for appropriate constants a, b, c, m. Of course, these examples suggest a more general sort of function, called a *polynomial function*, which has the form

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are all given constants. The *degree* of a polynomial $p(x)$ is the highest power of x occuring in the expression $y = p(x)$. So, we see that linear and quadratic functions are just the degree one and degree two polynomial functions, respectively. Modeling with polynomials of degree greater than two does arise. Here are some examples:

- Understanding acid-base equilibria (titrations) in chemistry involves polynomial functions having degree three; this becomes important in modeling the "strength" of acid rain.
- A basic model of epilepsy exists that involves describing the fraction of nerve cells firing in a neural network in terms of a third degree polynomial.
- General approximation problems, like *"...how can I estimate $\sqrt{3}$ to 10 decimal places?..."*, lead to polynomials of quite large degree. These are topics you will encounter in your future studies, but they require a substantial amount of specialized knowledge in chemistry, biology, medicine, or whatever, plus additional mathematical tools. We will just focus on a couple of the most basic ideas related to polynomial modeling.

Polynomials.

Here are some examples of polynomial functions:

$p(x) = x^2 - 4x + 7$	$p(x) = 34x + 1$	$p(t) = \pi t^3 - t + 2$
$p(x) = 3x^5 - 7x^2 + x + \sqrt{3}$	$p(x) = x^{112}$	$p(u) = 45$
$p(x) = x$	$p(s) = (1 + (2s + 4)^2)^3$	$p(\theta) = \theta^4 + \theta^2 + 1$
$p(x) = (1 + x^2)^3 - 4x^2 + 7x$	$p(z) = (z + 2)^{13}$	$p(t) = (t + 1)(t + 4)(t^3 + t^2 + t - 1)$

In contrast, these expressions are not polynomial functions:

$$f(x) = \sqrt{x} \qquad f(x) = \sqrt[3]{1 + x^3 + x^4} \qquad f(x) = \frac{\sqrt{r^2 - (x-h)^2}}{x}, \ r \text{ and } h \text{ constants.}$$

It is natural to establish a "dictionary" relating the graphs of polynomial functions with certain curves in the plane; this has been done in three cases:

A DICTIONARY OF CURVES AND POLYNOMIALS			
Degree	Name	Form of the Function	Graph
0	constant function	$y = c$; c a constant	horizontal line thru $(0, c)$
1	linear function	$y = mx + b$; m, b constants	line of slope m and y-intercept b
2	quadratic function	$y = ax^2 + bx + c$; a, b, c constants	parabola; see 2.3 for features.

A particularly simple type of polynomial has the form $p(x) = ax^n$, where n is a positive integer ($n = 1, 2, \dots$) and a is a constant; we call this a *monomial.* The graphs of monomials are fairly easy to describe. If $a = 1$, then there are two possible basic shapes:

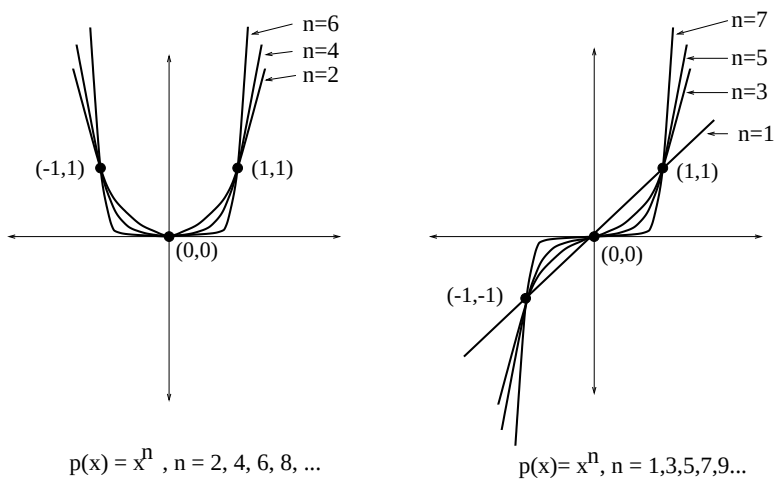


Figure 2.7.1: Graph of $y = x^n$

In the case of an even power (left-hand case), as the power becomes larger, the concavity can change, but the overall shape is a concave upward curve. Also, in the case of an odd power (right-hand case), as the power becomes larger, the concavity can change, but the overall shape is a half-concave upward and half-concave downward curve. (Note: Concavity is a concept which is hard to precisely measure using the tools of precalculus; it is best studied using calculus.) Using the general tools of shifting and dilation, we can easily go from the graph of $y = x^n$ in Figure 2.7.1 to the graph of $y = g(x) = a(x - h)^n + k$, where a, h, k are fixed constants.

For general polynomials of degree 3 and higher, the situation rapidly complicates. We call degree three polynomials *cubic polynomials* or *cubic functions.* For example, begin with the polynomials

$$\begin{aligned} p(x) &= x^3 + x^2 - x - 1, \\ q(x) &= -x^3 - 2x^2 + x + 2, \\ r(x) &= x^3 - 3x^2 + 3x - 1. \end{aligned}$$

These three polynomials are all built from the common monomials $1, x, x^2, x^3$, with different coefficients. The resulting graphs are very different, as pictured below:

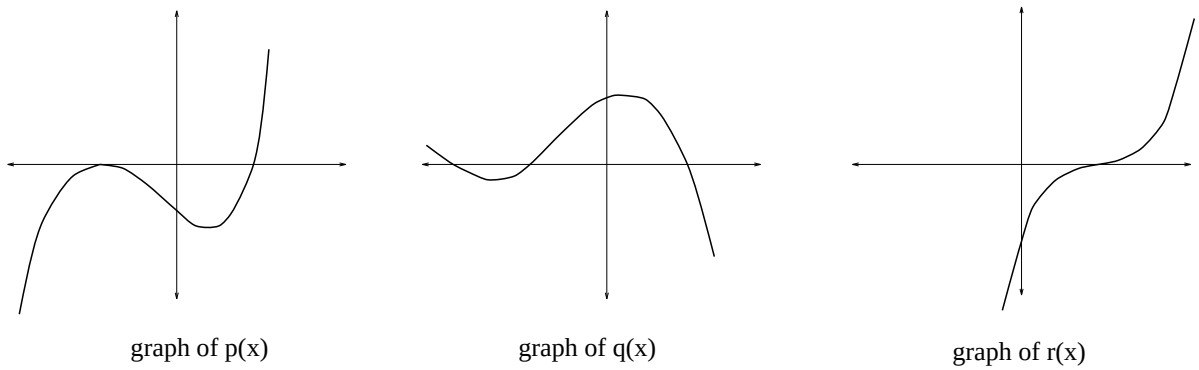


Figure 2.7.2: Three cubic polynomials

The moral of the story is that there is no *simple* way to predict the graph of the polynomial by knowing the monomials from which it is constructed. Not so long ago, this observation would mark the launch point for a long excursion into polynomial graphing techniques. The good news is that graphing devices now make a discussion of the mechanical aspects of graphing unnecessary in this course.

Roots and Factors of Polynomials.

If $y = p(x)$ is a polynomial whose graph crosses the x-axis at $x = c$, then $p(c) = 0$; we refer to c as a *root* (or *zero*) of p. From elementary algebra, we know that c is a root of a polynomial $p(x)$ if and only if $(x - c)$ is a *factor* of $p(x)$. This means we can find a polynomial $g(x)$ so that

$$p(x) \quad = \quad (x - c)g(x) \text{ and degree}(g(x)) < \text{degree}(p(x)).$$

We will describe how to actually find $g(x)$ in a moment. These comments tie together three important points.

Roots and Factors of Polynomials 2.7.2: *Let $p(x)$ be a polynomial and c a real number. The following three statements are equivalent :*

(a) The graph of $p(x)$ crosses the x-axis at $x = c$.

(b) c is a root of $p(x)$.

(c) $(x - c)$ is a factor of $p(x)$; i.e. we can find a polynomial $g(x)$ so that $p(x) = (x - c)g(x)$ and degree($g(x)$)< degree($p(x)$).

We can interpret this result as a link between a geometric phenomenon (crossing the x-axis) and a symbolic condition (conditions (b) and (c)).

Example 2.7.3: *If the product of two successive integers is 20, what are these integers?*

Solution. If a and b are two successive integers with product 20, then we have the equation $ab = 20$. As it stands, this is not an equation we can work with, since it involves two variables. Using the information in the problem, we can reduce this to an equation in one variable. Suppose we assume that a is the smaller of the two integers, then the fact they are successive tells us that $b = a + 1$. The equation to solve now becomes

$$\begin{aligned} a(a+1) &= 20 \\ a^2 + a - 20 &= 0 \\ (a-4)(a+5) &= 0. \end{aligned}$$

Since the equation is in fully factored form, we can read off the roots of the polynomial $p(a) = a^2 + a - 20$: the roots are $a = 4$ and $a = -5$. There will be two solutions to the problem. If $a = 4$, then the successive integers are $\{4, 5\}$; if $a = -5$, the successive integers are $\{-5, -4\}$. □

When a mathematical model involves a polynomial function, odds are we will need to factor it as fully as possible; this means finding as many roots as possible. A graphing device can be useful in this regard. We will now describe an algorithm for doing this.

Step 1: A polynomial $y = p(x)$ of degree n has at most n distinct linear factors; it can happen that there are strictly less than n distinct factors. In theory, every polynomial factors as a product of degree 1 and degree 2 factors using only real numbers.

For example, check that the following factorizations are valid. Each polynomial has degree 3; the first has three distinct factors, the second has only one distinct factor and the last has both a degree 1 and degree 2 factor.:

$$\begin{aligned} p(x) &= x^3 - 2x^2 - x + 2 = (x-1)(x+1)(x-2) \\ q(x) &= x^3 - 3x^2 + 3x - 1 = (x-1)(x-1)(x-1) = (x-1)^3 \\ m(x) &= x^3 - x^2 + x - 1 = (x^2+1)(x-1) \end{aligned}$$

It may be the case that we cannot factor the polynomial without introducing complex numbers. For example, $p(x) = x^2 + 1$ has no real roots, so can't be factored. There are two ways to see this: (i) apply the quadratic formula and you are forced to take the square root of a negative number (not allowed in this course!); (ii) plot the graph using a graphing device and notice it never crosses the x-axis, so it can't have a root.

Step 2: Any polynomial of degree 2 can be fully factored using the quadratic formula. For example, if $p(x) = 2x^2 + 2x - 6$, then the quadratic formula gives us the roots are $\alpha_1 = \frac{-1+\sqrt{13}}{2}, \alpha_2 = \frac{-1-\sqrt{13}}{2}$. This means that both $(x - \alpha_1)$ and $(x - \alpha_2)$ are factors, so if we use (2.7.2)(c),

$$2x^2 + 2x - 6 \quad = \quad a\left(x - \left[\frac{-1+\sqrt{13}}{2}\right]\right)\left(x - \left[\frac{-1-\sqrt{13}}{2}\right]\right),$$

for some constant a. If you multiply things out, the only way the equation can be true is if $a = 2$.

Step 3: For polynomials $p(x)$ of degree 3 or larger it can be very difficult to factor the polynomial (or equivalently to find the roots). There do exist complicated analogs of the quadratic formula in degrees 3 and 4, but it is an interesting fact that there cannot be an analogous formula for a general degree 5 or larger polynomial. Here is a procedure to follow:

- LOOK FOR EASY ROOTS. If, after dividing through by the leading coefficient, all the coefficients are integers, test to see if $\pm$ "divisors of the constant term" are roots. If you find an easy root, say c, that means we can write

  $$p(x) \quad = \quad (x - c)g(x)$$

  for some polynomial $g(x)$. You can find the polynomial $g(x)$ using the *long division algorithm*, which is just the polynomial analogue of long division of integers; we illustrate this procedure in Example 2.7.4 below. Now, you repeat this procedure, by trying to find roots (which give factors) of $g(x)$. If you don't find an easy root, resort to numerical (calculator) methods, described next.
- CALCULATOR METHODS: There are several ways to find numerical estimates for roots using a calculator.
  - *Tracing and zooming.* Graph the function, then use the "TRACE" feature to home in on the root. Use the "ZOOM" feature for extra accuracy.
  - *Zero finder.* Some calculators have a zero finder which may be located on a menu associated with graphing.
  - *Equation solver.* Some calculators have an equation solver. These will usually solve an equation in the form $f(x) = 0$. Of course, any equation in one variable can be put into that form.

Example 2.7.4: *Factor the polynomial* $p(x) = x^3 - 4x + 3$ *into degree one terms.*

Solution. To begin with, use a graphing device to find $x = 1$ is one root, so $(x - 1)$ is a factor of $p(x)$. We can now write

$$p(x) \quad = \quad (x - 1)g(x),$$

for some polynomial $g(x)$. To find $g(x)$, the idea is to mimic the technique we use to long divide two integers. For example, since 3218 is even, it must be divisible by 2:

$$
\begin{array}{r|l}
 & 1\ 6\ 0\ 9 \\ \cline{2-2}
2 & 3\ 2\ 1\ 8 \\
 & 2 \\ \cline{2-2}
 & 1\ 2\ 1\ 8 \\
 & 1\ 2 \\ \cline{2-2}
 & \quad\ 1\ 8 \\
 & \quad\ 0 \\ \cline{2-2}
 & \quad\ 1\ 8 \\
 & \quad\ 1\ 8 \\ \cline{2-2}
 & \quad\ \ 0 \quad \longleftarrow \textbf{remainder is 0, so divides evenly}
\end{array}
$$

The same idea works if we want to use long division on polynomials. Here is how we would proceed:

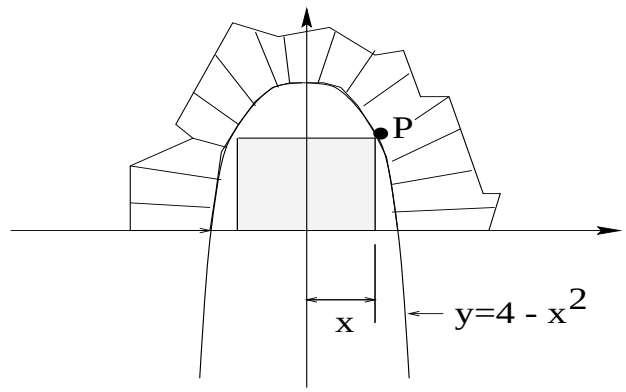
$$
\begin{array}{r|l}
 & x^2 + x - 3 \\ \cline{2-2}
x - 1 & x^3 + 0x^2 - 4x + 3 \\
 & x^3 - x^2 \\ \cline{2-2}
 & \quad x^2 - 4x + 3 \\
 & \quad x^2 - x \\ \cline{2-2}
 & \qquad\quad - 3x + 3 \\
 & \qquad\quad - 3x + 3 \\ \cline{2-2}
 & \qquad\qquad\qquad 0 \quad \longleftarrow \textbf{remainder 0 means divides evenly}
\end{array}
$$

This sequence of calculations tells us that $p(x) = (x-1)(x^2+x-3)$ and the quadratic formula can be applied to the degree two factor, arriving a factorization into degree one terms:

$$p(x) \;=\; (x-1)\left(x - \left[\frac{-1+\sqrt{13}}{2}\right]\right)\left(x - \left[\frac{-1-\sqrt{13}}{2}\right]\right).$$

□

Example 2.7.5: *An arch is in the shape of the graph of the polynomial $y = 4 - x^2$, as pictured below. A rectangular doorway fits under the arch and the units of measure will be feet. Verify that a door 2 feet wide and 3 feet high fits under the arch. Find a second rectangular door under the arch with different dimensions but having the same area as the first door.*



Solution. Imposing a coordinate system as pictured, we let x be the horizontal distance right of the pictured centerline; this means the door is $2x$ feet wide. The point P indicated is on the graph of $y = 4 - x^2$, so it must have coordinates $(x, 4 - x^2)$. This means that the height of the inscribed door is $4 - x^2$ feet. If we let $x = 1$, then $P = (1, 3)$. This means the door is 2 feet wide by 3 feet high, verifying the first claim.

The 2×3 door has area 6 sq. ft. and we want to find another door with the same area inscribed under the arch. To do this, we need to bring in an equation involving the area of the door. We have

$$A = A(x) = \text{area of the door} = (\text{width})(\text{height}) = 2x(4 - x^2) = 8x - 2x^3.$$

We are constrained to have area 6, which means we must solve the third degree polynomial equation $6 = A = 8x - 2x^3$:

$$\begin{aligned} 8x - 2x^3 - 6 &= 0 \\ x^3 - 4x + 3 &= 0. \end{aligned}$$

By the first part of the problem, we know that $x = 1$ is a root of this polynomial, so $(x - 1)$ is a factor. Applying long division, we have

$$x^3 - 4x + 3 = (x-1)(x^2 + x - 3) = (x-1)(x - [\frac{-1+\sqrt{13}}{2}])(x - [\frac{-1-\sqrt{13}}{2}]).$$

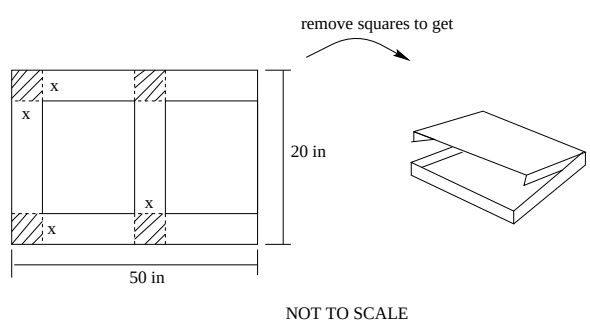
Conclude that there are three solutions to the equation $A(x) = 6 : x = 1, x = \frac{-1+\sqrt{13}}{2} = 1.303, x = \frac{-1-\sqrt{13}}{2} = -2.303$. In our problem, these solutions are interpreted as dimensions, so must be non-negative quantities. This tells us that the second inscribed door has dimensions:

$$\begin{aligned} \text{width} &: 2(\frac{-1+\sqrt{13}}{2}) = -1 + \sqrt{13} = 2.606\,\text{feet} \\ \text{height} &: 4 - (\frac{-1+\sqrt{13}}{2})^2 = 2.303\,\text{feet}. \end{aligned}$$

□

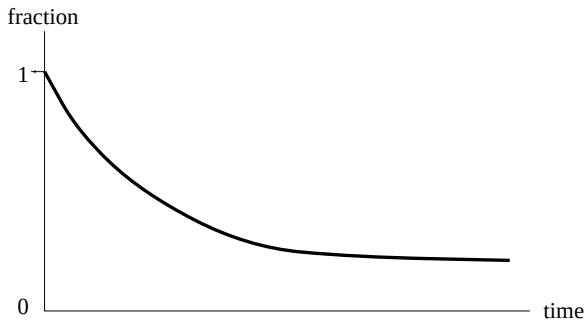
Problems

1. An open box is made out of rectangular material 20×30 inches by cutting squares from each corner.
   (a) Find a polynomial $V(x)$ in the variable x, for the volume of the box, if x represents the side length of a cutout square.
   (b) If the cutouts are 2 inches on each side, what is the volume of the box?
   (c) Find the largest domain of x values for the function $V(x)$ which makes physical sense.
   (d) If the volume is 1000 cubic inches, what are the feasible dimensions of the box.
2. Fully factor each of these polynomials, find all roots and graphically indicate where the graphs are above and below the x-axis:
   (a) $y = x^3 - 2x + 4x^2 - 8$.
   (b) $y = x^3 + 2x^2 - 6x - 7$.
   (c) $y = x^3 + 2x^2 - 6x + 7$.
   (d) $y = x^2 + 2x - 11$.
3. *Pagliacci Pizza* has designed a cardboard delivery box from a single piece of cardboard, as pictured.
   (a) Find a function $v(x)$ that computes the volume of the box in terms of x.
   (b) If you want the box to enclose 500 cu. inches, what size pie can be delivered in the box?
   (c) If you want the box to enclose the largest total volume, what size pie can be delivered in the box?



2.8 Inverse Functions

The experimental sciences are loaded with examples of functions relating time and some measured quantity. Here, time represents our "input" and the quantity we are measuring is the "output". For example, maybe you have just mixed together some chemical reactants in a vessel. As time goes by, you measure the fraction of reactants remaining and tabulate your results:



Viewing the input value as "time" and the output value as "fraction of product", we could try to find a function $y = f(t)$ modeling this data. Using this function, you can easily compute the fraction of reactants remaining at any time in the future. However, it is probably just as interesting to know how to predict the time when a given fraction of reactants exists. In other words, we would like a *new* function that allows us to input a "fraction of reactants" and get out the "time" when this occurs. This "reverses" the input/output roles in the original function. Is there a systematic way to find the new function if we know $y = f(t)$? The answer is yes and depends upon the general theory of inverse functions.

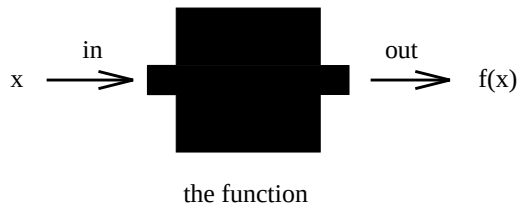
Concept of an Inverse Function.

Suppose you are asked to solve the following three equations for x. How do you proceed?

$$\begin{aligned}(x+2) &= 64\\ (x+2)^2 &= 64\\ (x+2)^3 &= 64.\end{aligned}$$

In the first equation, you add "-2" to each side, then obtain $x = 62$. In the third equation, you take the cube root of both sides of the equation, giving you $x + 2 = 4$, then subtract 2 getting $x = 2$. In second equation, you take a square root of both sides, BUT you need to remember **both** the positive and negative results when doing this. So, you are reduced down to $x + 2 = \pm 8$ or that $x = -10$ or 6. Why is it that in two of these cases you obtain a single solution, while in the remaining case there are two different answers? We need to sort this out, since the underlying ideas will surface when we address the *inverse circular functions* in section 3.6.

Let's recall the conceptual idea of a function: A function is a process which takes a number x and outputs new number $f(x)$.



So far, we've only worked with this process from "left to right"; i.e. given x, we simply put it into a symbolic rule and out pops a new number $f(x)$. This is all pretty mechanical and straightforward.

An example.

Let's schematically interpret what happens for the specific concrete example $y = f(x) = 3x - 1$, when $x = -1, -1/2, 0, 1/2, 1, 2$:

We could try to understand the function process in this example in "reverse order", going "right to left"; namely, you might ask what x value can be run through the process so you end up with the number 11? This is somewhat like the "Jeopardy" game show: You know what the answer is, you want to find the question. For our example, if we start out with some given y values, then we can define a "reverse process" $x = \frac{1}{3}(y + 1)$ which returns the x value required so that $f(x) = y$:

A second example.

If we begin with a linear function $y = f(x) = mx + b$, where $m \neq 0$, then we can always find a "reverse process" for the function. To do it, you must solve the equation $y = f(x)$ for x in terms of y:

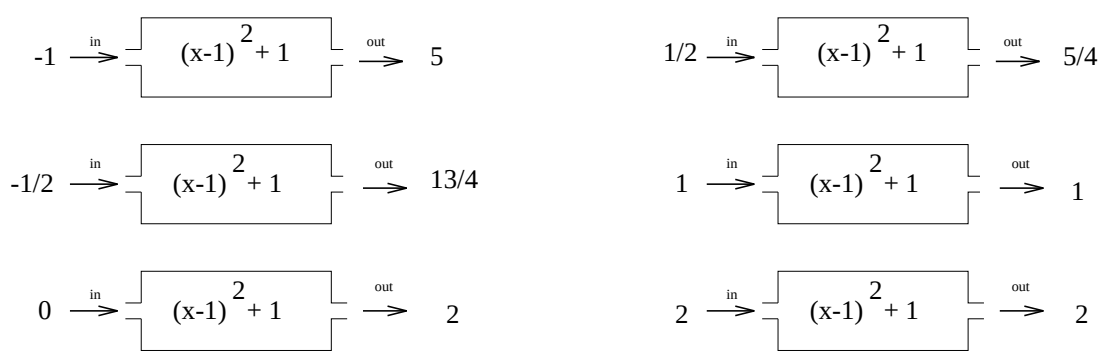
$$\begin{aligned} y &= mx + b \\ y - b &= mx \\ \frac{1}{m}(y - b) &= x \end{aligned}$$

So, if $m = 3$ and $b = -1$, we just have the first example above.

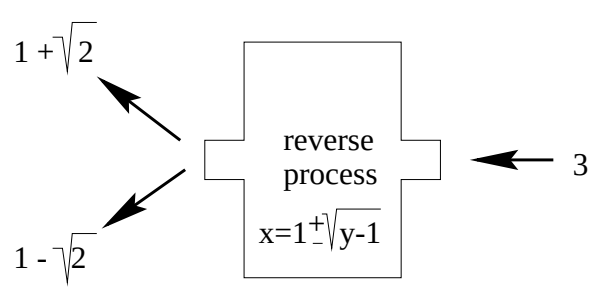
As another example, suppose $y = -0.8x + 2$; then $m = -0.8$ and $b = 2$. In this case, the reverse process is $-1.25(y - 2) = x$. If we are given the value $y = 11$, we simply compute that $x = -11.25$; i.e. $f(-11.25) = 11$.

A third example.

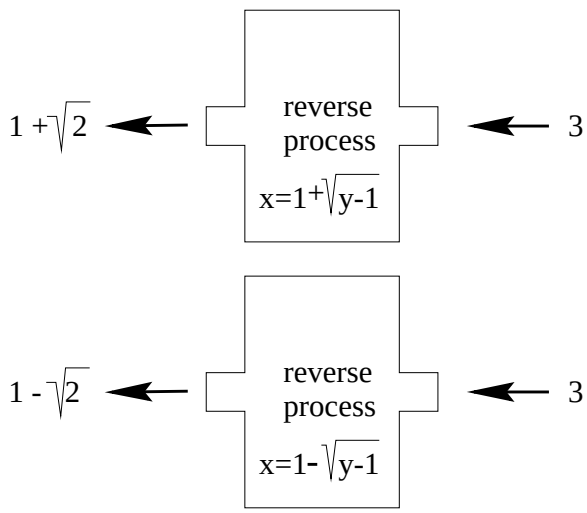
The previous examples hide a subtle point that can arise when we try to understand the "reverse process" for a given function. Suppose we begin with the function $y = f(x) = (x - 1)^2 + 1$. Here is a schematic of how the function works when we plug in $x = -1, -1/2, 0, 1/2, 1, 2$. As a "forward process", each input generates a unique output:



For this example, if we start out with some given y values, then we can try to define a "reverse process" $x =$??? which returns an x value required so that $f(x) = y$. Unfortunately, there is no way to obtain a single formula for this reverse process; here is what happens if you are given $y = 3$ and you try to solve for x:



The conclusion is that the "reverse process" has two outputs. This violates the rules required for a function, so this is NOT a function. The solution is to create two new "reverse processes":

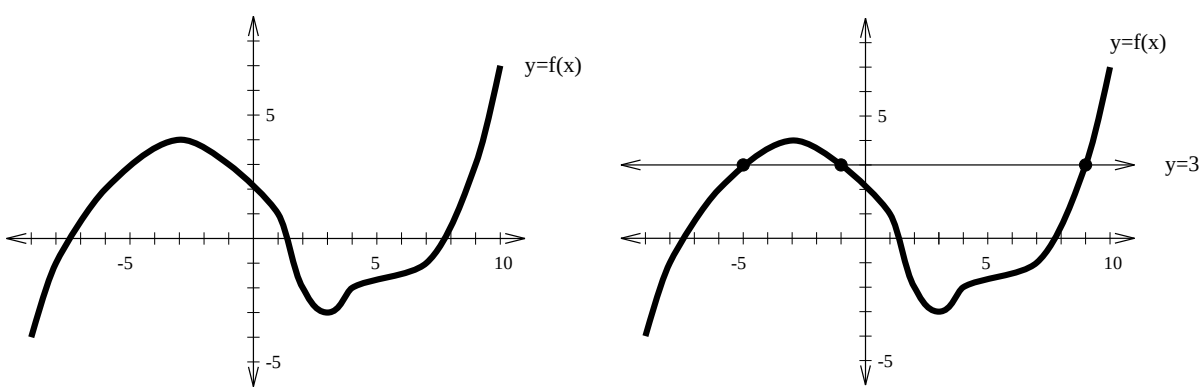


Each of these "reverse processes" has a unique output; in other words, each of these "reverse processes" defines a function.

So, given $y = 3$, there are TWO possible x values, namely $x = 1 \pm \sqrt{2}$, so that $f(1+\sqrt{2}) = 3$ and $f(1-\sqrt{2}) = 3$. In other words, the reverse process is not given by a single equation; there are TWO POSSIBLE reverse processs.

Graphical Idea of an Inverse.

We have seen that finding inverses is related to solving equations. However, so far, the discussion has been symbolic; we have pushed around a few equations and in the end generated some confusion. Let's use the tools of §2.2 to visualize what is going on here. Suppose we are given the graph of a function $f(x)$ as in the left-hand picture below. Now, what input x values result in an output value of 3? This involves finding all x such that $f(x) = 3$. Graphically, this means we are trying to find points on the graph of $f(x)$ so that their y-coordinates are 3. The easiest way to to do this is to draw the line $y = 3$ and find where it intersects the graph.



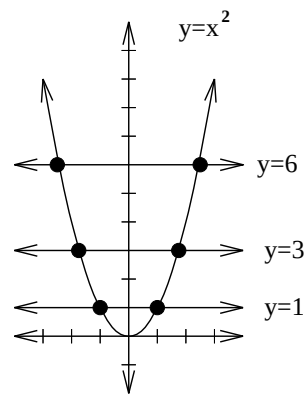
We can see the points of intersection are (-5,3), (-1,3), and (9,3). That means that $x = -5, -1, 9$ produce the output value 3; i.e., $f(-5) = f(-1) = f(9) = 3$.

This leads to our first important fact when trying to study the "reverse process" for a function:

Fact 2.8.1: *Given a number c, the x values such that $f(x) = c$ can be found by finding the x-coordinates of the intersection points of the graphs of $y = f(x)$ and $y = c$.*

Example 2.8.2: *Graph $y = f(x) = x^2$ and discuss the meaning of (2.8.1) when $c = 3, 1, 6$.*

Solution: We graph $y = x^2$ and the lines $y = 1$, $y = 3$ and $y = 6$. Let's use $c = 6$ as an example. For that, we need to simultaneously solve the equations $y = x^2$ and $y = 6$. Putting these together, we get $x^2 = 6$ or $x = \pm\sqrt{6} \approx \pm 2.449$; i.e. $f(\pm 2.449) = 6$. If $c = 3$, we get $x = \pm 1.732$; i.e. $f \pm 1.732) = 3$. Finally, if $c = 1$, we get $x = \pm 1$; i.e. $f(\pm 1) = 1$. □

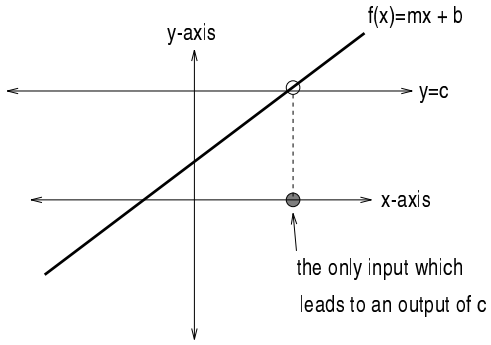


The pictures so far indicate another very important piece of information. For any number c, we can tell exactly "how many" input x values lead to the same output value c, just by counting the number of times the graphs of $y = f(x)$ and $y = c$ intersect.

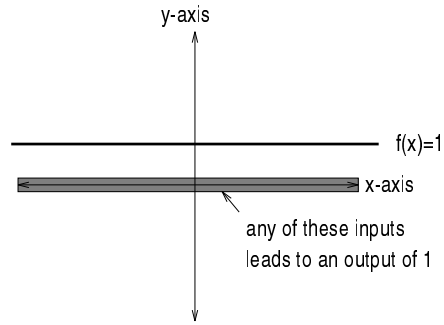
Fact 2.8.3: *For any function $f(x)$ and any number c, the number of x values so that $f(x) = c$ is the number of times the graphs of $y = c$ and $y = f(x)$ intersect.*

Examples 2.8.4: (i) If $f(x)$ is a linear function $f(x) = mx + b, m \neq 0$, then the graph of $f(x)$ intersects a given horizontal line $y = c$ EXACTLY once; i.e. the equation $c = f(x)$ always has a unique solution.

(ii) If $f(x) = d$ is a constant function and $c = d$, then every input x value in the domain leads to the output value c. On the other hand, if $c \neq d$, then no input x value will lead to the output value c. For example, if $f(x) = 1$ and $c = 1$, then every real number can be input to produce an output of 1; if $c = 2$, then no input value of x will lead to an output of 2.



Linear functions



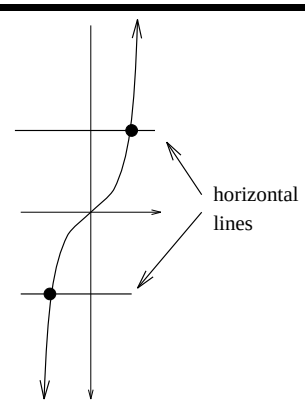
Constant functions

One-to-one Functions.

For a specified domain, *one-to-one functions* are functions with the property: Given any number c, there is *at most one* input x value in the domain so that $f(x) = c$. Among our examples thus far, linear functions (degree 1 polynomials) are always one-to-one. However, $f(x) = x^2$ is not one-to-one; we've already seen that it can have two values for some of its inverses. By Fact 2.8.3, we quickly can come up with what's called the horizontal line test.

Horizontal Line Test 2.8.5: *On a given domain of x-values, if the graph of some function $f(x)$ has the property that every horizontal line crosses the graph at most once, then the function is one-to-one on this domain.*

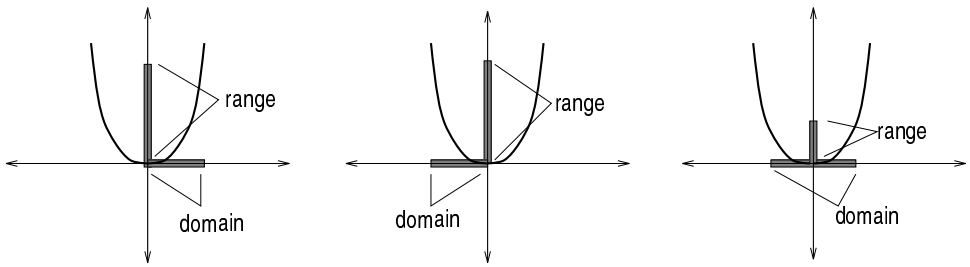
Example 2.8.6: *By the horizontal line test, it is easy to see that $f(x) = x^3$ is one-to-one on the domain of all real numbers.*



Although it isn't common, it's quite nice when a function is one-to-one because we don't need to worry as much about the number of input x values producing the same output y value. In effect, this is saying that we can define a "reverse process" for the function $y = f(x)$ which will also be a function; this is the key theme of the next section.

Inverse Functions.

Let's now come face to face with the problem of finding the "reverse process" for a given function $y = f(x)$. It is important to keep in mind that the domain and range of the function will both play an important role in this whole development. For example, here is the function $f(x) = x^2$ with three different domains specified and the corresponding range values:



These comments set the stage for a third important fact. Since the domain and range of the function and its inverse rule are going to be intimately related, we want to use notation which will highlight this fact. We have been using the letters x and y for the domain (input) and range (output) variables of $f(x)$ and the "reverse process" is going to reverse these roles. It then seems natural to simply write y (instead of c) for the input values of the "reverse process" and x for it's output values.

Fact 2.8.7: *Suppose a function $f(x)$ is one-to-one on a domain of x values. Then define a NEW FUNCTION by the rule*

$$f^{-1}(y) = \text{the } x \text{ value so that } f(x) = y.$$

The domain of y values for the function $f^{-1}(y)$ is equal to the range of the function $f(x)$.

The rule defined here is the "reverse process" for the given function. It is referred to as the *inverse function* and we read $f^{-1}(y)$ as "...eff inverse of y...".

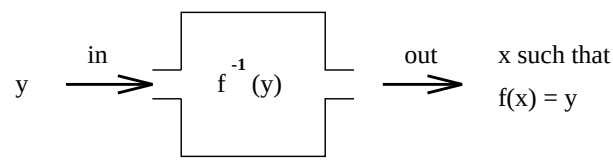
!!! CAUTION !!!

Both the "domain" of $f(x)$ and the "rule" $f(x)$ have equal influence on whether the inverse rule is a function. Keep in mind, you do NOT get an inverse function automatically from functions which are not one-to-one!

!!! CAUTION !!!

Schematic Idea of an Inverse Function.

Suppose that $f(x)$ is one-to-one, so that $f^{-1}(y)$ is a function. As a result, we can model $f^{-1}(y)$ as a black box. What does it do? If we put in y in the input side, we should get out the x such that $f(x) = y$

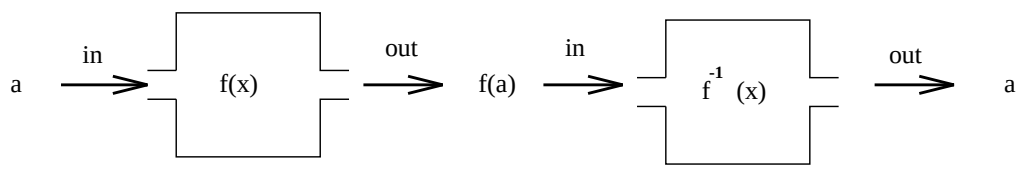


Now, let's try to unravel something very special that is happening on a symbolic level. What would happen if we plugged $f(a)$ into the inverse function for some number a? Then the the inverse rule $f^{-1}(f(a))$ tells us that we want to find some x so that $f(x) = f(a)$. But, we already know $x = a$ works and since $f^{-1}(y)$ is a function (hence gives us unique answers), the output of $f^{-1}(f(a))$ is just a. Symbolically, this means that

For every a value in the domain of $f(x)$, we have

$$f^{-1}(f(a)) = a. \tag{2.8.8}$$

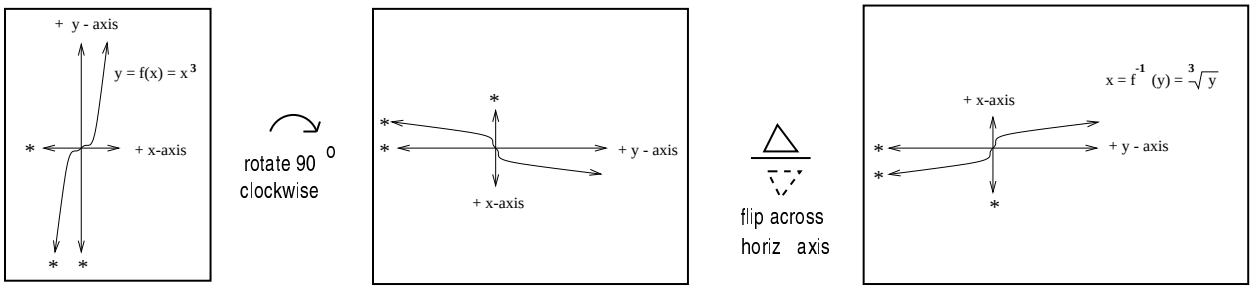
This is better shown in the following black box picture:



A good way to get an idea of what an inverse function is doing is to remember that $f^{-1}(y)$ *reverses the process* of $f(x)$. We can think of $f^{-1}(y)$ as a "black box" running $f(x)$ backwards.

Graphing Inverse Functions.

How can we get the graph of an inverse function? The idea is to manipulate the graph of our original one-to-one function in some prescribed way, ending up with the graph of $f^{-1}(y)$. This isn't as hard as it sounds, but some confusion with the variables enters into play. Remember that a typical point on the graph of a function $y = f(x)$ looks like $(x, f(x))$. Now let's take a look at the inverse function $x = f^{-1}(y)$. Given a number y in the domain of $f^{-1}(y)$, $y = f(x)$ for some x in the domain of $f(x)$; i.e. we are using the fact that the domain of f^{-1} equals the range of f. The function $f^{-1}(y)$ takes the number $f(x)$ and sends it to x, by (2.8.8). So when $f(x)$ is the input value, x becomes the output value. Conclude a point on the graph of $f^{-1}(y)$ looks like $(f(x), x)$. It's similar to the graph of $y = f(x)$, only the x and y coordinates have reversed! What does that do to the graph? Essentially, you reorient the picture so that the positive x-axis and positive y-axis are interchanged. Here is the process for the function $y = f(x) = x^3$ and it's inverse function $x = f^{-1}(y) = \sqrt[3]{y}$. We place some $*$ symbols on the graph to help keep track of what is happening:



Trying to Invert a Non one-to-one Function.

Suppose we blindly try to show that $\sqrt{y}$ is the inverse function for $y = x^2$, without worrying about all of this one-to-one stuff. We'll start out with the number -7. If $f^{-1}(y) = \sqrt{y}$, then we know that

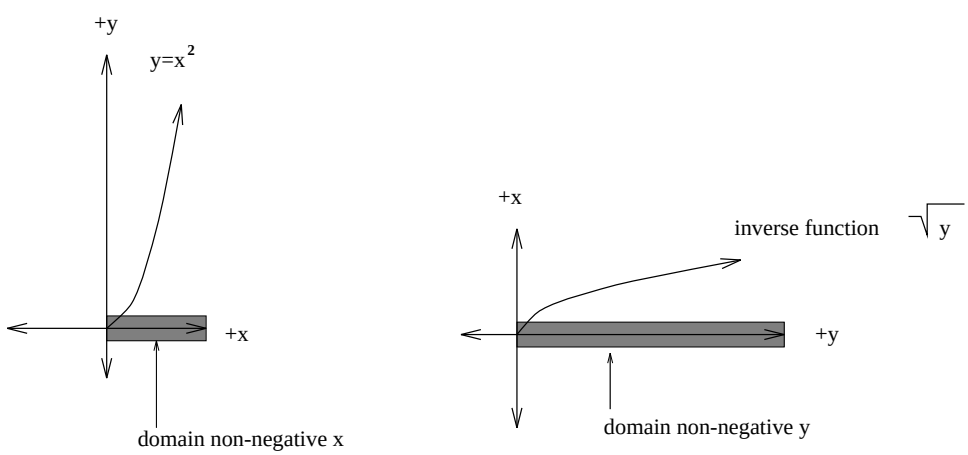
$$f^{-1}(f(-7)) \quad = \quad f^{-1}(49) = 7.$$

On the other hand, the formula in (2.8.8) tells us that we must have

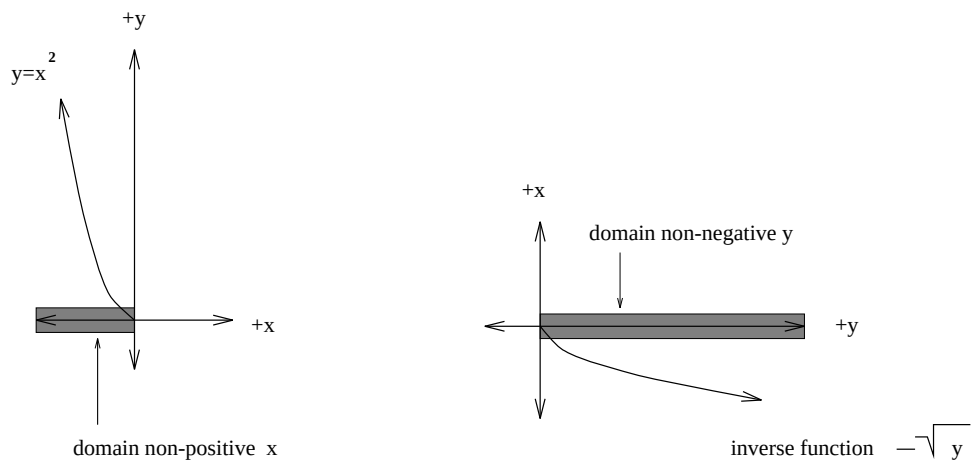
$$f^{-1}(f(-7)) \quad = \quad -7,$$

so we have just shown 7=-7! So clearly $f^{-1}(y) \neq \sqrt{y}$. Even if we try $f^{-1}(y) = -\sqrt{y}$, we produce a contradiction. It seems that if you didn't have to worry about negative numbers, things would be all right. Then you could say that $f^{-1}(y) = \sqrt{y}$. Let's try to see what this means graphically.

Let's set $f(x) = x^2$, **but** only for non-negative x-values. That means that we want to erase the graph to the left of the y-axis (so remember - no negative x-values allowed). The graph would then look like this:



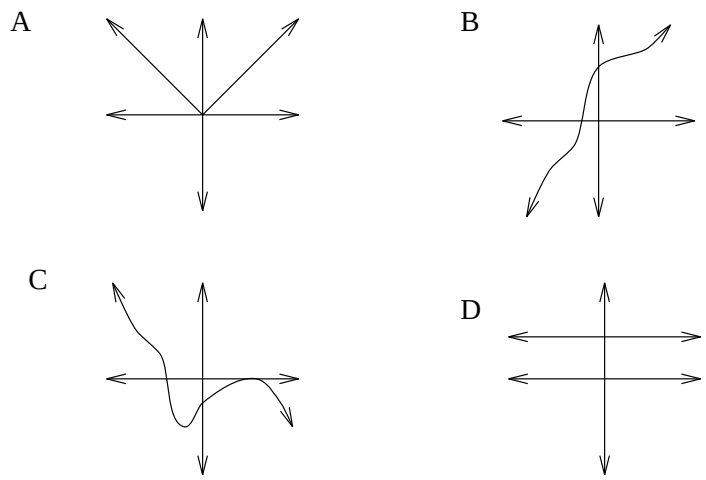
This is a now a one-to-one function! And now, one can see that its inverse function is $\sqrt{y}$. Similarly, we could have taken $f(x) = x^2$ but only for the non-positive x-values. In that case, $f^{-1}(y) = -\sqrt{y}$. In effect, we have split the graph of $y = x^2$ into two parts, each of which is the graph of a one-to-one function:



It is precisely this splitting into two cases that leads us multiple solutions of an equation like $x^2 = 5$. We obtain $x = \sqrt{5}$ and $x = -\sqrt{5}$; one solution comes from the side of the graph to the left of the y-axis, and the other from the side to the right of the y-axis. This is because we have separate inverse functions for the left and right side of the graph of $y = x^2$.

Problems

1. Show that the functions $y = x^3$, $y = 2x + \pi$ and $y = -2x$ are one-to-one by checking their graphs.
2. A function is called *onto* if every horizontal line will intersect the graph at least once.
   (a) Show that x^3 is onto and that x^2 is not onto.
   (b) Draw pictures of a graph (you do not need an algebraic formula - just the graph) which is (i) one-to-one and onto (don't use x^3) (ii) one-to-one but not onto (iii) not one-to-one but onto (iv) neither one-to-one nor onto (don't use x^2)
3. Which of the following graphs are one-to-one. If they are not one-to-one, section the graph up into parts that are one-to-one.



4. For each of the following functions: (1) sketch the function, (2) find the inverse function(s), and (3) sketch the inverse function(s). In each case, indicate the correct domains and ranges. Finally, make sure you test each of the functions you propose as an inverse with the following compositions:

$$f(f^{-1}(x)) \stackrel{?}{=} x$$

and

$$f^{-1}(f(x)) \stackrel{?}{=} x.$$

   (a) $f(x) = 3x - 2$
   (b) $f(x) = \frac{1}{3x-2}$
   (c) $f(x) = -x^2 + 3$
   (d) $f(x) = 2 + 5x - 3x^2$
   (e) $f(x) = \sqrt{4 - x^2}$, where $0 \leq x \leq 2$
5. Clovis is standing at the edge of a cliff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. Also, with

the origin of the coordinate system located where he is standing, the path of the rocket is described by the formula $y = -2x^2 + 120x$.

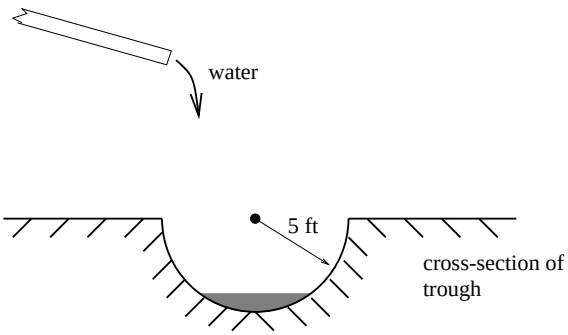
(a) Give a function $h = f(x)$ relating the height h of the rocket above the sloping ground to its x-coordinate.

(b) Find the maximum height of the rocket above the sloping ground. What is its x-coordinate when it is at its maximum height?

(c) While the rocket is still going up, Clovis measures its height h above the sloping ground. Give a function $x = g(h)$ relating the x-coordinate of the rocket to h.

(d) Does this function still work when the rocket is going down? Explain.

6. A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour.



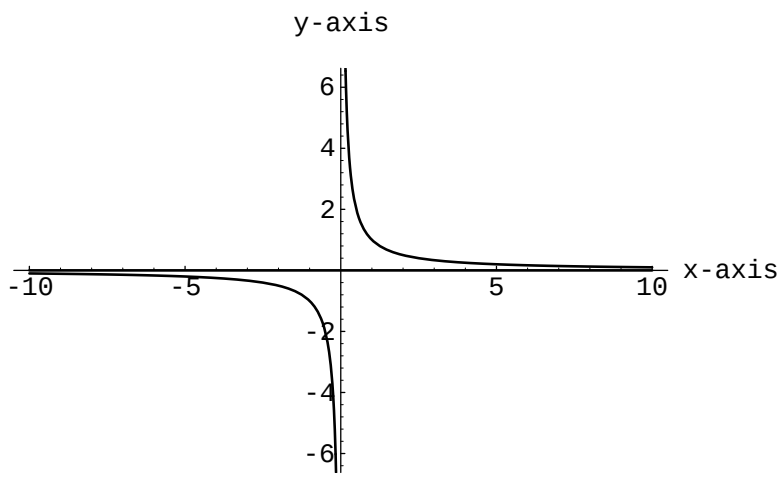
(a) Give a function $w = f(t)$ relating the width w of the surface of the water to the time t, in hours. Make sure to specify the domain and compute the range too.

(b) After how many hours will the surface of the water have width of 6 feet?

(c) Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.

2.9 Rational Functions

A *rational function* is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. The easiest example is $f(x) = \frac{1}{x}$ which we graph below.



An *asymptote* is a line that the graph of a function gets very close to, without ever quite reaching it. Note that as x gets very large, the graph of $y = \frac{1}{x}$ becomes very close to the x-axis. The line $y = 0$ is a *horizontal asymptote* for the function. When x is close to 0, the graph is very close to the y-axis. The line $x = 0$ is a *vertical asymptote* for this function. Note that the graph goes up the vertical asymptote on the right, and down it on the left. Now let's consider a general rational function $f(x) = \frac{p(x)}{q(x)}$. Our first question about any function is "What are its zeros?"

Examples 2.9.1: *Let* $h(x) = \frac{2x^2-x-1}{x^2-1}$. *Describe the zeros for this function.*

Solution. We wish to solve $0 = h(x)$ for x.

$$\begin{aligned} 0 &= h(x) \\ &= \frac{2x^2 - x - 1}{x^2 - 1} \\ 0 \cdot (x^2 - 1) &= 2x^2 - x - 1 \\ 0 &= (x-1)(2x+1). \end{aligned}$$

It appears that there are two answers, $x = 1$ and $-\frac{1}{2}$. But looking closer we see that $h(1)$ is undefined, because the bottom is zero. So $x = -\frac{1}{2}$ is really the only answer. □

The problem in the last example can be avoided if we reduce our rational function to lowest terms. In general, let $f(x) = \frac{p(x)}{q(x)}$ be a rational function which is reduced to lowest terms. Then the zeros of $f(x)$ are the x values where $p(x) = 0$. Furthermore, the vertical asymptotes of $y = f(x)$ occur at the values where $q(x) = 0$.

Examples 2.9.2: *Put the rational function $h(x) = \frac{2x^2-x-1}{x^2-1}$ into lowest form. Describe the zeros and asymptotes.*

Solution.

$$\begin{aligned} h(x) &= \frac{(x-1)(2x+1)}{(x-1)(x+1)} \\ &= \frac{2x+1}{x+1} \end{aligned}$$

Thus we have a zero at $x = -\frac{1}{2}$ and the line $x = -1$ is a vertical asymptote. To find the horizontal asymptote for this function, note that the value of the fraction does not change if we multiply top and bottom by $\frac{1}{x}$.

$$\begin{aligned} h(x) &= \frac{2x+1}{x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\ &= \frac{2 + \frac{1}{x}}{1 + \frac{1}{x}} \end{aligned}$$

When x is very large, $\frac{1}{x}$ is very small and the value of $y = h(x)$ is very close to $\frac{2}{1}$. Thus $y = 2$ is the horizontal asymptote for $h(x)$. □

For a general function $f(x) = \frac{p(x)}{q(x)}$, if $\deg q(x) < \deg p(x)$ then there is no horizontal asymptote. If $\deg p(x) = \deg q(x)$ then the horizontal asymptote is given by the ratio of the leading coefficients in $p(x)$ and $q(x)$, as in the last example. If $\deg q(x) > \deg p(x)$ then $y = 0$ is a horizontal asymptote. The next example explains why.

Examples 2.9.3: *Let* $g(x) = \frac{x+1}{x^2-9}$. *Describe the zeros and asymptotes.*

Solution. As before, the value does not change if we multiply both the top and the bottom by $\frac{1}{x^2}$.

$$\begin{aligned} g(x) &= \frac{x+1}{x^2-9} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{9}{x^2}} \end{aligned}$$

When x is very large, the top is close to 0 but the bottom is close to 1. Thus the ratio is close to 0 and $y = 0$ is a horizontal asymptote. Note that $g(x) = \frac{x+1}{x^2-9}$ is in lowest terms. The top vanishes at $x = -1$. This is the only zero of $g(x)$. The bottom is zero at $x = \pm 3$. Thus this function has two vertical asymptotes: $x = 3$ and $x = -3$. □

Now we are almost ready to graph $y = g(x) = \frac{x+1}{x^2-9}$. But first it is useful to figure out where the function is positive and where it is negative. Of course, the graph can cross the x-axis at the zero of the function. But it can also jump across at the vertical asymptotes. The function $g(x)$ has one zero, $x = -1$, and two vertical asymptotes, $x = -3$ and $x = 3$. Mark these three values on a number line; see Figure 2.9.1 below. These points divide the number line into four intervals. Check the function at a point in each interval. Here I use $x = -4$, -2, 0 and 4. Other values would be just as good. I conclude that $g(x)$ is negative when $x < -3$ or $-1 < x < 3$ and positive when $-3 < x < -1$ or $x > 3$.

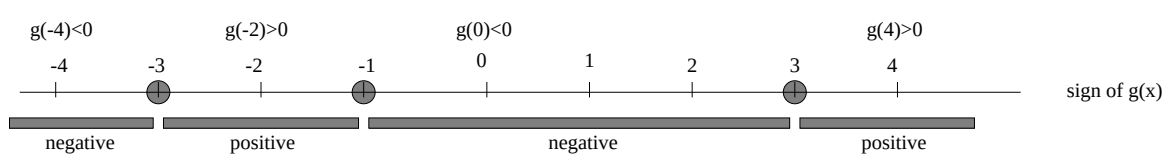
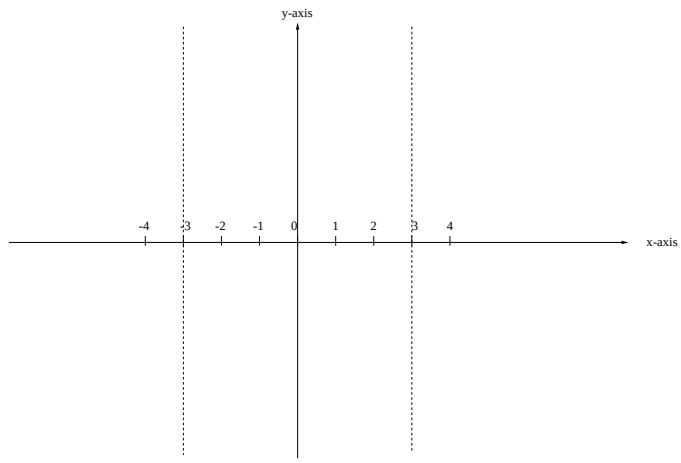
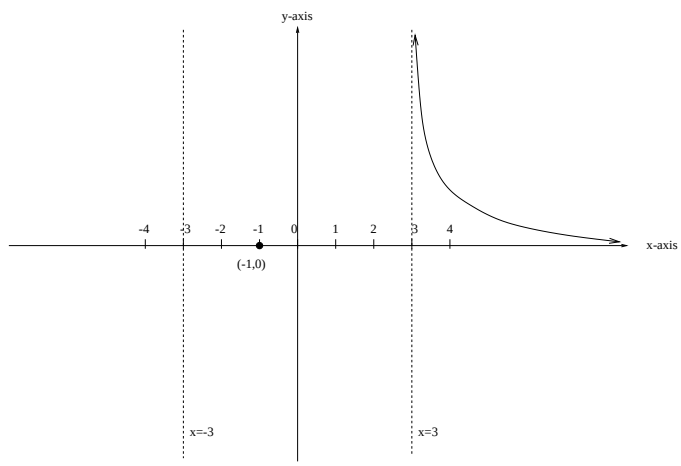


Figure 2.9.1

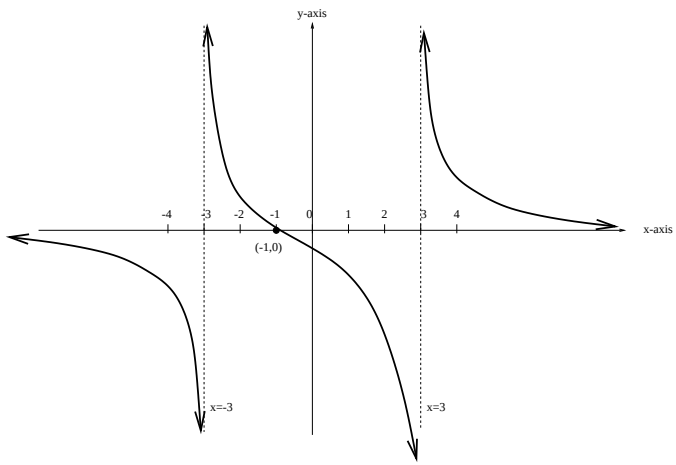
Now let's graph $y = g(x) = \frac{x+1}{x^2-9}$. First draw a pair of axes and sketch in the vertical asymptotes $x = -3$ and $x = 3$. We know that the x-axis is a horizontal asymptote. Note that $(-1, 0)$ is the only place where the graph crosses the x-axis.



Coming in from the right we have $x > 3$ and so $g(x) > 0$. Thus the graph must come in above the horizontal asymptote and go up $x = 3$ on the right.



Next, we are in the interval $-1 < x < 3$ so $g(x)$ is negative. Thus the graph must go down $x = 3$ on the left. Continue this reasoning to get the rest of the graph.



A rational function is called *linear-to-linear* if both the top and the bottom have degree one. This is a very important class of functions, with many applications. In general, these look like

$$f(x) = \frac{ax+b}{cx+d}$$

where a, b, c and d are real numbers. In fact, we can always take $c = 1$. Here's an example that shows how.

Examples 2.9.4: Let $f(x) = \frac{2x+3}{5x-7}$. Then

$$\begin{aligned} f(x) &= \frac{2x+3}{5x-7} \cdot \frac{\frac{1}{5}}{\frac{1}{5}} \\ &= \frac{\frac{2}{5}x + \frac{3}{5}}{x - \frac{7}{5}} \end{aligned}$$

This can be useful when we are trying to model something using linear-to-linear rational functions.

Examples 2.9.5:*Clyde makes extra money scalping tickets in front of the Kingdome. The amount he charges for a ticket depends on how many he has. If he only has one ticket, he charges \$100 for it. If he has 10 tickets, he charges \$80 a piece. But if he has a large number of tickets, he will sell them for \$50 each. How much will he charge for a ticket if he holds 20 tickets?*

Solution. We want to give a linear-to-linear rational function relating the price of a ticket y to the number of tickets x that Clyde is holding. As we saw above, we can assume the function

is of the form

$$y = \frac{ax+b}{x+c}$$

where a, b and c are numbers. Note that $y = a$ is the horizontal asymptote. When x is very large, y is close to 50. This means the line $y = 50$ is a horizontal asymptote. Thus $a = 50$ and

$$y = \frac{50x+b}{x+c}.$$

Next we plug in the point $(1, 100)$ to get a linear equation in b and c.

$$\begin{aligned} 100 &= \frac{50 \cdot 1 + b}{1+c} \\ 100 \cdot (1+c) &= 50 + b \\ 50 &= b - 100c \end{aligned}$$

Similarly, plugging in $(10, 80)$ and doing a little algebra (do it now!) gives $300 = b - 80c$. Solving these two linear equations simultaneously gives $c = 12.5$ and $b = 1300$. Thus our function is

$$y = \frac{50x + 1300}{x + 12.5}$$

and, if Clyde holds 20 tickets, he will charge

$$y = \frac{50 \cdot 20 + 1300}{20 + 12.5} = \$70.77$$

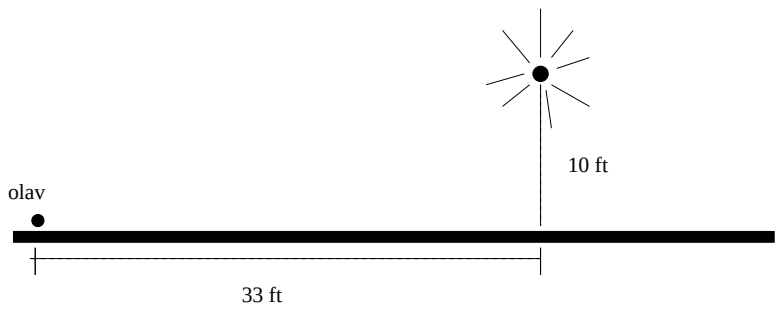
per ticket.

Problems

1. Give the domain of each of the following functions. Find the zeros. Sketch a graph and indicate any vertical or horizontal asymptotes. Give equations for the asymptotes.
   (a) $f(x) = \frac{2x}{x-1}$ (b) $g(x) = \frac{3x+2}{2x-5}$ (c) $h(x) = \frac{x^2-x-6}{x+2}$

   (d) $h(x) = \frac{x-3}{x^2-3x-4}$ (e) $g(x) = \frac{2x^2-7x+3}{x^2-x-2}$ (f) $q(x) = \frac{5}{x^2+2x+3}$
2. Isobel is producing and selling casette tapes of her rock band. When she had sold 10 tapes, her net profit was \$6. When she had sold 20 tapes, however, her net profit had shrunk to \$4 due to increased production expenses. But when she had sold 30 tapes, her net profit had rebounded to \$8.
   (a) Give a quadratic model relating Isobel's net profit y to the number of tapes sold x.
   (b) Divide the profit function in part (a) by the number of tapes sold x to get a model relating average profit w per tape to the number of tapes sold.
   (c) How many tapes must she sell in order to make \$1.20 per tape in net profit?

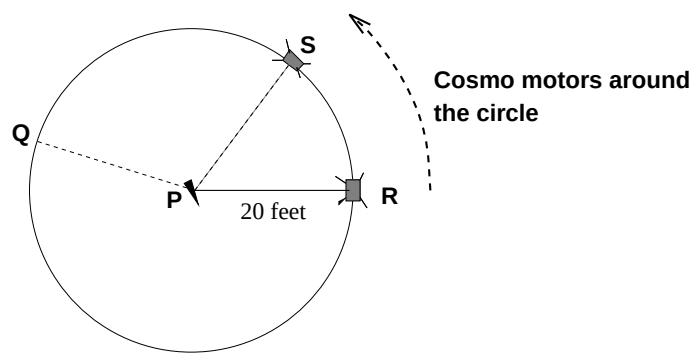
3. A street light is 10 feet North of a straight bike path that runs East-West. Olav is bicycling down the path at a rate of 15 MPH. At noon, Olav is 33 feet West of the point on the bike path closest to the street light. (See the picture). The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \frac{k}{d^2}$, where k is a constant depending on the light source.
   (a) From 20 feet away, the street light has an intensity of 1 candle. What is k?
   (b) Find a function which gives the intensity of the light shining on Olav as a function on time, in seconds.
   (c) When will the light on Olav have maximum intensity?
   (d) When will the intensity of the light be 2 candles?



Chapter 3

Circular Functions

So far, the equations we have studied have an algebraic character, involving the variables x and y, arithmetic operations and maybe extraction of roots. Restricting our attention to such equations would limit our ability to describe certain natural phenomena. An important example involves understanding motion around a circle, and it can be motivated by analyzing a very simple scenario: Cosmo the dog, tied by a 20 foot long tether to a post, begins walking around a circle.



-TOP VIEW-

A number of very natural questions arise:

Questions: *How can we measure the angles $\angle SPR, \angle QPR$ and $\angle QPS$? How can we measure the arc lengths $arc(RS), arc(SQ)$ and $arc(RQ)$? How can we measure the rate Cosmo is moving around the circle? If we know how to measure angles, can we compute the coordinates of R, S and Q? Turning this around, if we know how to compute the coordinates of R, S and Q, can we then measure the angles $\angle SPR, \angle QPR$ and $\angle QPS$? Finally, how can we specify the direction Cosmo is traveling?*

We will answer all of these questions and see how the theory which evolves can be applied to a variety of problems. The definition and basic properties of the *circular functions* will emerge as a central theme in this Chapter. The full problem-solving power of these functions will become apparent in our discussion of *sinusoidal functions* in §3.5 and *parameterized motion* in Chapter 4.

3.1 Measuring an Angle

The xy-coordinate system is well equipped to study straight line motion between two locations. For problems of this sort, the important tool is the distance formula. However, as Cosmo has illustrated, not all two-dimensional motion is along a straight line. In this section, we will describe how to calculate length along a circular arc, which requires a quick review of angle measurement.

Standard and Central Angles.

An *angle* is the union of two rays emanating from a common point called the *vertex* of the angle. A typical angle can be dynamically generated by rotating a single ray from one position to another, sweeping *counterclockwise* or *clockwise*, as pictured below. We often insert a curved arrow to indicate the direction in which we are sweeping out the angle. The ray ℓ_1 is called the *initial side* and ℓ_2 the *terminal side* of the angle $\angle AOB$.

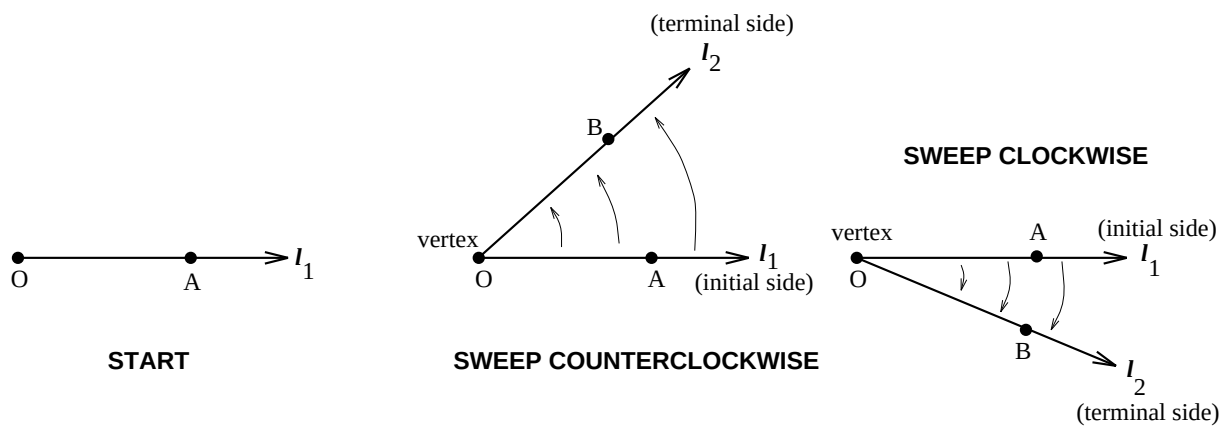


Figure 3.1.1: Angle $\angle AOB$

Working with angles, we need to agree on a standard frame of reference for viewing them. Within the usual xy-coordinate system, imagine a model of $\angle AOB$ in Figure 3.1.1 constructed from two pieces of rigid wire, welded at the vertex. Sliding this model around inside the xy-plane will not distort its shape, only its position relative to the coordinate axis. So, we can slide the angle into position so that the initial side is coincident with the positive x-axis and the vertex is the origin. Whenever we do this, we say the angle is in *standard position*. Once an angle is in standard position, we can construct a circle centered at the origin and view our standard angle as cutting out a particular "pie shaped wedge" of the corresponding disc.

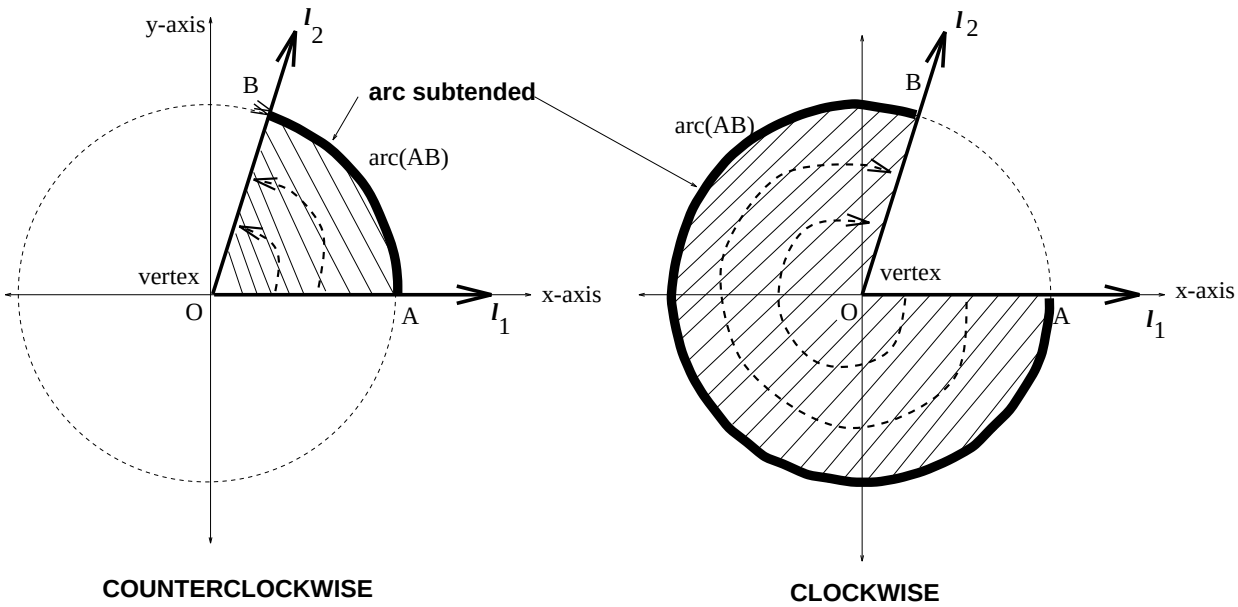


Figure 3.1.2: Standard angles and arcs

Notice, the shaded regions in Figure 3.1.2 depend on whether we sweep the angle counter-clockwise or clockwise from the initial side. The portion of the "pie wedge" along the circle edge, which is an arc, is called the *arc subtended* by the angle. We can keep track of this arc using the notation $arc(AB)$. A *central angle* is any angle with vertex at the center of a circle, but it's initial side may or may not be the positive x-axis. For example, $\angle QPS$ in the Figure beginning this Chapter is a central angle which is not in standard position.

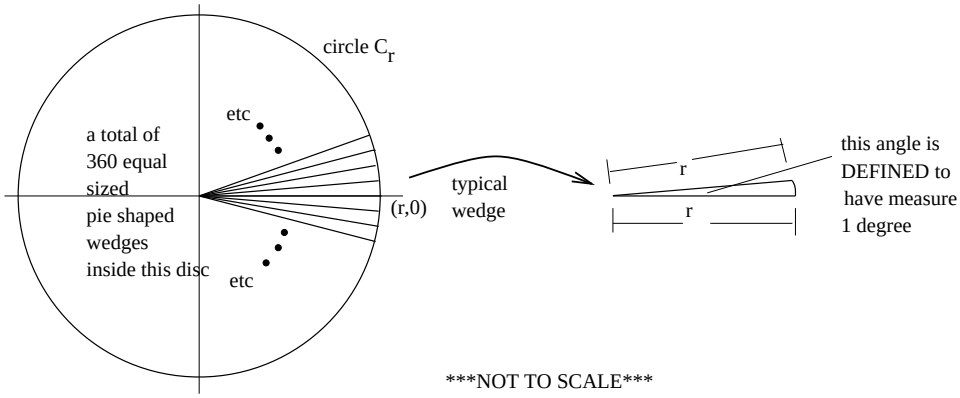
An Analogy.

To measure the dimensions of a box you would use a ruler. In other words, you use an instrument (the ruler) as a standard against which you measure the box. The ruler would most likely be divided up into either English units (inches) or metric units (centimeters), so we could express the dimensions in a couple of different ways, depending on the units desired.

By analogy, to measure the size of an angle, we need a standard against which any angle can be compared. In this section, we will describe two standards commonly used: the *degree method* and the *radian method* of angle measurement. The key idea is this: Beginning with a circular region, describe how to construct a "basic" pie shaped wedge whose interior angle becomes the standard unit of angle measurement.

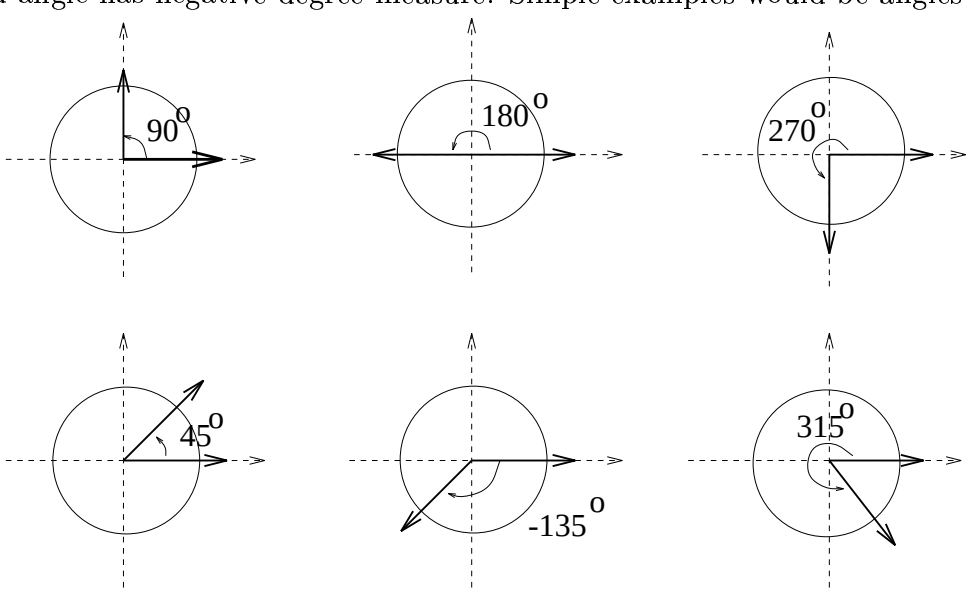
Degree Method.

Begin by drawing a circle of radius r, call it C_r, centered at the origin. Divide this circle into 360 equal sized pie shaped wedges, beginning with the the point $(r, 0)$ on the circle; i.e. the place where the circle crosses the x-axis.



We will refer to the arcs along the outside edges of these wedges as *one-degree arcs*. Why 360 equal sized arcs? The reason for doing so is historically tied to the fact that the ancient Babylonians did so as they developed their study of astronomy. (There is actually an alternate system based on dividing the circular region into 400 equal sized wedges; see Exercise 3.1.16.) Any central angle which subtends one of these 360 equal sized arcs is defined to have a measure of *one degree*, denoted 1°.

We can now use one-degree arcs to measure any angle: Begin by sliding the angle $\angle AOB$ into standard central position, as in Figure 3.1.2. Piece together consecutive one-degree arcs in a counterclockwise or clockwise direction, beginning from the initial side and working toward the terminal side, approximating the angle $\angle AOB$ to the nearest degree. If we are allowed to divide a one-degree arc into a fractional portion, then we could precisely determine the number m of one-degree arcs which consecutively fit together into the given arc. If we are sweeping counterclockwise from the initial side of the angle, m is defined to be the *degree measure* of the angle. If we sweep in a clockwise direction, then $-m$ is defined to be the *degree measure* of the angle. So, in Figure 3.1.2, the left-hand angle has positive degree measure while the right-hand angle has negative degree measure. Simple examples would be angles like these:



Notice, with our conventions, the rays determining an angle with measure -135° sit inside the circle in the same position as those for an angle of measure 225°; the minus sign keeps track of sweeping the positive x-axis clockwise (rather than counterclockwise).

We can further divide a one-degree arc into 60 equal arcs, each called a *one minute arc*. Each one-minute arc can be further divided into 60 equal arcs, each called a *one second arc*. This then leads to angle measures of one minute, denoted $1'$ and one second, denoted $1''$:

$$1^\circ \quad = \quad 60\,\text{minutes} = 3600\,\text{seconds}.$$

For example, an angle of measure θ= 5 degrees 23 minutes 18 seconds is usually denoted $5^o23'18''$. We could express this as a decimal of degrees:

$$5^o23'18'' \quad = \quad 5 + \frac{23}{60} + \frac{18}{3600}\,\text{degrees} = 5.3883.$$

As another example, suppose we have an angle with measure 75.456^o and we wish to convert this into degree/minute/second units. First, since $75.456^o = 75^o + 0.456^o$, we need to write 0.456^o in minutes by the calculation:

$$0.456\,\text{degree} \times 60\,\frac{\text{minutes}}{\text{degree}} \quad = \quad (27.36)'.$$

This tells us that $75.456^o = 75^o(27.36)' = 75^o27' + (0.36)'$. Now we need to write $(0.36)'$ in seconds via the calculation:

$$0.36\,\text{minutes} \times 60\,\text{seconds/minute} \quad = \quad (21.6)''.$$

In other words, $75.456^o = 75^o27'(21.6)''$.

Degree measurement of an angle is very closely tied to direction in the plane, explaining its use in map navigation. With some additional work, it is also possible to relate degree measure and lengths of circular arcs. To do this carefully, first go back to Figure 3.1.2 and recall the situation where an arc $arc(AB)$ is subtended by the central angle $\angle AOB$. In this situation, the *arc length* of $arc(AB)$, commonly denoted by the letter s, is the distance from A to B computed along the circular arc; keep in mind, this is NOT the same as the straight line distance between the points A and B.

For example, consider the six angles pictured above, of measures 90°, 180°, 270°, 45°, -135° and 315°. If the circle is of radius r and we want to compute the lengths of the arcs subtended by these six angles, then this can be done using the formula for the circumference of a circle (on the back of this text) and the following general principle:

(length of a part) = (fraction of the part)$\times$(length of the whole)

For example, the circumference of the entire circle of radius r is $2\pi r$; this is the "length of the whole" in the general principle. The arc subtended by a 90° angle is $\frac{90}{360} = \frac{1}{4}$ of the entire

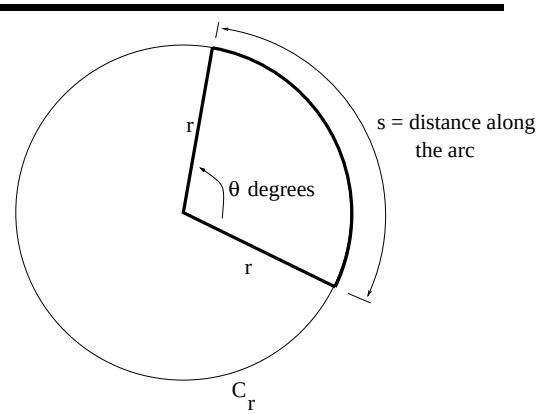
circumference; this is the "fraction of the part" in the general principle. The boxed formula implies:

$$s \quad = \quad \text{arc length of the } 90^o \text{ arc} = \left(\frac{1}{4}\right) 2\pi r = \frac{\pi r}{2}.$$

Similarly, a 180° angle subtends an arc of length $s = \pi r$, a 315° angle subtends an arc of length $s = \left(\frac{315}{360}\right) 2\pi r = \frac{7\pi r}{4}$, etc. In general, we arrive at this formula:

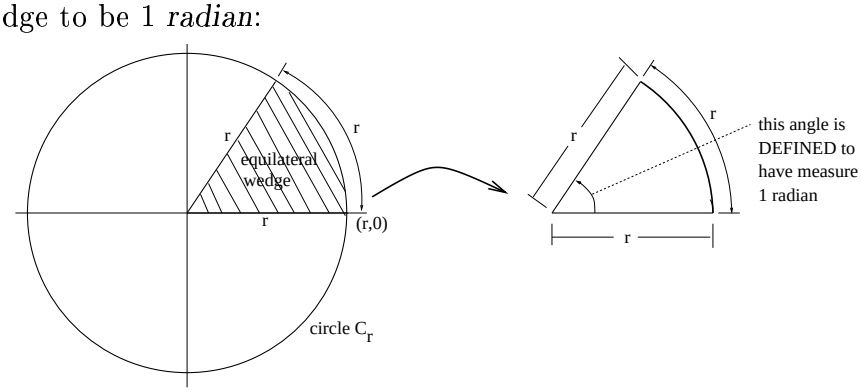
Arc Length 3.1.1: *Start with a central angle of measure θ degrees inside a circle of radius r. Then this angle will subtend an arc of length*

$$\boxed{s = \left(\frac{2\pi}{360}\right) r\theta}$$

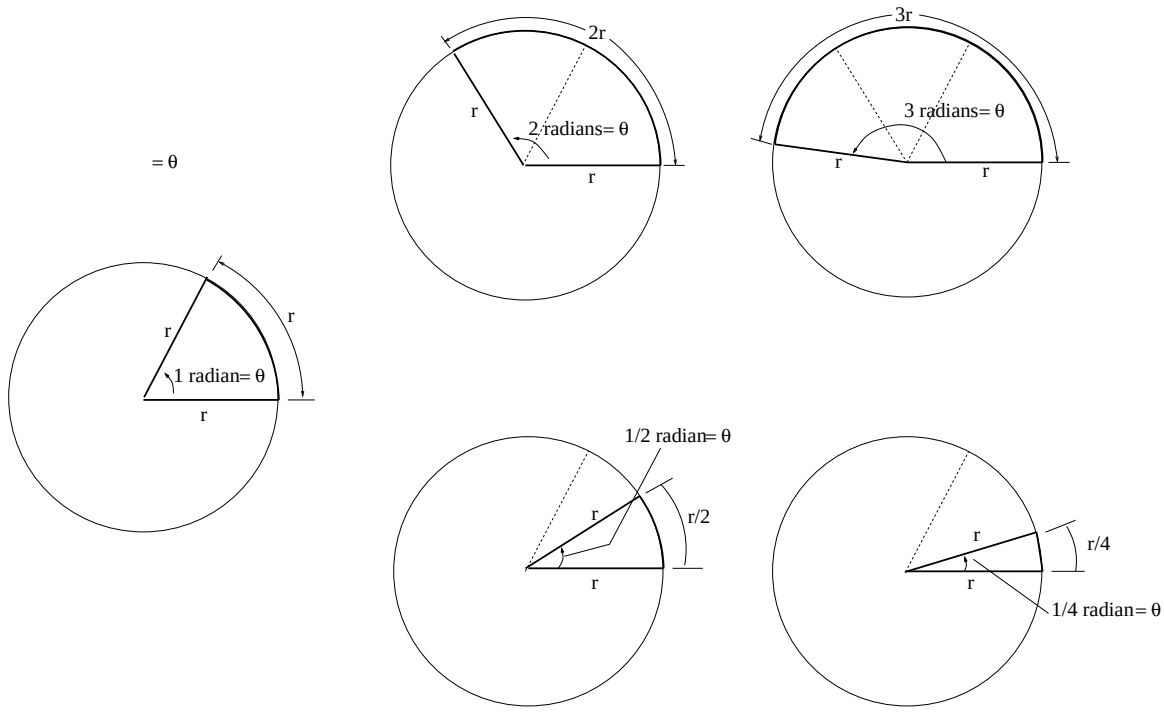


<u>Radian Method</u>. The key to understanding degree measurement was the description of a "basic wedge" which contained an interior angle of measure 1°; this was straightforward and familiar to all of us. Once this was done, we could proceed to measure any angle in degrees and compute arc lengths as in (3.1.1). However, the formula for the length of an arc subtended by an angle measured in degrees is sort of cumbersome, involving the curious factor $\frac{2\pi}{360}$. Our next goal is to introduce an alternate angle measurement scheme called *radian measure* that begins with a different "basic wedge". As will become apparent, a big selling point of radian measure is that arc length calculations become easy.

As before, begin with a circle C_r of radius r. Construct an *equilateral wedge* with all three sides of equal length r; see picture below. We define the measure of the interior angle of this wedge to be 1 *radian*:

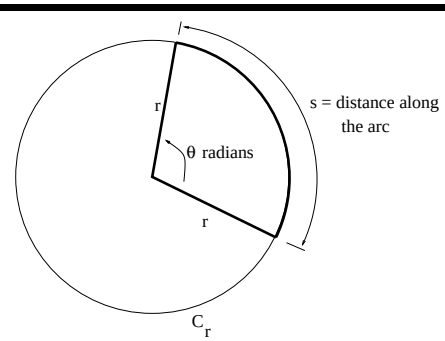


Once we have defined an angle of measure 1 radian, we can define an angle of measure 2 radians by putting together two equilateral wedges. Likewise, an angle of measure $\frac{1}{2}$ radian is obtained by symmetrically dividing an equilateral wedge in half, etc.

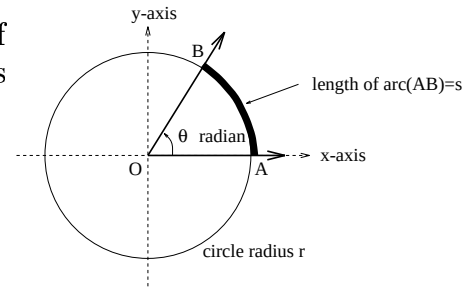


Reasoning in this way, we can piece together equilateral wedges or fractions of such to compute the radian measure of any angle. It is important to notice an important relationship between the radian measure of an angle and arc length calculations. In the five angles pictured above, 1 radian, 2 radian, 3 radian, $\frac{1}{2}$ radian and $\frac{1}{4}$ radian, the length of the arcs subtended by these angles θ are r, $2r$, $3r$, $\frac{1}{2}r$ and $\frac{1}{4}r$. In other words, a pattern emerges that gives a very simple relationship between the length s of an arc and the radian measure of the subtended angle:

Arc Length 3.1.2: *Start with a central angle of measure θ radians inside a circle of radius r. Then this angle will subtend an arc of length $s = \theta r$.*



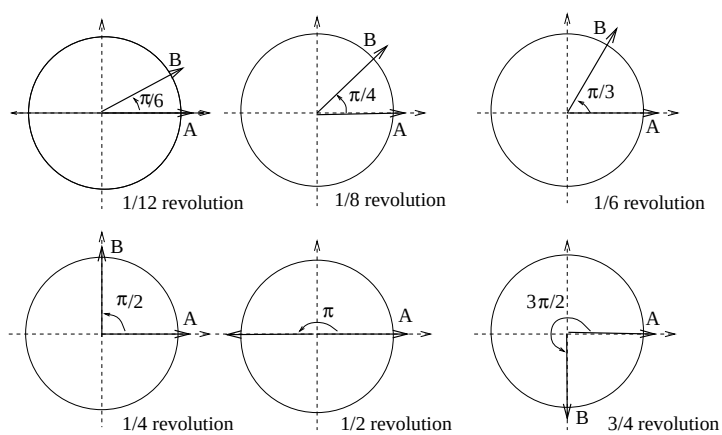
These remarks allow us to summarize the definition of the *radian measure* θ of $\angle AOB$ inside a circle of radius r by the formula:



$$\theta = \begin{cases} \frac{s}{r} & \text{if angle is swept counterclockwise} \\ -\frac{s}{r} & \text{if angle is swept clockwise} \end{cases}$$

The units of θ are sometimes abbreviated as *rad*. It is important to appreciate the role of the radius of the circle C_r when using radian measure of an angle: An angle of radian measure θ will subtend an arc of length $|\theta|$ on the *unit circle*. In other words, radian measure of angles is (up to $\pm$) **exactly the same** as arc length on the unit circle; we couldn't hope for a better connection!

The difficulty with radian measure versus degree measure is really one of familiarity. Let's view a few common angles in radian measure. It is easiest to start with the case of angles in central standard position within the unit circle. Examples of basic angles would be fractional parts of one complete revolution around the unit circle; for example, $\frac{1}{12}$ revolution, $\frac{1}{8}$ revolution, $\frac{1}{6}$ revolution, $\frac{1}{4}$ revolution, $\frac{1}{2}$ revolution and $\frac{3}{4}$ revolution. One revolution around the unit circle describes an arc of length 2π and so the subtended angle (1 revolution) is 2π radians. We can now easily find the radian measure of these six angles. For example, $\frac{1}{12}$ revolution would describe an angle of measure $(\frac{1}{12})2\pi$ rad$=\frac{\pi}{6}$ rad. Similarly, the other five angles pictured below have measures $\frac{\pi}{4}$ rad, $\frac{\pi}{3}$ rad, $\frac{\pi}{2}$ rad, π rad and $\frac{3\pi}{2}$ rad.



All of these examples have positive radian measure. For an angle with negative radian measure, such as $\theta = -\frac{\pi}{2}$ radians, we would locate B by rotating $\frac{1}{4}$ revolution *clockwise*, etc. From these calculations and our previous examples of degree measure we find that

$$180\,\text{degrees} = \pi\,\text{radians}. \tag{3.1.3}$$

Solving this equation for degrees or radians will provide conversion formulas relating the two types of angle measurement. The formula also helps explain the origin of the curious conversion factor $\frac{\pi}{180} = \frac{2\pi}{360}$ in formula (3.1.1).

Areas of Wedges.

The beauty of radian measure is that it is rigged so that we can easily compute lengths of arcs and areas of circular sectors (i.e. "pie-shaped regions"). This is a key reason why we will almost always prefer to work with radian measure.

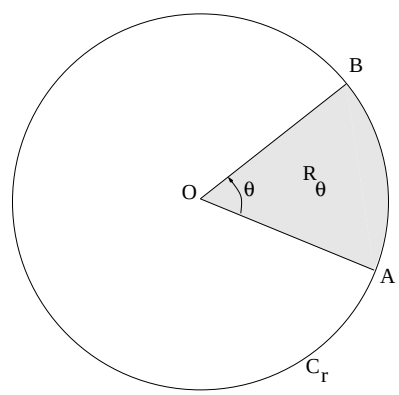
If a 16 inch pizza is cut into 12 equal slices, what is the area of a single slice?

This can be done using a general principle:

(Area of a part) = (area of the whole)×(fraction of the part)

So, for our pizza: (area one slice)= (area whole pie)×(fraction of pie) $=(8^2\pi)(\frac{1}{12}) = \frac{16\pi}{3}$.

Let's apply the same reasoning to find the area of a circular sector. We know the area of the circular disc bounded by a circle of radius r is πr^2. Let R_θ be the "pie shaped wedge" cut out by an angle $\angle AOB$ with positive measure θ radians. Using the above principle



$$\begin{aligned} \text{area}(R_\theta) &= (\text{area of disc bounded by } C_r) \times (\text{portion of disc accounted for by } R_\theta) \\ &= (\pi r^2)(\frac{\theta}{2\pi}) = \frac{1}{2}r^2\theta. \end{aligned}$$

For example, if $r = 3$ in. and $\theta = \frac{\pi}{4}$ rad, then the area of the pie shaped wedge is $\frac{9}{8}\pi$ sq. in.

Wedge Area 3.1.4: *Start with a central angle with positive measure θ radians inside a circle of radius r. The area of the "pie shaped region" bounded by the angle is $\frac{1}{2}r^2\theta$.*

Example 3.1.5: *A water drip irrigation arm 1200 feet long rotates around a pivot P once every 12 hours. How much area is covered by the arm 1 hour? in 37 minutes? How much time is required to drip irrigate 1000 square feet?*

Solution. The irrigation arm will complete one revolution in 12 hours. The angle swept out by one complete revolution is 2π radians, so after t hours the arm sweeps out an angle $\theta(t)$ given by

$$\theta(t) = \frac{2\pi \text{ radians}}{12 \text{ hours}} \times t \text{ hours} = \frac{\pi}{6} t \text{ radians.}$$

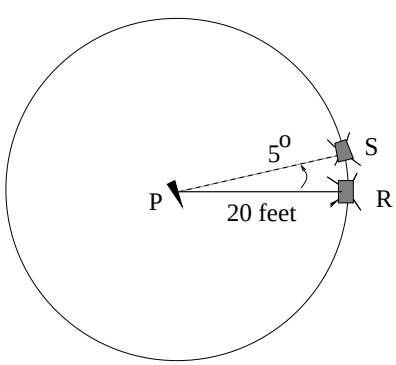
Consequently, by (3.1.4), the area $A(t)$ of the irrigated region after t hours is

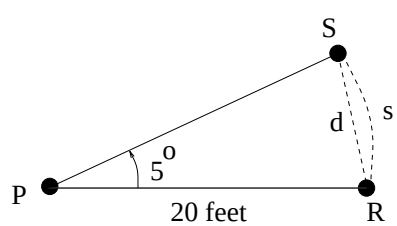
$$A(t) = \frac{1}{2}(1200)^2\theta(t) = \frac{1}{2}(1200)^2\frac{\pi}{6}t = 120,000\pi t \text{ sq. ft.}$$

After 1 hour, the irrigated area is $A(1) = 120,000\pi = 376,991$ sq. ft. Likewise, after 37 minutes, which is $\frac{37}{60}$ hours, the area of the irrigated region is $A(\frac{37}{60}) = 120,000\pi(\frac{37}{60}) = 232,500$ sq. ft. To answer the final question, we need to solve the equation $A(t) = 1000$; i.e., $120,000\pi t = 1000$, so $t = \frac{1}{120\pi}\text{hours}\frac{3600 \text{ seconds}}{\text{hour}} = 9.55$ seconds. □

Chord Approximation.

Our ability to compute arc lengths can be used as an estimating tool for distances between two points. Let's return to the situation posed at the beginning of this section: Cosmo the dog, tied by a 20 foot long tether to a post in the ground, begins at location R and walks counterclockwise to location S. Furthermore, let's suppose you are standing at the center of the circle determined by the tether and you measure the angle from R to S to be 5°; see the left-hand figure. Because the angle is small, notice that the straight line distance d from R to S is approximately the same as the arc length s subtended by the angle $\angle RPS$; the right-hand figure is a blow-up:



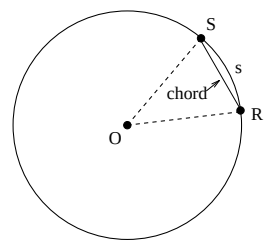


Example 3.1.6: *Estimate the distance from R to S.*

Solution. We first convert the angle into radian measure via (3.1.3): $5^o = 0.0873$ radians. By (3.1.4), the arc s has length 1.745 feet = 20.94 inches. This is approximately equal to the distance from R to S, since the angle is small. □

We call a line segment connecting two points on a circle a *chord* of the circle. The above example illustrates a general principal for approximating the length of any chord. As the size of the angle decreases, the accuracy of the arc length approximation will improve.

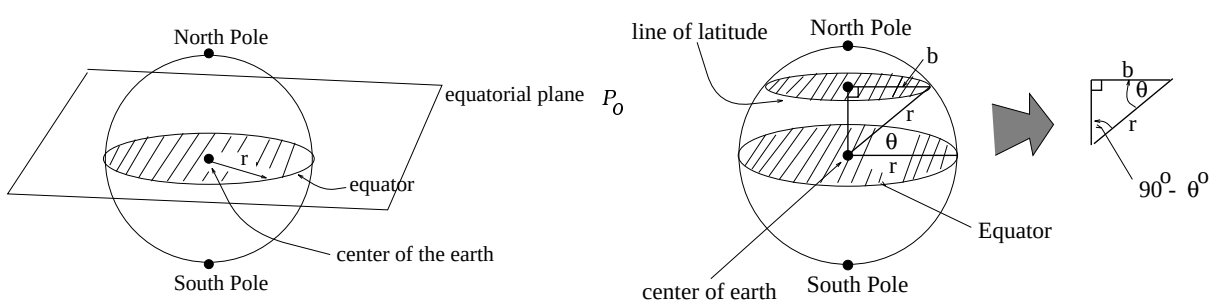
Chord Approximation 3.1.7: *In the Figure, if the central angle is small, then* $s \approx |\overline{RS}|$.



Great Circle Navigation (an introduction).

A basic problem is to find the shortest route between any two locations on the earth. We will review how to coordinatize the surface of the earth and recall the fact that the shortest path between two points is measured along a *great circle*.

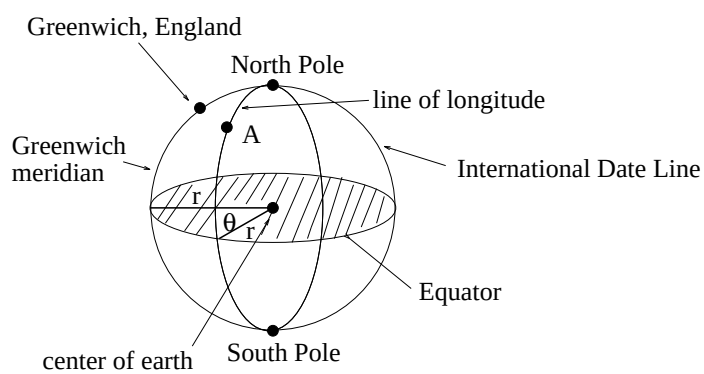
View the earth as a sphere of radius $r = 3960$ miles. We could slice the earth with a two-dimensional plane $\mathcal{P}_0$ which is both perpendicular to a line connecting the North and South poles and passes through the center of the earth. Of course, the resulting intersection will trace out a circle of radius $r = 3960$ miles on the surface of the earth, which we call the *equator*. We call the plane $\mathcal{P}_0$ the *equatorial plane*. Slicing the earth with any other plane $\mathcal{P}$ parallel to $\mathcal{P}_0$, we can consider the right triangle pictured below and the angle θ:



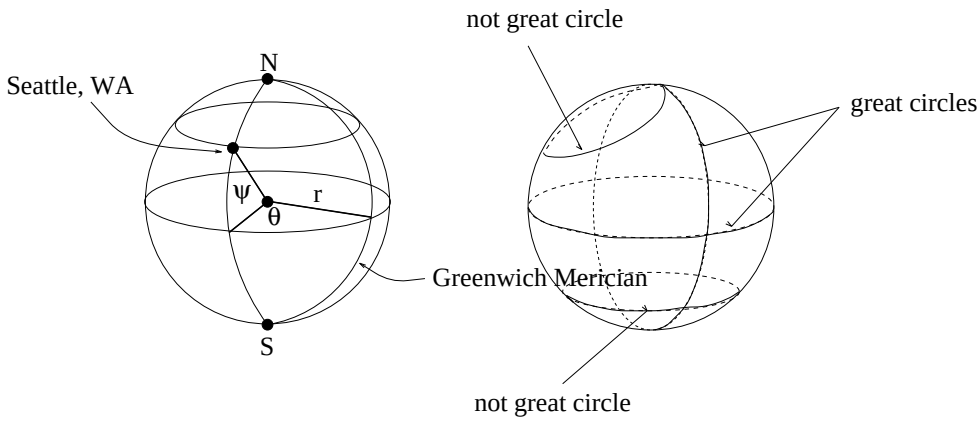
Essentially two cases arise, depending on whether or not the plane $\mathcal{P}$ is above or below the equatorial plane. The plane $\mathcal{P}$ slices the surface of the earth in a circle, which we call a line of *latitude*. This terminology is somewhat incorrect, since these lines of latitude are actually

circles on the surface of the earth, but the terminology is by now standard. Depending on whether this line of latitude lies above or below the equatorial plane, we refer to it as the θ^o North line of latitude (denoted θ^o N) or the θ^o South line of latitude (denoted θ^o S). Notice, the radius b of a line of latitude can vary from a maximum of 3960 miles (in the case of $\theta = 0^o$), to a minimum of 0 miles, (when $\theta = 90^o$). When $b = 0$, we are at the North or South poles on the earth.

In a similar spirit, we could imagine slicing the earth with planes $\mathcal{Q}$ which are perpendicular to both the equatorial plane and passing through the center of the earth. The resulting intersection will trace out a circle of radius 3960 miles on the surface of the earth, which is called a line of *longitude*. Half of a line of longitude from the North Pole to the South Pole is called a *meridian*. We distinguish one such meridian; the one which passes through Greenwich, England as the *Greenwich meridian*. Longitudes are measured using angles East or West of Greenwich. Pictured below, the longitude of A is θ. Because θ is east of Greenwich, θ measures longitude East, typically written θ^o E; west longitudes would be denoted as θ^o W. All longitudes are between 0° and 180°. The meridian which is 180° West (and 180° E) is called the *International Date Line*.



Introducing the grid of latitude and longitude lines on the earth amounts to imposing a coordinate system. In other words, any position on the earth can be determined by providing the longitude and latitude of the point. The usual convention is to list longitude first. For example, Seattle has coordinates $123^o7'$ W, $48^o0'$ N. Since the labels "N and S" are attached to latitudes and the labels "E and W" are attached to longitudes, there is no ambiguity here. This means that Seattle is on the line of longitude $123^o7'$ West of the Greenwich meridian and on the line of latitude $48^o0'$ North of the equator. In the figure below, we indicate the key angles $\psi = 48^o0'$ and $\theta = 123^o7'$ by inserting the three indicated radial line segments.



Now that we have coordinatized the earth, it is natural to study the distance between two locations. A *great circle* of a sphere is defined to be a circle lying on the sphere with the same center as the sphere. For example, the equator and any line of longitude are great circles. However, lines of latitude are not great circles (except the special case of the equator). Great circles are very important because they are used to find the shortest distance between two points on the earth. The important fact from geometry is summarized below.

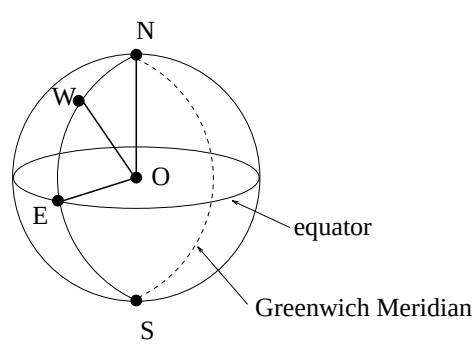
Great Circles 3.1.8: *The shortest distance between two points on the earth is measured along a great circle connecting them.*

Example 3.1.9: *What is the shortest distance from the North Pole to Seattle, WA ?*

Solution. The line of longitude $123°7'$ W is a great circle connecting the North Pole and Seattle. So, the shortest distance will be the arc length s subtended by the angle $\angle NOW$ pictured below. Since the latitude of Seattle is $48^o0'$, the angle $\angle EOW$ has measure $48^o0'$. Since $\angle EON$ is a right angle (i.e., $90°$), $\angle NOW$ has measure $42°0'$. By (3.1.3) and (3.1.4),

$$s = (3960\,\text{miles})(42°)(0.01745\,\text{radians/degree}) = 2903\,\text{miles},$$

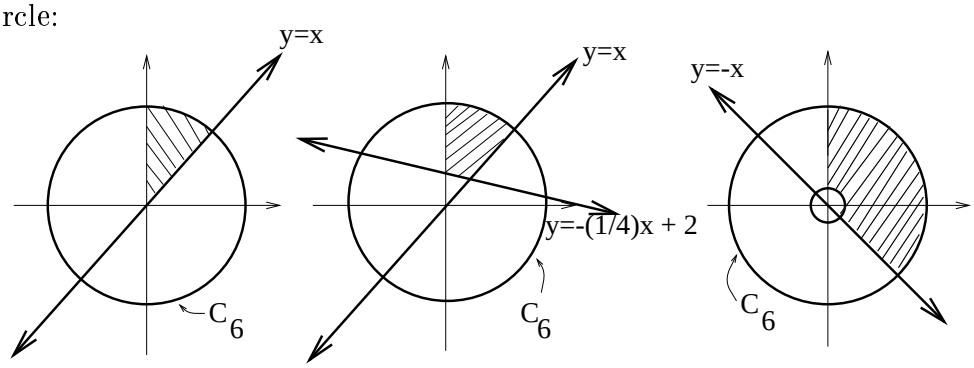
which is the shortest distance from the pole to Seattle.



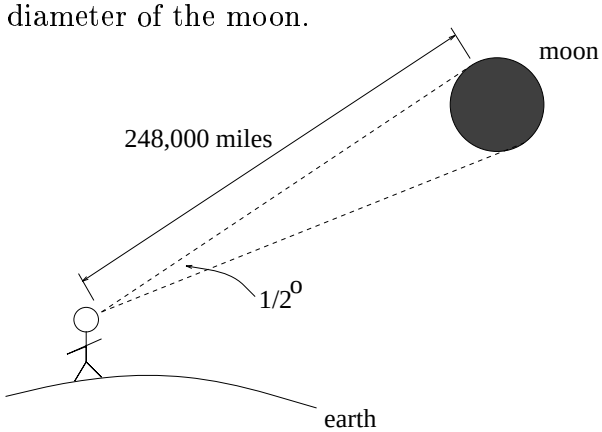
□

Problems

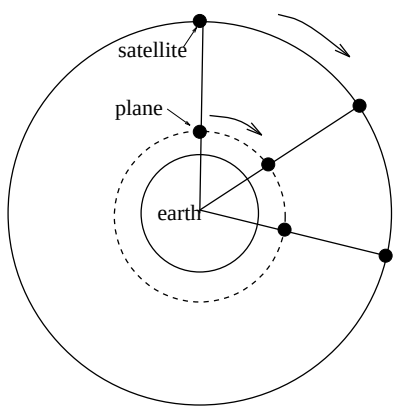
1. Let $\angle AOB$ be an angle of measure θ.
   (a) Convert $\theta = 13.4^o$ into degrees/minutes/seconds and into radians.
   (b) Convert $\theta = 1^o 4' 44''$ into degrees and radians.
   (c) Convert $\theta = 0.1$ radian into degrees and degrees/minutes/seconds.
2. A water treatment facility operates by dripping water from a 60 foot long arm whose end is mounted to a central pivot. The water then filters through a layer of charcoal. The arm rotates once every 8 minutes.
   (a) Find the area of charcoal covered with water after 1 minute.
   (b) Find the area of charcoal covered with water after 1 second.
   (c) How long would it take to cover 100 square feet of charcoal with water?
   (d) How long would it take to cover 3245 square feet of charcoal with water?
3.* Let C_6 be the circle of radius 6 centered at the origin in the xy-coordinate system. Compute the areas of the shaded regions in the picture below; the inner circle in the rightmost picture is the unit circle:



4. Astronomical measurements are often made by computing the small angle formed by the extremities of a distant object and using the estimating technique in (3.1.7). In the picture below, the full moon is shown to form an angle of $\frac{1}{2}^o$ when the distance indicated is 248,000 miles. Estimate the diameter of the moon.



5. During aerial spraying of insecticide to combat the Gypsy moth, a helicopter crew is assigned a sector of forest with central angle of $\frac{3}{7\pi^2}$ radians and a radius of 3.8 miles. How many square miles of forest will the crew spray?
6. A aircraft is flying at the speed of 500 mph at an elevation of 10 miles above the earth, beginning at the North pole and heading South along the Greenwich meridian. A spy satellite is orbiting the earth at an elevation of 4800 miles above the earth in a circular orbit in the same plane as the Greenwich meridian. Miraculously, the plane and satellite always lie on the same radial line from the center of the earth. Assume the radius of the earth is 3960 miles.



   (a) When is the plane directly over a location with latitude $74^o30'18''$ N?
   (b) How fast is the satellite moving?
   (c) When is the plane directly over the equator?
   (d) How far has the plane traveled (beginning over the North pole) when it is directly over the equator?
   (e) How far has the satellite traveled (beginning over the North pole) when it is directly over the equator?

3.2 Measuring Circular Motion

If Cosmo begins at location R and walks counterclockwise, always maintaining a tight tether, how can we measure Cosmo's speed?

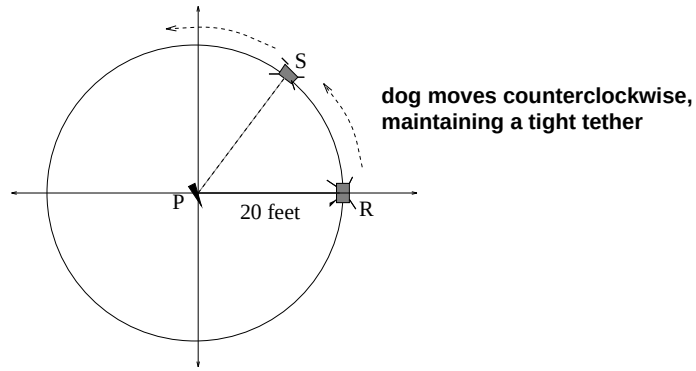


Figure 3.2.1: How fast is Cosmo moving?

This is a "dynamic question" and requires that we discuss ways of measuring circular motion. In contrast, if we take a snapshot and ask to measure the specific angle $\angle RPS$, this is a "static question", which we answered in the previous section.

Different ways to measure Cosmo's speed.

If Cosmo starts at location R and arrives at location S after some amount of time, we could study

$$\omega = \frac{\text{measure } \angle ROS}{\text{time required to go from } R \text{ to } S}.$$

The funny greek letter "ω" on the left of side of the equation is pronounced "oh-meg-a". We will refer to this as an *angular speed*. Typical units are "degrees/minute, "degrees/second", "radians/minute", etc. For example, if the angle swept out by Cosmo after 8 seconds is 40°, then Cosmo's angular speed is $\frac{40^o}{8 \text{ seconds}} = 5^o$/sec. Using (3.1.3), we can convert to radian units and get $\omega = \frac{\pi}{36}$ rad/sec $= \frac{5}{3}\pi$ rad/min. This is a new example of a *rate* and we can ask to find the total change, in the spirit of (1.1.3). If we are given ω in units of "rad/time" or "deg/time", we have

$$\theta \;=\; \omega t,$$

which computes the measure of the angle θ swept out after time t (i.e. the total change in the angle). Angular speed places emphasis upon the "size of the angle being swept out per unit time" by the moving object, starting from some initial position. We need to somehow indicate the direction in which the angle is being swept out. This can be done by indicating "clockwise" our "counterclockwise". Alternatively, we can adopt the convention that the positive rotational direction is counterclockwise, then insert a minus sign to indicate rotation clockwise. For example, saying that Cosmo is moving at an angular speed of $\omega = -\frac{\pi}{2}$ rad/sec means he is moving clockwise $\frac{\pi}{2}$ rad/sec.

Another way to study the rate of a circular motion is to count the number of complete circuits of the circle per unit time. This sort of rate has the form

$$\frac{\text{number of revolutions}}{\text{unit time}};$$

we will also view this as an angular speed. If we take "minutes" to be the preferred unit of time, we arrive at the common measurement called *revolutions per minute*, usually denoted RPM or rev/min. For example, if Cosmo completes one trip around the circle every 2 minutes, then Cosmo is moving at a rate of $\frac{1}{2}\ RPM$. If instead, Cosmo completes one trip around the circle every 12 seconds, then we could first express Cosmo's speed in units of *revolutions/second* as $\frac{1}{12}$ rev/second, then convert to RPM units:

$$(\frac{1}{12}\,\text{rev/sec})(60\,\text{sec/min}) \quad = \quad 5\,RPM.$$

As a variation, if we measure that Cosmo completed $\frac{3}{7}$ of a revolution in 2 minutes, then Cosmo's angular speed is computed by

$$\frac{\frac{3}{7}\text{rev}}{2\,\text{min}} \quad = \quad \frac{3}{14}\,RPM.$$

The only possible ambiguity involves the direction of revolution: the object can move clockwise or counterclockwise.

The one shortcoming of using angular speed is that we are not directly keeping track of the distance the object is traveling. This is fairly easy to remedy. Returning to Figure 3.2.1, the circumference of the circle of motion is $2\pi(20) = 40\pi$ feet. This is the distance traveled per revolution, so we can now make conversions of angular speed into "distance traveled per unit time"; this is called the *linear speed*. If Cosmo is moving $\frac{1}{2}RPM$, then he has a linear speed of

$$v \quad = \quad \left(\frac{1\text{rev}}{2\text{min}}\right)\left(\frac{40\pi\text{ft}}{\text{rev}}\right) = 20\pi\,\text{ft/min}.$$

Likewise, if Cosmo is moving $\frac{\pi}{7}$ rad/sec, then

$$v \quad = \quad \left(\frac{\pi\text{rad}}{7\text{sec}}\right)\left(\frac{1\text{rev}}{2\pi\text{rad}}\right)\left(\frac{40\pi\text{ft}}{\text{rev}}\right) = \frac{20\pi}{7}\,\text{ft/sec}.$$

Note: This discussion is an example of what is usually called "units analysis". The key idea we have illustrated is how to convert between two different types of units:

$$\left(\frac{\text{rev}}{\text{min}}\right) \overset{convertsto}{\rightarrow} \left(\frac{\text{ft}}{\text{min}}\right)$$

Different Ways to Measure Circular Motion.

The discussion of Cosmo applies to circular motion of any object. As a matter of convention, we usually use the Greek letter ω to denote angular speed and v for linear speed. If an object is moving around a circle of radius r at a constant rate, then we can measure it's speed in two ways:

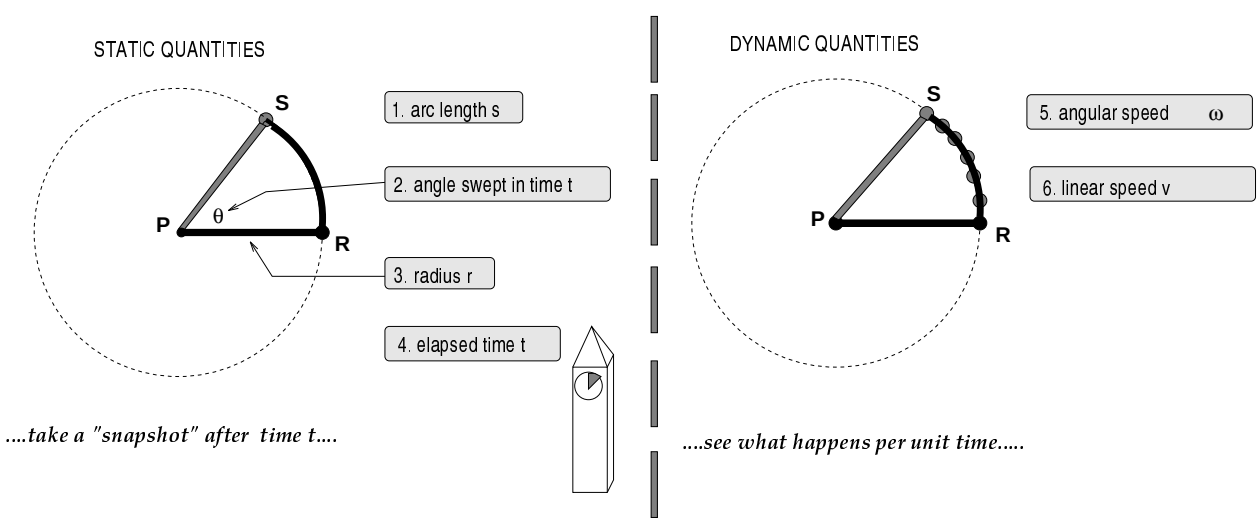
- The angular speed
$$\omega = \frac{\text{“revolutions”}}{\text{“unit time”}} \text{ or } \frac{\text{“degrees swept”}}{\text{“unit time”}} \text{ or } \frac{\text{“radians swept”}}{\text{“unit time”}}.$$
- The linear speed
$$v = \frac{\text{“distance traveled”}}{\text{“per unit time”}}.$$

Measuring and Converting 3.2.1: *We can convert between angular and linear speeds using these facts:*

- *1 revolution = 360º = 2π radians;*
- *The circumference of a circle of radius r units is $2\pi r$ units.*

Three Key Formulas.

If an object begins moving around a circle, there are a number of quantities we can try to relate. Some of these are “static quantities”: Take a visual “snapshot” of the situation after a certain amount of time has elapsed, then we can measure the *radius, angle swept, arc length* and *time* elapsed. Other quantities of interest are “dynamic quantities”: This means something is CHANGING with respect to time; in our case, the *linear speed* (which measures distance traveled per unit time) and *angular speed* (which measures angle swept per unit time) fall into this category.



We now know two general relationships for circular motion:

(i) $s = r\theta$, where s=arclength (a linear distance), r=radius of the circular path and θ=angle swept in RADIAN measure; this was the content of (3.1.2) of the previous section.

(ii) $\theta = \omega t$, where θ is the measure of an angle swept, ω= angular speed and t represents time elapsed. This is really just a consequence of units manipulation.

Notice how the units work in these formulas. If r=20 feet and $\theta = 1.3$ radians, then the arc length $s = 20(1.3)$ feet= 26 feet; this is the length of the arc of radius 20 feet that is subtending the angle θ. If $\omega = 3$ rad/second and $t = 5$ seconds, then $\theta = 3\frac{\text{rad}}{\text{seconds}} \times 5\,\text{seconds} = 15$ radians.

If we replace "θ" in $s = r\theta$ of (i) with $\theta = \omega t$ in (ii), then we get

$$s = r\omega t.$$

This gives us a relationship between arclength s (a distance) and time t. Plug in the fact that the linear speed is defined to be $v = \frac{\text{"distance"}}{t}$ and we get

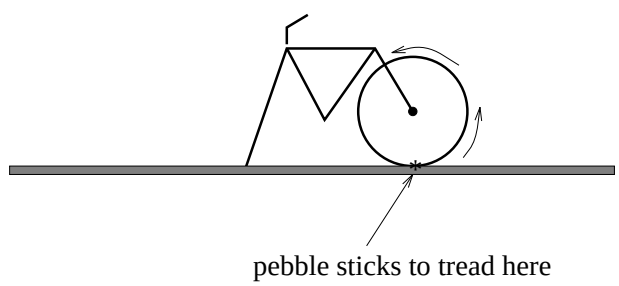
$$v = \frac{s}{t} = \frac{r\omega t}{t} = r\omega.$$

All of these observations are summarized below.

Three Really Useful Formulas 3.2.2: *If we measure angles θ in <u>RADIANS</u> and ω in units of radians per unit time, we have these three formulas:*

$$\begin{aligned} s &= r\theta \\ \theta &= \omega t \\ v &= r\omega \end{aligned}$$

Example 3.2.3: *You are riding a stationary exercise bike and the speedometer reads a steady speed of 40 mph (miles per hour). If the rear wheel is 28 inches in diameter, determine the angular speed of a location on the rear tire. A pebble becomes stuck to the tread of the rear tire. Describe the location of the pebble after 1 second and 0.1 second.*



Solution. The tires will be rotating in a counterclockwise direction and the radius $r = \frac{1}{2}28 = 14$ inches. The other given quantity, "40 mph", involves miles, so we need to decide which common units to work with. Either will work, but since the problem is focused on the wheel, we will utilize inches.

If the speedometer reads 40 mph, this is the linear speed of a specified location on the rear tire. We need to convert this into an angular speed, using unit conversion formulas. First, the linear speed of the wheel is

$$v = (40\,\text{miles/hr})(5280\,\text{ft/mile})(12\,\text{in/ft})(1\text{ hr}/60\text{ min})(1\,\text{min}/\ 60\text{ sec}) = 704\,\text{in/sec}.$$

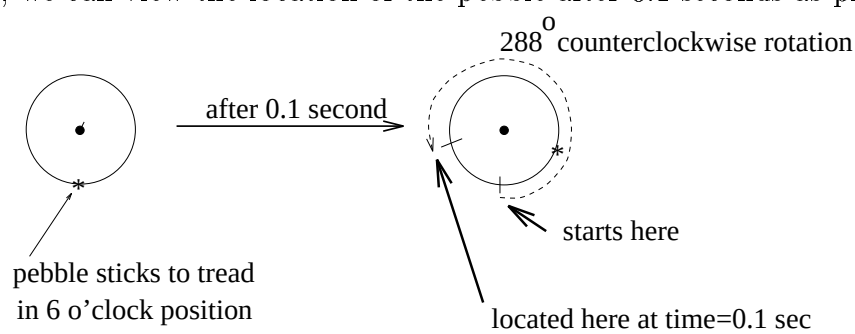
Now, the angular speed ω of the wheel will be

$$\omega = \frac{704\,\text{inches/second}}{2(14)\pi\,\text{inches/rev}} = 8\,\text{rev/sec} = 480\,RPM$$

It is then an easy matter to convert this to

$$\omega = (8\,\text{rev/sec})(360\,\text{degrees/revolution}) = 2880\,\text{degrees/second}.$$

If the pebble begins at the "6 o'clock" position (the place the tire touches the ground on the wheel), then after 1 second the pebble will go through 8 revolutions, so will be in the "6 o'clock" position again. After 0.1 seconds, the pebble will go through (8 rev/sec)(0.1 second)= 0.8 revolutions = (0.8 rev)(360 degrees/rev) = 288°. Keeping in mind that the rotation is counterclockwise, we can view the location of the pebble after 0.1 seconds as pictured below:



□

We solved the previous problem using the "unit conversion method". There is an alternate approach available, which uses one of the formulas in (3.2.2). Here is how you could proceed: First, as above, we know the linear speed is $v = 704$ in/sec. Using the "$v = \omega r$" formula, we have

$$\begin{aligned} 704\,\text{in/sec} &= \omega(14\,\text{in}) \\ \omega &= 50.28\,\text{rad/sec}. \end{aligned}$$

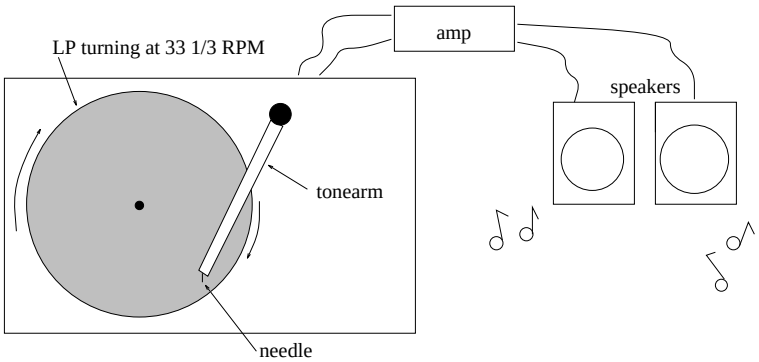
Notice how the units worked out in the calculation: the "time" unit comes from v and the "angular" unit will always be radians. As a comparison with the solution above, we can convert ω into RPM units: $\omega = (50.28\,\text{rad/sec})(1\,\text{rev}/2\pi\,\text{rad}) = 8\,\text{rev/sec}$.

All of the problems in this section can be worked using either the "unit conversion method" or the "$v = \omega r$ method".

Music Listening Technology (old and new).

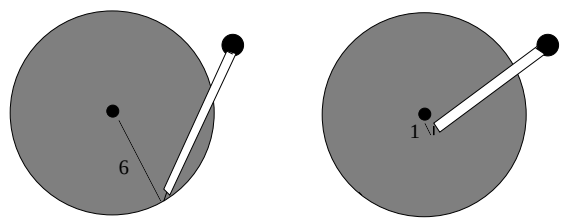
The technology of reproducing music has gone through a revolution since the early 1980's. The stereo long playing record (the *LP*) and the digital compact disc (the *CD*) are two methods of storing musical data for later reproduction in a home stereo system. These two technologies adopt different perspectives as to which notion of circular speed is best to work with.

Long playing stereo records are thin vinyl plastic discs of radius 6 inches onto which small spiral grooves are etched into the surface; we can approximately view this groove as a circle. The *LP* is placed on a flat 12 inch diameter platter which turns at a constant angular speed of $33\frac{1}{3}RPM$. An arm on a pivot (called the tone arm) has a needle mounted on the end (called the cartridge), which is placed in the groove on the outside edge of the record. Because the grooves wobble microscopically from side-to-side, the needle will mimic this motion. In turn, this sets a magnet (mounted on the opposite end of the needle) into motion. This moving magnet sits inside a coil of wire, causing a small varying voltage; the electric signal is then fed to your stereo, amplified and passed onto your speakers, reproducing music!



This is known as *analogue* technology and is based upon the idea of maintaining a constant angular speed of $33\frac{1}{3}RPM$ for the storage medium (our *LP*). (Older analogue technologies used $45RPM$ and $78RPM$ records. However, $33\frac{1}{3}RPM$ became the consumer standard for stereo music.) With an *LP*, the beginning of the record (the *lead-in groove*) would be on the outermost edge of the record and the end of the record (the *exit groove*) would be close to the center. Placing the needle in the lead-in groove, the needle gradually works its way to the exit groove. However, whereas the angular speed of the LP is a constant $33\frac{1}{3}RPM$, the linear speed at the needle can vary quite a bit, depending on the needle location.

Analogue LP's 3.2.4: *The "lead-in groove" is 6 inches from the center of an LP, while the "exit groove" is 1 inch from the center. What is the linear speed (mph) of the needle in the "lead-in groove"? What is the linear speed (mph) of the needle in the "exit groove"? Find the location of the needle if the linear speed is 1 mph.*



Solution. This is a straightforward application of (3.2.1). Let v_6 (resp. v_1) be the linear speed at the lead in groove (resp. exit groove); the subscript keeps track of the needle radial location. Since the groove is approximately a circle,

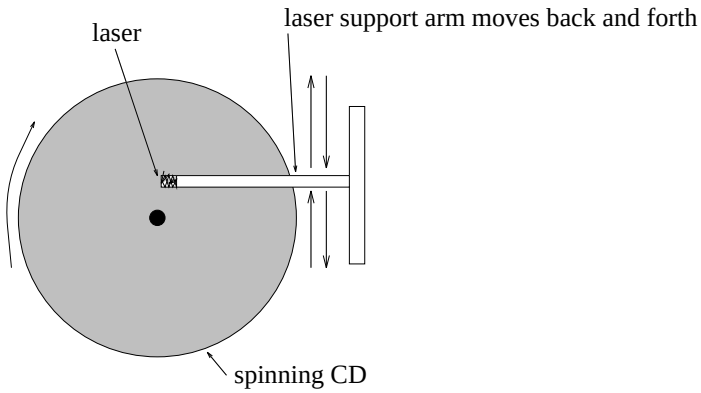
$$\begin{aligned} v_6 &= (33\frac{1}{3}\,\text{rev/min})(2(6)\pi\,\text{inches/rev}) \\ &= 1257\,\text{in/min} = \frac{(1257\,\text{in/min})(60\,\text{min/hour})}{(5280\,\text{ft/mile})(12\,\text{in/ft})} = 1.19\,\text{mph} \end{aligned}$$

Similarly, $v_1 = 0.2$ mph. To answer the remaining question, let r be the radial distance from the center of the LP to the needle location on the record. If $v_r = 1$ mph:

$$1\,\text{mile/hour} = v_r = (33\frac{1}{3}\ \text{rev/min})(2r\pi\,\text{in/rev})(60\,\text{min/hr})(\frac{1\,\text{ft}}{12\,\text{in}})(\frac{1\ \text{mile}}{5280\,\text{ft}})$$

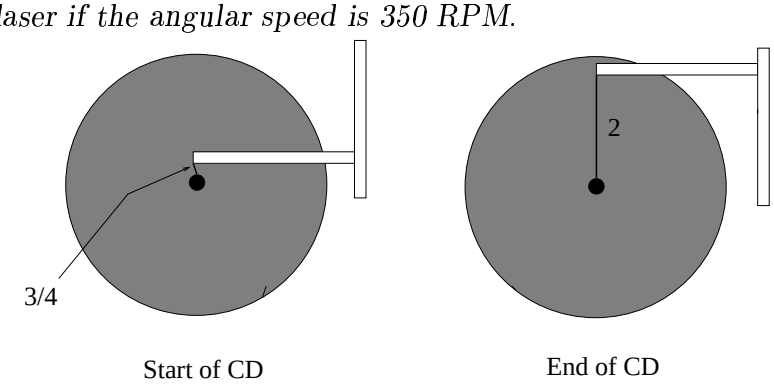
So, when the needle is $r = 5.04$ inches from the center, the linear speed is 1 mph. □

In the early 1980's, a new method of storing and reproducing music was introduced; this medium is called the *digital compact disc*, referred to as a CD for short. This is a thin plastic disc of diameter 4.5 inches, which appears to the naked eye to have a shiny silver coating on one side. Upon microscopic examination one would find concentric circles of pits in the silver coating. This disc is placed in a CD player, which spins the disc. A laser located above the spinning disc will project onto the spinning disc. The pits in the silver coating will cause the reflected laser light to vary in intensity. A sensor detects this variation, converting it to a digital signal (the *analogue to digital or AD conversion*). This is fed into a *digital to analogue or DA conversion* device, which sends a signal to your stereo, again producing music.



The technology of $CD's$ differs from that of $LP's$ in two crucial ways. First, the circular motion of the spinning CD is controlled so that the target on the disc below the laser is always moving at a constant linear speed of 1.2 meters/sec = 2835 inches/minute. Secondly, the beginning location of the laser will be on the inside portion of the disc, working its way outward to the end. In this context, it makes sense to study how the angular speed of the CD is changing, as the laser position changes.

Digital CD's 3.2.5: *What is the angular speed (in RPM) of a CD if the laser is at the beginning, located $\frac{3}{4}$ inches from the center of the disc ? What is the angular speed (in RPM) of a CD if the laser is at the end, located 2 inches from the center of the disc ? Find the location of the laser if the angular speed is 350 RPM.*



Solution. This is an application of (3.2.1). Let $\omega_{3/4}$ be the angular speed at the start and ω_2 the angular speed at the end of the CD; the subscript is keeping track of the laser distance from the CD center.

$$\omega_2 = \frac{(2835\,\text{inches/min})}{(2(2)\pi\,\text{inches/rev})} = 225.6\,RPM \qquad \omega_{3/4} = \frac{2835\,\text{inches/min})}{(2(0.75)\pi\,\text{inches/rev})} = 601.6\,RPM$$

To answer the remaining question, let r be the radial distance from the center of the CD to the laser location on the CD. If the angular speed ω_r at this location is $350RPM$, we have

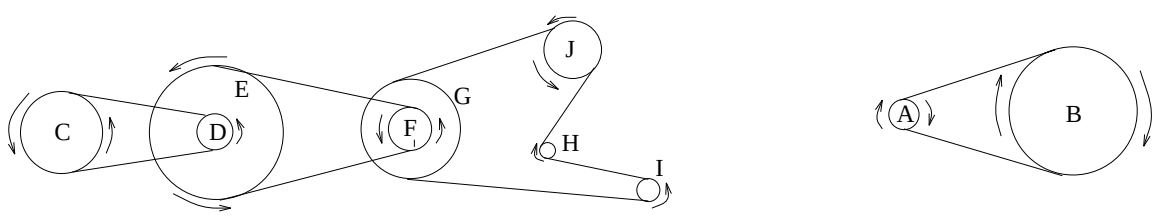
the equation

$$\begin{aligned} 350\,RPM &= \omega_r = \frac{2835\,\text{inches/min}}{(2r\pi\,\text{inches/rev})} \\ 1.289\,\text{inches} &= r. \end{aligned}$$

So, when the laser 1.289 inches from the center, the CD is moving $350RPM$. □

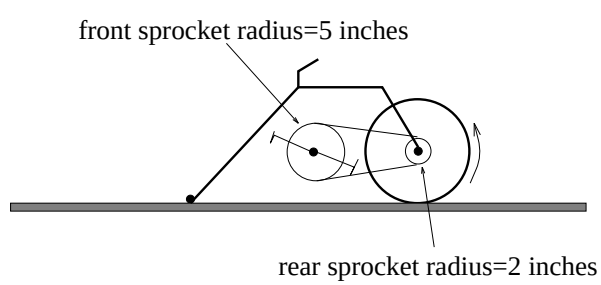
Belt and Wheel Problems.

The industrial revolution spawned a number of elaborate machines involving systems of belts and wheels. Computing the speed of various belts and wheels in such a system may seem complicated at first glance. The situation can range from a simple system of two wheels with a belt connecting them, to more elaborate designs.

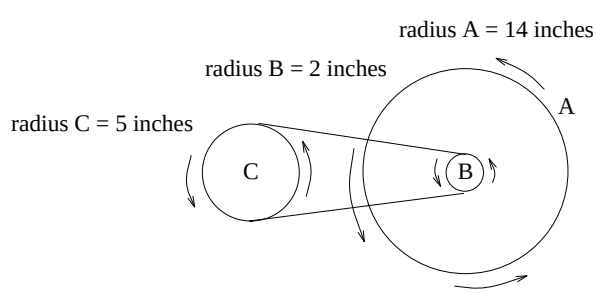


We call problems of this sort *belt/wheel problems*, or more generally, *connected wheel problems*. Solving problems of this type always uses the same strategy, which we will first highlight by way of an example.

Example 3.2.6: *You are riding a stationary exercise bike. Assume the rear wheel is 28 inches in diameter, the rear sprocket has radius 2 inches and the front sprocket has radius 5 inches. How many revolutions per minute of the front sprocket produces a forward speed of 40 mph on the bike (miles per hour)?*



Solution. There are 3 wheels involved with a belt (the bicycle chain) connecting two of the wheels. In this problem, we are provided with the linear speed of wheel A (which is 40 mph) and we need to find the angular speed of wheel C=front sprocket.



Denote by v_A, v_B, v_C the linear speeds of each of the wheels A, B and C, respectively. Likewise, let $\omega_A, \omega_B, \omega_C$ denote the angular speeds of each of the wheels A, B and C, respectively. In addition, the chain connecting the wheels B and C will have a linear speed, which we will denote by v_{chain}. The strategy is broken into a sequence of steps which leads us from the known linear speed v_A to the angular speed ω_C of wheel C:

- Step 1: Given v_A, find ω_A. Use the fact $\omega_A = \frac{v_A}{r_A}$.
- Step 2: Observe $\omega_A = \omega_B$; this is because the wheel and rear sprocket are both rigidly mounted on a common axis of rotation.
- Step 3: Given ω_B, find v_B. Use the fact $v_B = r_B\omega_B = r_B\omega_A = \left(\frac{r_B}{r_A}\right) v_A$.
- Step 4: Observe $v_B = v_{chain} = v_C$; this is because the chain is directly connecting the two sprockets and assumed not to slip.
- Step 5: Given v_C, find ω_C. Use the fact $\omega_C = \frac{v_C}{r_C} = \frac{v_B}{r_C} = \left(\frac{r_B}{r_A r_C}\right) v_A$.

Saying that the speedometer reads 40 mph is the same as saying that the linear speed of a location on the rear wheel is $v_A = 40\,mph$. Converting this into angular speed was carried out in our solution to (3.2.3) above; we found that $\omega_A = 480RPM$. This completes Step 1 and so by Step 2, $\omega_A = \omega_B = 480RPM$. For Step 3, we convert $\omega_B = 480RPM$ into linear speed following (3.2.1):

$$v_B = (480\,\text{rev/minute})((2(2)\pi)\,\text{inches/rev}) = 6032\,\text{inches/min}.$$

By Step 4, conclude that the linear speed of wheel C is $v_C = 6032\,\text{inches/min}$. Finally, to carry out Step 5, we convert the linear speed into angular speed:

$$\omega_C = \frac{(6032\,\text{inches/min})}{2(5)\pi\,\text{inches/rev}} = 192\,RPM = 3.2\,\text{rev/sec}.$$

In conclusion, the bike rider must pedal the front sprocket at the rate of 3.2 rev/sec. □

This example indicates the basic strategy used in all belt/wheel problems.

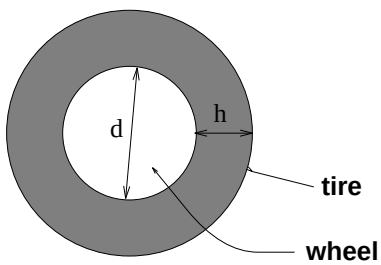
Belt Wheel Strategy 3.2.7: *Three basic facts are used in all such problems:*

- *Using "unit conversion" or (3.2.2) allows us to go from linear speed v to angular speed ω, and vice versa.*
- *If two wheels are fastened rigidly to a common axle, then they have the same angular speed. (Caution: two wheels fastened to a common axle typically do not have the same linear speed!)*
- *If two wheels are connected by a belt (or chain), the linear speed of the belt coincides with the linear speed of each wheel.*

Problems

1. (a) A wheel of radius 22 ft. is rotating 11 RPM counterclockwise. What is the angular speed ω and the linear speed v?
   (b) A wheel of radius 8 in. is rotating 15°/sec. What is the linear speed v, the angular speed in RPM and the angular speed in rad/sec?
   (c) You are standing on the equator of the earth (radius 3960 miles). What is your linear and angular speed?
   (d) An auto tire has radius 12 inches. If you are driving 65 mph, what is the angular speed in rad/sec and the angular speed in RPM?

2. Your car's speedometer is geared to accurately give your speed using a certain size tire. Suppose your car has d =14 inch diameter wheels and the height of the tire is h =4.5 inches.



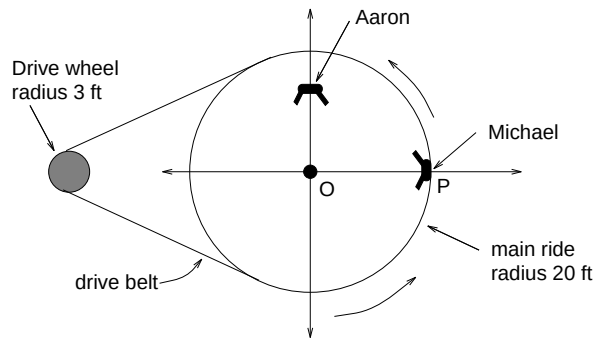
   (a) You buy a new set of tires with $h = 5.5$ inches and $d = 14$ inches. On a trip to Spokane, you maintain a constant speed of 65 mph, according to your speedometer. However, as luck would have it, you are stopped for speeding. Explain how this could happen. What did the radar gun display as your true speed?
   (b) You are furious over the speeding ticket and return to the tire dealer, demanding new tires which are the correct size. The dealer only has "low profile tires" in stock, which are $h = 3.75$ inches high. If you accept these and drive away from the dealer with your speedometer reading 35 mph, how fast are you really going?

3. You are riding a bicycle along a level road. Assume each wheel is 26 inches in diameter, the rear sprocket has radius 3 inches and the front sprocket has radius 7 inches. How many revolutions per minute of the front sprocket produces a speed of 35 mph?

4. A *laser video disc*, called an LVD, is the video counterpart of the musical CD; it is the digital version of the VCR tape player. Again, a LVD is a thin plastic disc with a silver coating on one side, but now the diameter is 12 inches. Just as with the CD, information is read off using a laser and the LVD is designed to spin at a constant linear speed below the laser; this speed is 1.2 meters/sec = 2835 inches/minute. The laser begins $\frac{3}{4}$ inch from the center of the spinning disc and works it way out to the end (5.5 inches from the center).
   (a) Find the angular speed of the LVD at the beginning and end of the LVD; i.e. when the laser is $\frac{3}{4}$ inch and 5.5 inches from the center.

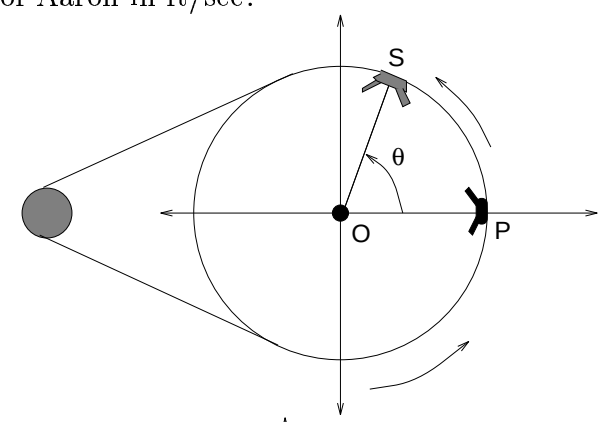
(b) Describe the location of the laser if the angular speed is 100 RPM.

5. Michael and Aaron are on the "UL-Tossum" ride at *Funworld*. This is a merry-go-round of radius 20 feet which spins counterclockwise 60 RPM. The ride is driven by a belt connecting the outer edge of the ride to a drive wheel of radius 3 feet:

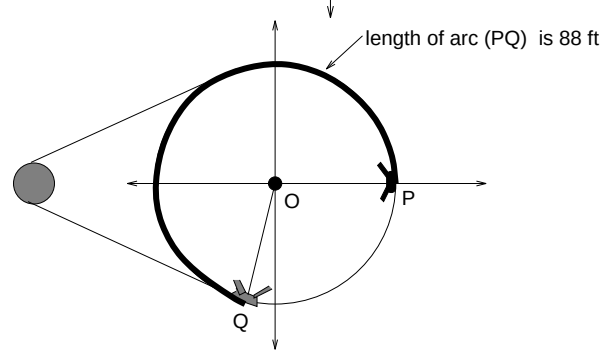


(a) Assume Michael is seated on the edge of the ride, as pictured. What is Michael's linear speed in mph and ft/sec?
(b) What is the angular speed of the drive wheel in RPM?
(c) Suppose Aaron is seated 16 feet from the center of the ride. What is the angular speed of Aaron in RPM? What is the linear speed of Aaron in ft/sec?

(d) After 0.23 seconds Michael will be located at S as pictured. What is the angle $\angle POS$ in degrees? What is the angle $\angle POS$ in radians? How many feet has Michael traveled?

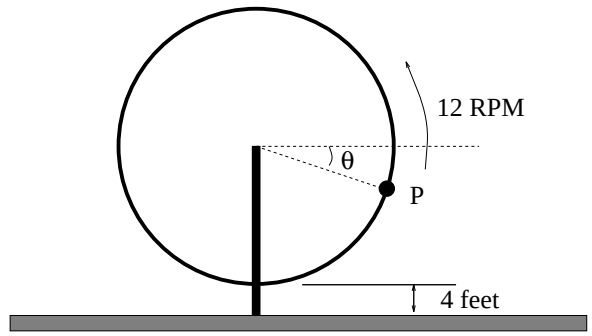


(e) Assume Michael has traveled 88 feet from the position P to a new position Q. How many seconds will this take? What will be the angle swept out by Michael?



6. Cherie is running around the perimeter of a circular track at a rate of 10 ft/sec. The track has a radius of 100 yards. After 10 seconds, Cherie turns and runs along a radial line to the center of the circle. Once she reaches the center, she turns and runs along a radial line to her starting point on the perimeter. Assume Cherie does not slow down when she makes these two turns.
(a) Sketch a picture of the situation.
(b) How far has Cherie traveled once she returns to her starting position?
(c) Find the area of the pie shaped sector enclosed by Cherie's path.

7. John has been hired to design an exciting carnival ride. Tiff, the carnival owner, has decided to create the worlds greatest ferris wheel. Tiff isn't into math; she simply has a vision and has told John these constraints on her dream: (i) the wheel should rotate counterclockwise with an angular speed of 12 RPM; (ii) the linear speed of a rider should be 200 mph; (iii) the lowest point on the ride should be 4 feet above the level ground.



(a) Find the radius of the ferris wheel.

(b) Once the wheel is built, John suggests that Tiff should take the first ride. Assume the wheel starts turning when Tiff is at the location P and it takes her 1.3 seconds to reach the top of the ride. Find the angle θ pictured.

(c) Poor engineering causes Tiff's seat to fly off in 6 seconds. Describe where Tiff is located (an angle description) the instant she becomes a human missile.

3.3 The Circular Functions

Suppose Cosmo begins at location R and walks in a counterclockwise direction, always maintaining a tight 20 ft long tether. As Cosmo moves around the circle, how can we describe his location at any given instant?

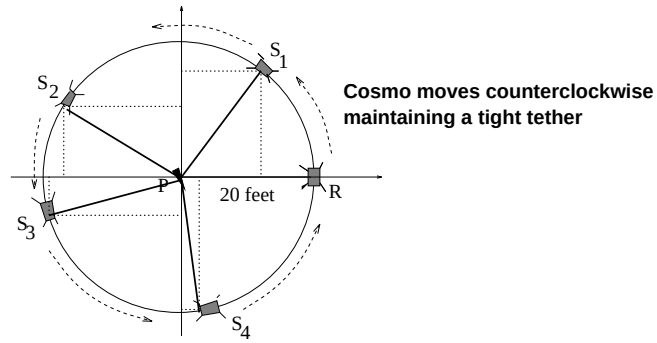
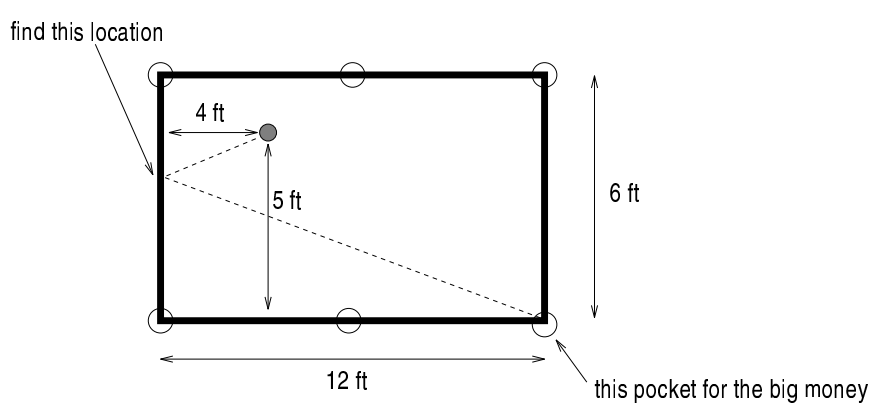


Figure 3.3.1: Cosmo's location?

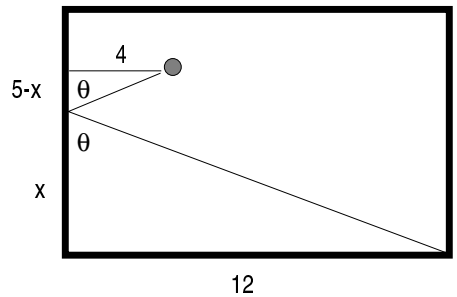
In one sense, we have already answered this question: The measure of $\angle RPS_1$ exactly pins down a location on the circle of radius 20 feet. But, we really might prefer a description of the horizontal and vertical coordinates of Cosmo; this would tie in better with the coordinate system we typically use. Solving this problem will require NEW functions, called the *circular functions*.

Relating Sides and Angles of a Right Triangle.

You are preparing to make your final shot at the British Pocket Billiard World Championships. The position of your ball is as in the figure below and you must play the ball off the left cushion into the lower-right corner pocket, as indicated by the dotted path. For the big money, where should you aim to hit the cushion?



The solution to this problem depends on two basic facts. First, the angles of entry and exit between the path the cushion will be equal. Secondly, the two obvious right triangles in this picture are similar triangles. Let x represent the distance from the bottom left corner to the impact point of the ball's path:



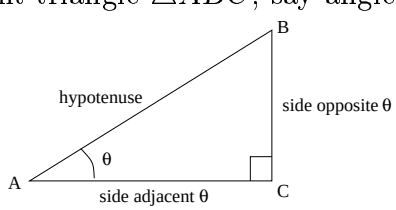
Properties of similar triangles tell us that the ratios of common sides are equal:

$$\frac{4}{5-x} = \frac{12}{x}.$$

If we solve this equation for x, we obtain $x = \frac{15}{4} = 3.75$ feet. This discussion is enough to win the tourney. But, of course, there are still other questions we can ask about this simple example: What is the angle θ? That is going to require substantially more work; indeed the bulk of this Chapter! It turns out, there is a lot of mathematical mileage in the idea of studying ratios of sides of right triangles. The first step, which will get the ball rolling, is to introduce new functions whose very definition involves relating sides and angles of right triangles.

The Trigonometric Ratios.

From elementary geometry, the sum of the angles of any triangle will equal 180°. Given a right triangle $\triangle ABC$, since one of the angles is 90°, the remaining two angles must be *acute angles*; i.e., angles of measure between 0° and 90°. If we specify one of the acute angles in a right triangle $\triangle ABC$, say angle θ, we can label the three sides using this terminology:



We then consider the following three ratios of side lengths, referred to as *trigonometric ratios*:

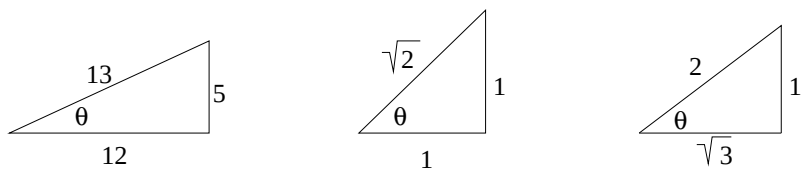
$$\sin(\theta) \stackrel{\text{def}}{=} \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}};$$

$$\cos(\theta) \stackrel{\text{def}}{=} \frac{\text{length of side adjacent } \theta}{\text{length of hypotenuse}};$$

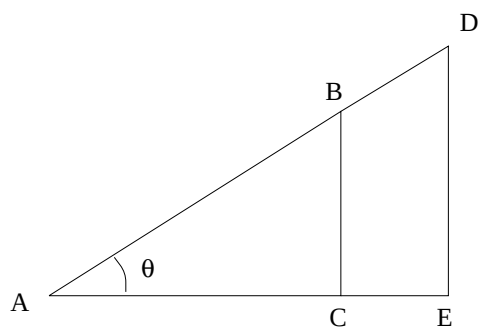
$$\tan(\theta) \stackrel{\text{def}}{=} \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta} = \frac{\sin(\theta)}{\cos(\theta)}.$$

For example, below we have pictured three right triangles; you can verify that the Pythagorean Theorem holds in each of the cases. In the left-hand triangle below, $\sin(\theta) = \frac{5}{13}$, $\cos(\theta) =$

$\frac{12}{13}$, $\tan(\theta) = \frac{5}{12}$. In the middle triangle below, $\sin(\theta) = \frac{1}{\sqrt{2}}$, $\cos(\theta) = \frac{1}{\sqrt{2}}$, $\tan(\theta) = 1$. In the right-hand triangle below, $\sin(\theta) = \frac{1}{2}$, $\cos(\theta) = \frac{\sqrt{3}}{2}$, $\tan(\theta) = \frac{1}{\sqrt{3}}$.



The symbols "sin, cos, tan" are abbreviations for the words *sine, cosine* and *tangent*, respectively. As we have defined them, the trigonometric ratios depend on the dimensions of the triangle. However, the same ratios are obtained for *any* right triangle with acute angle θ:



This follows from the properties of similar triangles: Notice $\triangle ABC$ and $\triangle ADE$ are similar. If we use $\triangle ABC$ to compute $\cos(\theta)$, then we find $\cos(\theta) = \frac{|\overline{AC}|}{|\overline{AB}|}$. On the other hand, if we use $\triangle ADE$, we obtain $\cos(\theta) = \frac{|\overline{AE}|}{|\overline{AD}|}$. Since the ratios of common sides of similar triangles must agree, we have $\cos(\theta) = \frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{AE}|}{|\overline{AD}|}$, which is what we wanted to be true. The same argument can be used to show that $\sin(\theta)$ and $\tan(\theta)$ can be computed using *any* right triangle with acute angle θ.

Except for some "rigged" right triangles, it is not easy to calculate the trigonometric ratios. Before the 1970's, approximate values of $\sin(\theta), \cos(\theta), \tan(\theta)$ were listed in long tables or calculated using a slide rule. Today, a scientific calculator saves the day on these computations. Most scientific calculators will give an approximation for the values of the trigonometric ratios. However, it is good to keep in mind we can compute the EXACT values of the trigonometric ratios when $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ radians or, equivalently, when $\theta = 0^o, 30^o, 45^o, 60^o, 90^o$.

—Special Trigonometric Ratios—					
Trig Ratio	θ				
	0	$30^o = \frac{\pi}{6}$ radians	$45^o = \frac{\pi}{4}$ radians	$60^o = \frac{\pi}{3}$ radians	$90^o = \frac{\pi}{2}$ radians
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Table 3.3.1: Exact Trigonometric Ratios

Some people make a big deal of "approximate" vs. "exact" answers; we won't worry about it here, unless we are specifically asked for an exact answer. However, here is something we will make a big deal about:

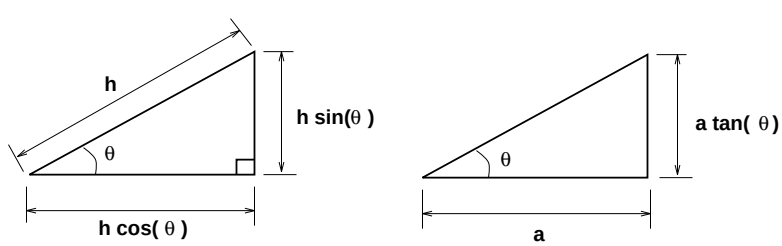
!!! CAUTION !!! When computing values of $\cos(\theta), \sin(\theta)$ and $\tan(\theta)$ on your calculator, make sure you are using the correct "angle mode" when entering θ; i.e. "degrees" or "radians". !!! CAUTION !!!

For example, if $\theta = 1^o$, then $\cos(1^o) = 0.9998, \sin(1^o) = 0.0175$ and $\tan(1^o) = 0.0175$. In contrast, if $\theta = 1$ radians, then $\cos(1) = 0.5403, \sin(1) = 0.8415$ and $\tan(1) = 1.5574$.

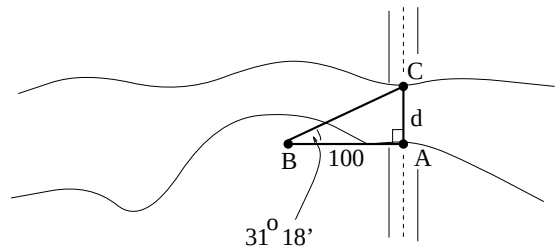
Applications.

When confronted with a situation involving a right triangle where the measure of one acute angle θ and one side are known, we can solve for the remaining sides using the appropriate trigonometric ratios. Here is the key picture to keep in mind:

The Meaning of Trig Ratios 3.3.1: *Given a right triangle, the trigonometric ratios relate the lengths of the sides as follows:*



Example 3.3.2: *To measure the distance across a river for a new bridge, surveyors placed poles at locations A, B and C. The length $|AB| = 100$ feet and the measure of the angle $\angle ABC$ is $31^o18'$. Find the distance to span the river. If the measurement of the angle $\angle ABC$ is only accurate within $\pm 2'$, find the possible error in $|\overline{AC}|$.*

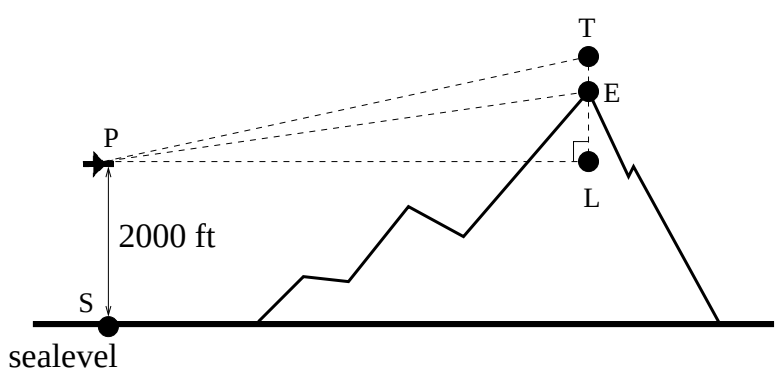


Solution. The trigonometric ratio relating these two sides would be the tangent and we can convert θ into decimal form, arriving at:

$$\tan(31^o 18') \quad = \quad \tan(31.3^o) = \frac{|\overline{AC}|}{|\overline{BA}|} = \frac{d}{100}; \quad \text{i.e. } d = 60.8 \text{ feet.}$$

This tells us that the bridge needs to span a gap of 60.8 feet. If the measurement of the angle was in error by $+2'$, then $\tan(31^o 20') = \tan(31.3333^o) = 0.6088$ and the span is 60.88 ft. On the other hand, if the measurement of the angle was in error by $-2'$, then $\tan(31^o 16') = \tan(31.2667^o) = 0.6072$ and the span is 60.72 ft. □

Example 3.3.3: *A plane is flying 2000 feet above sea level toward a mountain. The pilot observes the top of the mountain to be* 18° *above the horizontal, then immediately flies the plane at an angle of* 20° *above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain?*



Solution. We can compute the hypotenuse of $\triangle LPT$ by using the speed and time information about the plane: $|\overline{PT}| = 100\,\text{mph}(5\,\text{minutes})(1\,\text{hour}/60\,\text{minutes}) = \frac{25}{3}$ miles. The definitions

of the trigonometric ratios show:

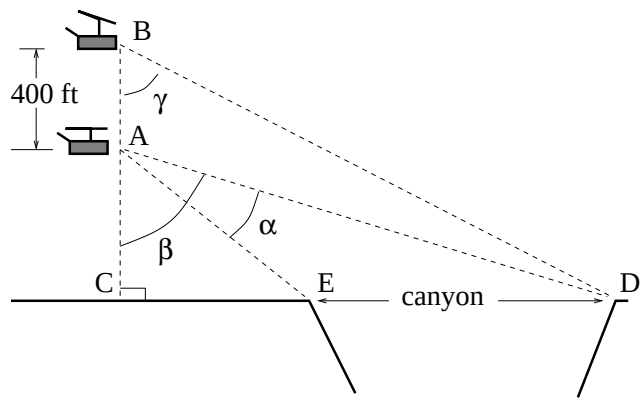
$$|\overline{TL}| \quad = \quad \frac{25}{3}\sin(20^o) = 2.850\,\text{miles};\ |\overline{PL}| = \frac{25}{3}\cos(20^o) = 7.831\,\text{miles}.$$

With this data, we can now find $|\overline{EL}|$:

$$|\overline{EL}| \quad = \quad |\overline{PL}|\tan(18^o) = 2.544\,\text{miles}.$$

The height of the plane above the peak is $|\overline{TE}| = |\overline{TL}| - |\overline{EL}| = 2.850 - 2.544 = 0.306$ miles= 1616 feet. The elevation of the peak above sea level is given by: Peak elevation = plane altitude + $|\overline{EL}| = |\overline{SP}| + |\overline{EL}|= = 2000 + (2.544)(5280) = 15{,}432$ feet. □

Example 3.3.4: *A Forest Service helicopter needs to determine the width of a deep canyon. While hovering, they measure the angle $\gamma = 48^o$ at position B (see picture), then descend 400 feet to position A and make two measurements of $\alpha = 13^o$ (the measure of $\angle EAD$), $\beta = 53^o$ (the measure of $\angle CAD$). Determine the width of the canyon to the nearest foot.*



Solution. We will need to exploit three right triangles in the picture: $\triangle BCD, \triangle ACD$ and $\triangle ACE$. Our goal is to compute $|\overline{ED}| = |\overline{CD}| - |\overline{CE}|$, which suggests more than one right triangle will come into play.

The first step is to use $\triangle BCD$ and $\triangle ACD$ to obtain a system of two equations and two unknowns involving some of the side lengths; we will then solve the system. From the definitions of the trigonometric ratios,

$$\begin{aligned} |\overline{CD}| &= (400 + |\overline{AC}|)\tan(48^o) \\ |\overline{CD}| &= |\overline{AC}|\tan(53^o). \end{aligned}$$

Plugging the second equation into the first and rearranging we get

$$|\overline{AC}| = \frac{400\tan(48^o)}{\tan(53^o) - \tan(48^o)} = 2053\,\text{feet}.$$

Plugging this back into the second equation of the system gives

$$|\overline{CD}| = (2053)\tan(53^o) = 2724\,\text{feet}.$$

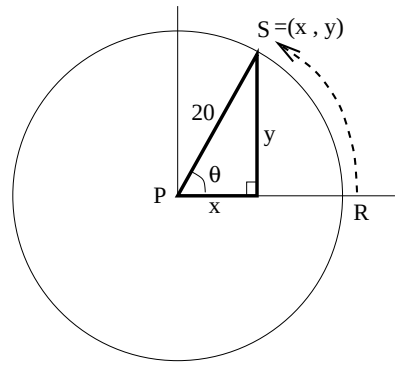
The next step is to relate $\triangle ACD$ and $\triangle ACE$, which can now be done in an effective way using the calculations above. Notice that the measure of $\angle CAE$ is $\beta - \alpha = 40^o$. We have

$$|\overline{CE}| = |\overline{AC}|\tan(40^o) = (2053)\tan(40^o) = 1723\,\text{feet}.$$

As noted above, $|\overline{ED}| = |\overline{CD}| - |\overline{CE}| = 2724 - 1723 = 1001$ feet is the width of the canyon. □

Circular Functions.

If *Cosmo* is located somewhere in the first quadrant of Figure 3.3.1, represented by the location S, we can use the trigonometric ratios to describe his coordinates. Impose the indicated xy-coordinate system with origin at P and extract the pictured right triangle with vertices at P and S.



The radius is 20 ft. and applying (3.3.1) gives

$$S = (x, y) = (20\cos(\theta), 20\sin(\theta)).$$

Unfortunately, we run into a snag if we allow *Cosmo* to wander into the second, third or fourth quadrant, since then the angle θ is no longer acute.

Are the trig ratios functions?

Recall that $\sin(\theta), \cos(\theta)$ and $\tan(\theta)$ are defined for acute angles θ inside a right triangle. We would like to say that these three equations actually define functions where the variable is an angle θ. Having said this, it is natural to ask if these three equations can be extended to be defined for ANY angle θ. For example, we need to explain how $\sin(\frac{2\pi}{3})$ is defined.

To start, we begin with the unit circle pictured in the xy-coordinate system. Let $\theta = \angle ROP$ be the angle in standard central position pictured below. If θ is positive (resp. negative), we adopt the convention that θ is swept out by counterclockwise (resp. clockwise) rotation of the initial side $\overline{OR}$.

The objective is to find the coordinates of the point P in this figure. Notice that each coordinate of P (the x-coordinate and the y-coordinate) will depend on the given angle θ. For this reason, we need to introduce two new functions involving the variable θ.

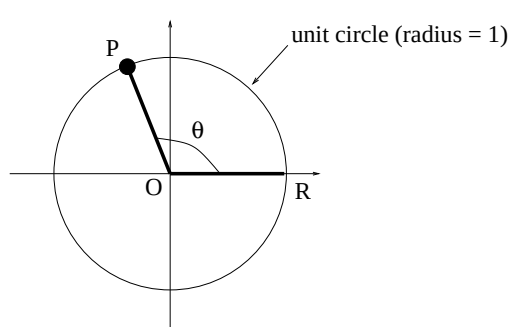


Figure 3.3.2: Coordinates of points on the unit circle

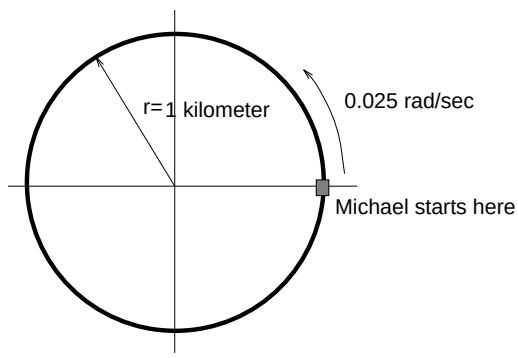
Official Definition 3.3.5: *Let θ be an angle in standard central position inside the unit circle, as in Figure 3.3.2. This angle determines a point P on the unit circle. Define two new functions,* $\cos(\theta)$ *and* $\sin(\theta)$*, on the domain of all θ values as follows:*

$$\cos(\theta) \stackrel{def}{=} \textit{horizontal } x\textit{-coordinate of } P \textit{ on unit circle}$$

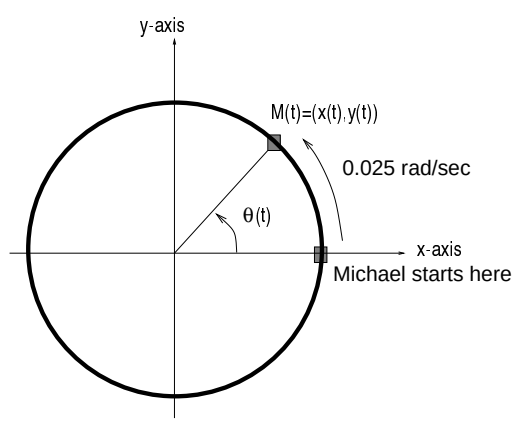
$$\sin(\theta) \stackrel{def}{=} \textit{vertical } y\textit{-coordinate of } P \textit{ on unit circle.}$$

We refer to $\sin(\theta)$ and $\cos(\theta)$ as the *basic circular functions.* Keep in mind that these functions have variables which are angles (either in degree or radian measure). These functions will be on your calculator. Again, BE CAREFUL to check the angle mode setting on your calculator ("degrees" or "radians") before doing a calculation.

Example 3.3.6: *Michael is test driving a vehicle counterclockwise around a desert test track which is circular of radius 1 kilometer. He starts at the location pictured, traveling 0.025 rad/sec. Impose coordinates as pictured. Where is Michael located (in xy-coordinates) after 18 seconds?*



Solution. Let $M(t)$ be the point on the circle of motion representing Michael's location after t seconds and $\theta(t)$ the angle swept out the by Michael after t seconds:



Since we are given the angular speed, we get

$$\theta(t) = 0.025t \quad \text{rad.}$$

Since the angle $\theta(t)$ is in central standard position, we get

$$M(t) = (\cos(\theta(t)), \sin(\theta(t))) = (\cos(0.025t), \sin(0.025t)).$$

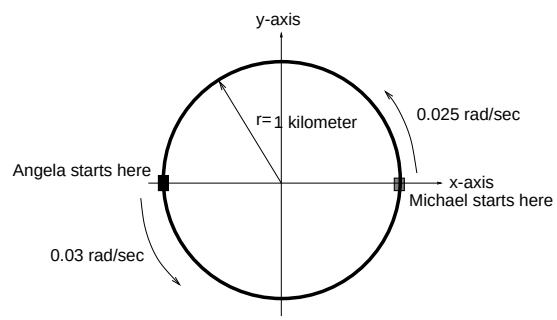
So, after 18 seconds Michael's location will be $M(18) = (0.9004, 0.4350)$. □

!!! CAUTION !!!

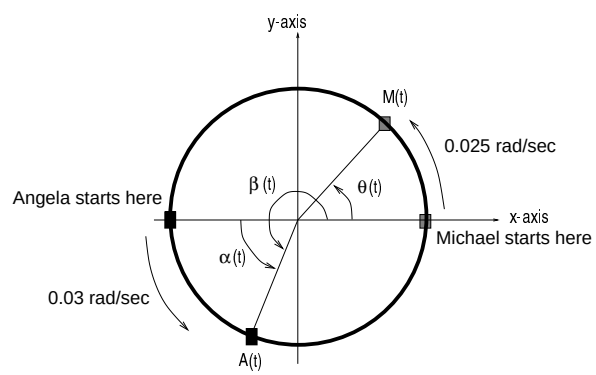
Interpreting the coordinates of the point $P = (\cos(\theta), \sin(\theta))$ in Figure 3.3.2 only works if the angle θ is viewed in central standard position. You must do some additional work if the angle is placed in a different position; see the next Problem.

!!! CAUTION !!!

Example 3.3.7: *Both Angela and Michael are test driving vehicles counter-clockwise around a desert test track which is circular of radius 1 kilometer. They start at the locations pictured. Michael is traveling 0.025 rad/sec and Angela is traveling 0.03 rad/sec. Impose coordinates as pictured. Where are the drivers located (in xy-coordinates) after 18 seconds?*



Solution. Let $M(t)$ be the point on the circle of motion representing Michael's location after t seconds. Likewise, let $A(t)$ be the point on the circle of motion representing Angela's location after t seconds. Let $\theta(t)$ be the angle swept out the by Michael and $\alpha(t)$ the angle swept out by Angela after t seconds.



Since we are given the angular speeds, we get

$$\theta(t) \;=\; 0.025t \;\; radians \quad and \quad \alpha(t) = 0.03t \;\; radians.$$

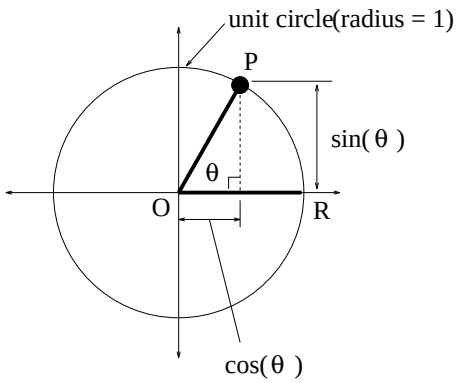
From the previous Example 3.3.6, $M(t) = (\cos(0.025t), \sin(0.025t))$ and $M(18) = (0.9004, 0.4350)$. Angela's angle $\alpha(t)$ is NOT in central standard position, so we must observe that $\alpha(t) + \pi = \beta(t)$, where $\beta(t)$ is in central standard position; see picture above. We conclude that

$$A(t) \;=\; (cos(\beta(t)), \sin(\beta(t))) = (cos(\pi + 0.03t), \sin(\pi + 0.03t)).$$

So, after 18 seconds Angela's location will be $A(18) = (-0.8577, -0.5141)$. □

Relating circular functions and right triangles.

If the point P on the unit circle is located in the first quadrant, then we can compute $\cos(\theta)$ and $\sin(\theta)$ using trigonometric ratios:



In general, it's useful to relate right triangles, the unit circle and the circular functions. To describe this connection, given θ we place it in central standard position in the unit circle, where $\angle ROP = \theta$. Draw a line through P perpendicular to the x-axis, obtaining an inscribed right triangle. Such a right triangle has hypotenuse of length 1, vertical side of length labeled b and horizontal side of length labeled a. There are four cases:

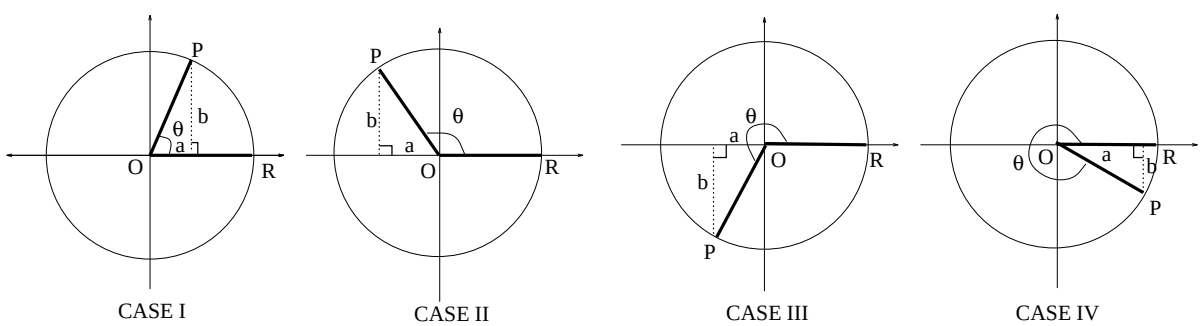


Figure 3.3.3: Possible positions of θ on the unit circle

Case I has already been discussed, arriving at $\cos(\theta) = a$ and $\sin(\theta) = b$. In Case II , we can interpret $\cos(\theta) = -a, \sin(\theta) = b$. We can reason similarly in the other Cases III and IV, using Figure 3.3.3, and we arrive at this conclusion:

Circular Functions and Triangles 3.3.8: *View θ as in Figure 3.3.3 and form the pictured inscribed right triangles. Then we can interpret $\cos(\theta)$ and $\sin(\theta)$ in terms of these right triangles as follows:*

$$\begin{aligned} \textit{Case I:} &\quad \cos(\theta) = a,\ \sin(\theta) = b \\ \textit{Case II:} &\quad \cos(\theta) = -a,\ \sin(\theta) = b \\ \textit{Case III:} &\quad \cos(\theta) = -a,\ \sin(\theta) = -b \\ \textit{Case IV:} &\quad \cos(\theta) = a,\ \sin(\theta) = -b. \end{aligned}$$

What About Other Circles?.

What happens if we begin with a circle C_r with radius r (possibly different than 1) and want to compute the coordinates of points on this circle?

The circular functions can be used to answer this more general question. Picture our circle C_r centered at the origin in the same picture with unit circle C_1 and the angle θ in standard central position for each circle. As pictured, we can view $\theta = \angle ROP = \angle SOT$. If $P = (x, y)$ is our point on the unit circle corresponding to the angle θ, then the calculation below shows how to compute coordinates on general circles:

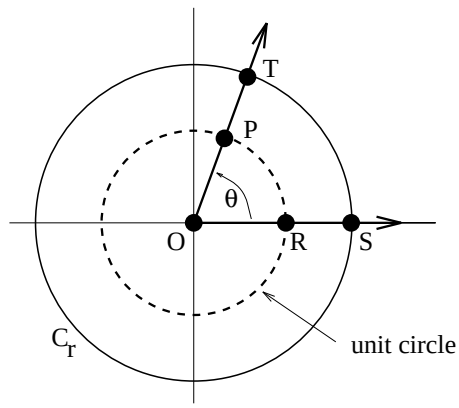
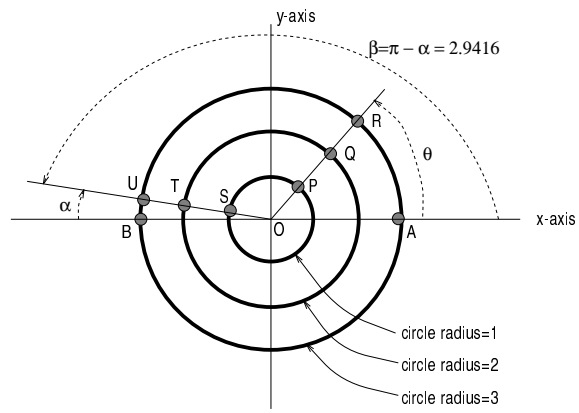


Figure 3.3.4: Points on other circles

$$\begin{aligned} P = (x, y) = (\cos(\theta), \sin(\theta)) \in C_1 \quad &\Leftrightarrow \quad x^2 + y^2 = 1 \\ &\Leftrightarrow \quad r^2x^2 + r^2y^2 = r^2 \\ &\Leftrightarrow \quad (rx)^2 + (ry)^2 = r^2 \\ &\Leftrightarrow \quad T = (rx, ry) = (r\cos(\theta), r\sin(\theta)) \in C_r. \end{aligned}$$

Point Coordinates on Circles 3.3.9: *Let C_r be a circle of radius r centered at the origin and $\theta = \angle SOT$ an angle in standard central position for this circle, as in Figure 3.3.4. Then the coordinates of $T = (r\cos(\theta), r\sin(\theta))$.*

Examples 3.3.10: *Consider the picture below, with $\theta = 0.8$ radians and $\alpha = 0.2$ radians. What are the coordinates of the labeled points?*

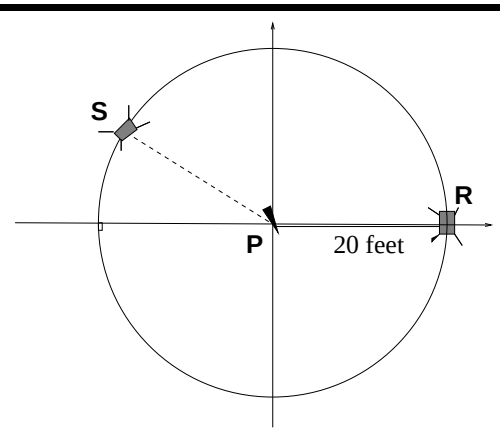


Solution. The angle θ is in standard central position; α is a central angle, but it is not in standard position. Notice, $\beta = \pi - \alpha = 2.9416$ is an angle in standard central position which locates the same points U, T, S as the angle α. Applying (3.3.10):

$$\begin{aligned}
P &= (\cos(0.8), \sin(0.8)) = (0.6967, 0.7174) \\
Q &= (2\cos(0.8), 2\sin(0.8)) = (1.3934, 1.4347) \\
R &= (3\cos(0.8), 3\sin(0.8)) = (2.0901, 2.1521) \\
S &= (\cos(2.9416), \sin(2.9416)) = (-0.9801, 0.1987) \\
T &= (2\cos(2.9416), 2\sin(2.9416)) = (-1.9602, 0.3973) \\
U &= (3\cos(2.9416), 3\sin(2.9416)) = (-2.9403, 0.5961).
\end{aligned}$$

□

Example 3.3.11: *Suppose Cosmo begins at the position R in the figure, walking around the circle of radius 20 feet with an angular speed of $\frac{4}{5}$ RPM counterclockwise. After 3 minutes have elapsed, describe Cosmo's precise location.*

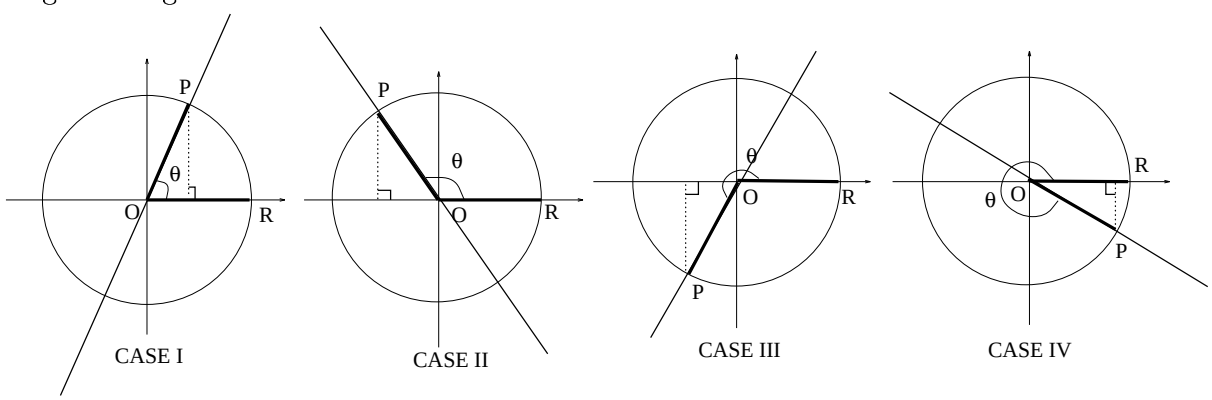


Solution. Cosmo has traveled $3\frac{4}{5} = \frac{12}{5}$ revolutions. If θ is the angle traveled after 3 minutes, $\theta = \frac{12}{5}\text{rev}(2\pi \text{ radians/rev}) = \frac{24\pi}{5}$ radians $= 15.08$ radians. By (3.3.9), we have $x = 20\cos(\frac{24\pi}{5}rad) = -16.18$ feet and $y = 20\sin(\frac{24\pi}{5}rad) = 11.76$ feet. Conclude that Cosmo is located at the point $S = (-16.18, 11.76)$. Using (3.1.3), $\theta = 864^o = 2(360^o) + 144^o$; this means that Cosmo walks counterclockwise around the circle two complete revolutions, plus 144°.

□

Other Basic Circular Function.

Given any angle θ, our constructions offer a concrete link between the cosine and sine functions and right triangles inscribed inside the unit circle:



The slope of the hypotenuse of these inscribed triangles is just the slope of the line through $\overline{OP}$. Since $P = (\cos(\theta), \sin(\theta))$ and $O = (0, 0)$:

$$slope \quad = \quad \frac{\Delta y}{\Delta x} = \frac{\sin(\theta)}{\cos(\theta)};$$

this would be valid as long as $\cos(\theta) \neq 0$. This calculation motivates a new circular function called the *tangent of* θ by the rule

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \text{provided} \quad \cos(\theta) \neq 0.$$

The only time $\cos(\theta) = 0$ is when the corresponding point P on the unit circle has x-coordinate 0. But, this only happens at the positions $(0, 1)$ and $(0, -1)$ on the unit circle, corresponding to angles of the form $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. These are the cases when the inscribed right triangle would "degenerate" to having zero width and the line segment $\overline{OP}$ becomes vertical. In summary, we then have this general fact to keep in mind:

line slope=$\tan(\theta)$

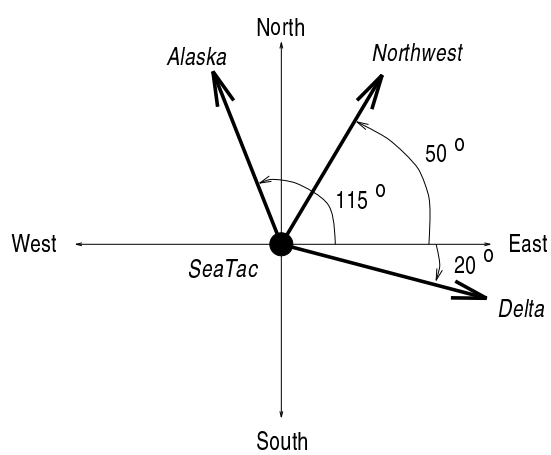
Three other commonly used circular functions come up from time to time. The *cotangent* function $y = \cot(\theta)$, the *secant* function $y = \sec(\theta)$ and the *cosecant* function $y = \csc(\theta)$ are

defined by the formulas:

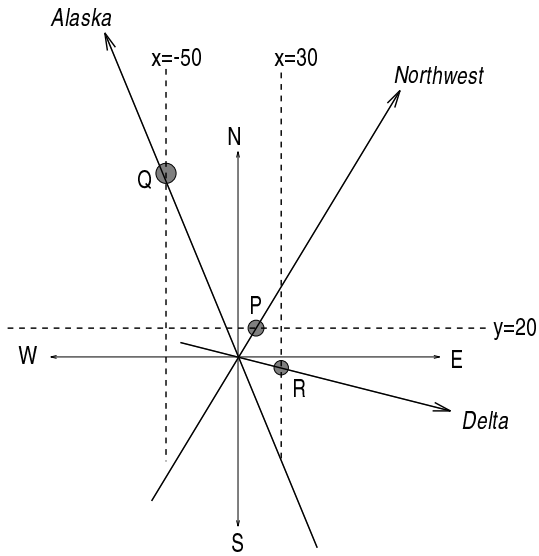
$$\sec(\theta) \stackrel{\text{def}}{=} \frac{1}{\cos(\theta)}, \qquad \csc(\theta) \stackrel{\text{def}}{=} \frac{1}{\sin(\theta)}, \qquad \cot(\theta) \stackrel{\text{def}}{=} \frac{1}{\tan(\theta)}.$$

Just as with the tangent function, one needs to worry about the values of θ for which these functions are undefined (due to division by zero. We will not need these functions in this text.

Example 3.3.12: *Three airplanes depart SeaTac Airport. A NorthWest flight is heading in a direction 50° counterclockwise from East, an Alaska flight is heading 115° counterclockwise from East and a Delta flight is heading 20° clockwise from East. Find the location of the Northwest flight when it is 20 miles North of SeaTac. Find the location of the Alaska flight when it is 50 miles West of SeaTac. Find the location of the Delta flight when it is 30 miles East of SeaTac.*



Solution. We impose a coordinate system as pictured, where "East" (resp. "North") points along the positive x-axis (resp. positive y-axis). To solve the problem, we will find the equation of the three lines representing the flight paths, then determine where they intersect the appropriate horizontal or vertical line. The Northwest and Alaska directions of flight are angles in standard central position; the Delta flight direction will be -20°. We can imagine right triangles with their hypotenuses along the directions of flight, then using the tangent function, we have these three immediate conclusions:



$$\begin{aligned} \text{slope NW line} &= \tan(50^o) = 1.19 \\ \text{slope Alaska line} &= \tan(115^o) = -2.14 \\ \text{slope Delta line} &= \tan(-20^o) = -0.364. \end{aligned}$$

All three flight paths pass through the origin $(0, 0)$ of our coordinate system, so the equations of the lines through the flight paths will be:

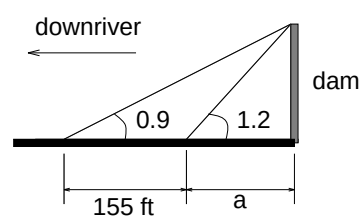
NW flight: $y = 1.19x$, Alaska flight: $y = -2.14x$, Delta flight: $y = -0.364x$.

The Northwest flight is 20 miles North of SeaTac when $y = 20$; plugging into the equation of the line of flight gives $20 = 1.19x$, so $x = 16.81$ and the plane location will be $P = (16.81, 20)$. Similarly, the Alaska flight is 50 miles West of SeaTac when $x = -50$; plugging into the equation of the line of flight gives $y = -2.14(-50) = 107$ and the plane location will be $Q = (-50, 107)$. Finally, check that the Delta flight is at $R = (30, -10.92)$ when it is 30 miles East of SeaTac. □

Problems

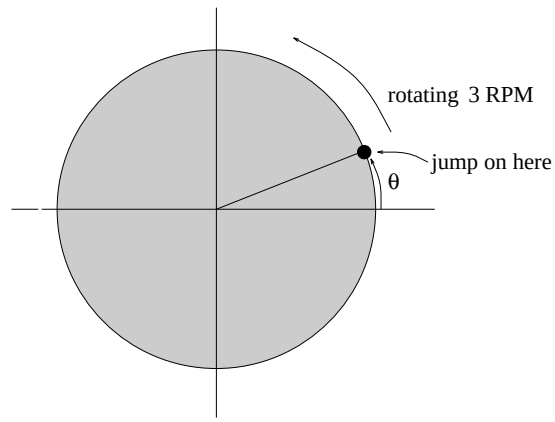
1. (a) Using the circular functions, compute the distance d in Example 3.1.6 in §3.1; compare this to the arc length s.
   (b) Go back to Chord Approximation in 3.1.7 and give a formula for the EXACT length of the chord in terms of the arc length s.
   (c) Return to Exercise 3.1.8 in §3.1 and compute the EXACT diameter of the moon.

2. The top of the *Boulder Dam* has an angle of elevation of 1.2 radians from a point on the Colorado River. Measuring the angle of elevation to the top of the dam from a point 155 feet farther down river is 0.9 radians; assume the two angle measurements are taken at the same elevation above sea level. How high is the dam?

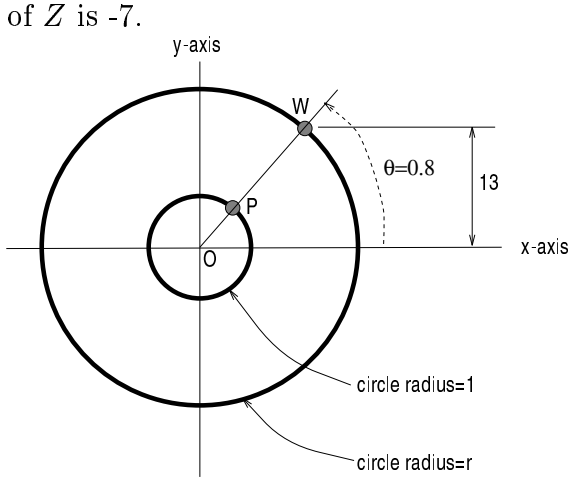


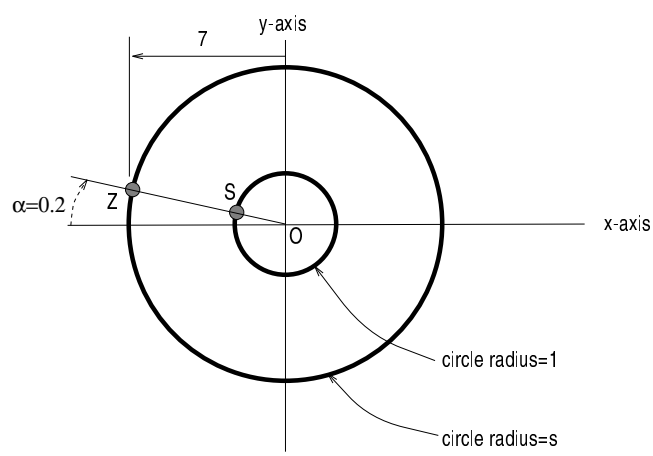
3. A merry-go-round is rotating at the constant angular speed of 3 RPM counterclockwise. The platform of this ride is a circular disc of radius 24 feet. You jump onto the ride at the location pictured below.

   (a) If $\theta = 34°$, then what are your xy-coordinates after 5 seconds?
   (b) If $\theta = -2.1$ rad, then what are your xy-coordinates after 2 hours and 7 seconds? Draw an accurate picture of the situation.

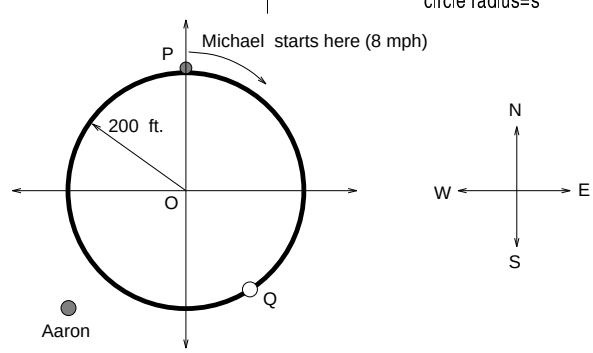


4. In the left-hand picture, find a circle of radius r so that the vertical coordinate of the point W is 13. In the right-hand picture, find a circle of radius s so that the horizontal coordinate of Z is -7.





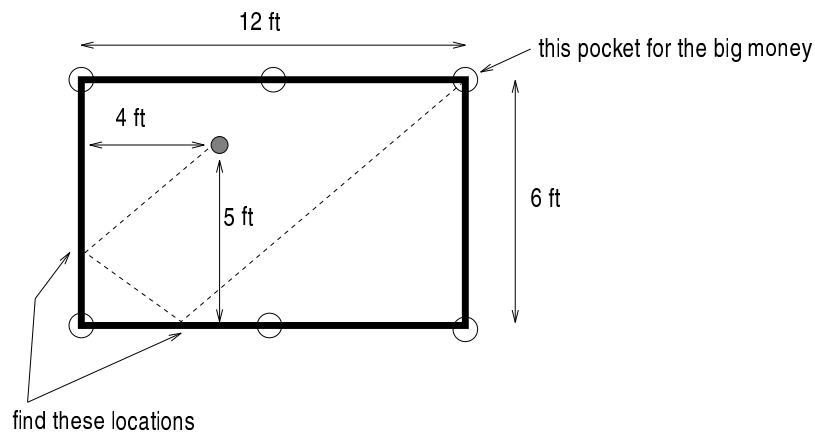
5.* Michael is running 8 mph clockwise around a circular track of radius 200 feet. Michael begins at the Northernmost point on the track. Aaron is located 200 feet West and 200 feet South of the center of the circular track.



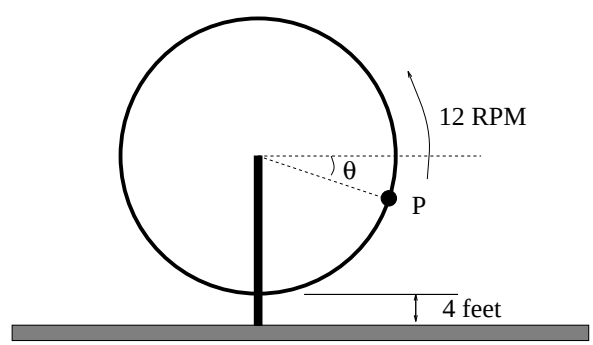
   (a) If Q is Michael's location after 45 seconds, what are the coordinates of Q?
   (b) What is the distance from Aaron's location to Q?

(c) Aaron starts running toward Q (constant speed in a straight line) at the instant Michael starts running toward Q. Aaron plans to tackle Michael the instant he arrives at Q. How fast should Aaron run?

6. You are defending your title as the *US Billiard Champion*. Your final shot requires playing off the left and bottom cushions into the top right corner pocket, as indicated by the dotted path. For the big money, where should you aim to hit the left cushion and where will the ball strike the bottom cushion? (We will return to this problem in Exercise 3.6.12 and determine the angles involved.)



7. John has been hired to design an exciting carnival ride. Tiff, the carnival owner, has decided to create the worlds greatest ferris wheel. Tiff isn't into math; she simply has a vision and has told John these constraints on her dream: (i) the wheel should rotate counterclockwise with an angular speed of 12 RPM; (ii) the linear speed of a rider should be 200 mph; (iii) the lowest point on the ride should be 4 feet above the level ground. Recall, we worked on this in the previous exercise section.

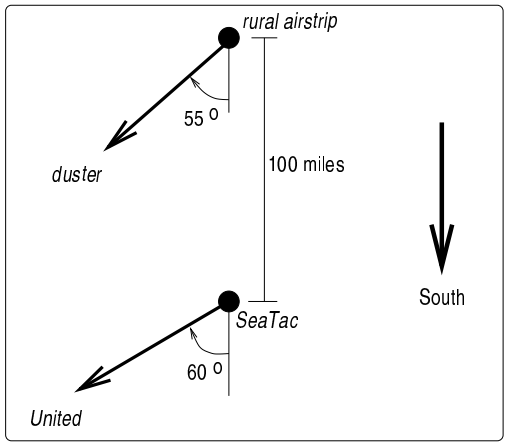


(a) Impose a coordinate system and find the coordinates $T(t) = (x(t), y(t))$ of Tiff at time t seconds after she starts the ride?

(b) Find Tiff's coordinates the instant she becomes a human missile.

(c) Find the equation of the tangential line along which Tiff travels the instant she becomes a human missile. Sketch a picture indicating this line and her initial direction of motion along it when the seat detaches.

8. A United flight departs SeaTac airport heading 60° clockwise from South. A crop duster leaves a rural airstrip located 100 miles due North of SeaTac and is heading 55° clockwise from South. Where will the two lines of flight cross?



3.4 Basic Properties and Trigonometric Functions

Our definitions of the circular functions are based upon the unit circle. This makes it easy to visualize many of their properties.

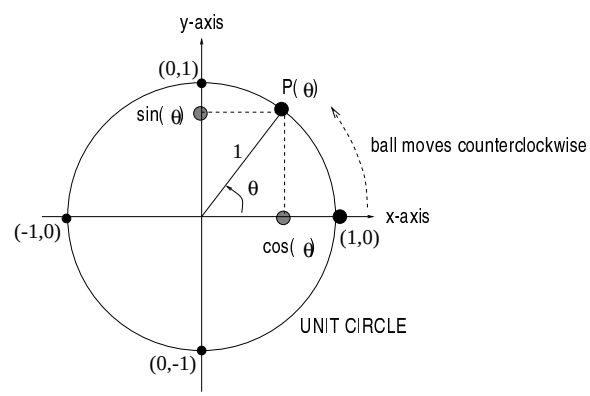
Easy Properties of Circular Functions.

How can we determine the range of function values for $\cos(\theta)$ *and* $\sin(\theta)$*?*

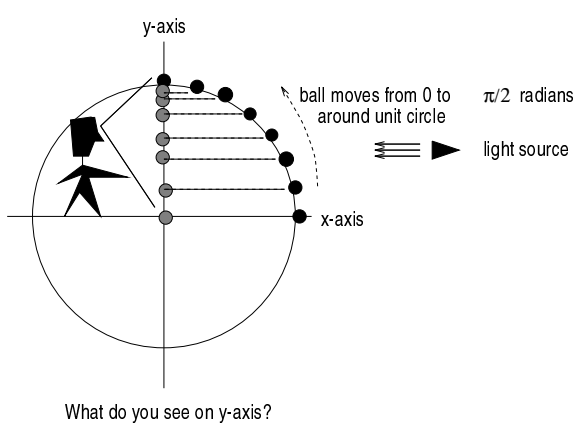
To begin with, recall the abstract definition for the range of a function $f(\theta)$:

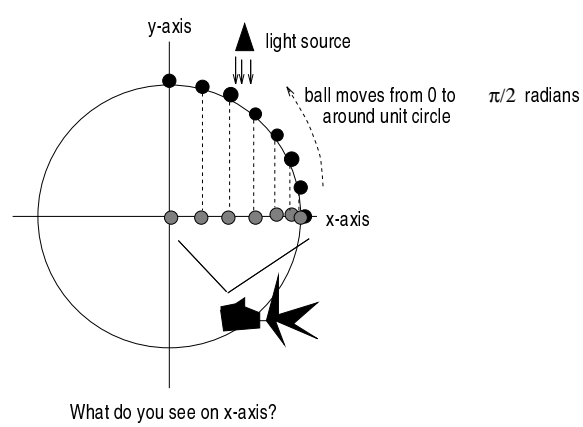
$$range\, of\, f \quad = \quad \{f(\theta) : \theta \text{ is in the domain}\}.$$

Using the unit circle constructions of the basic circular functions, it is easy to visualize the range of $\cos(\theta)$ and $\sin(\theta)$. Beginning at the position $(1, 0)$, imagine a ball moving counterclockwise around the unit circle. If we "freeze" the motion at any point in time, we will have swept out an angle θ and the corresponding position $P(\theta)$ on the circle will have coordinates $P(\theta) = (\cos(\theta), \sin(\theta))$:



By studying the coordinates of the ball as it moves in the first quadrant, we will be studying $\cos(\theta)$ and $\sin(\theta)$, for $0 \leq \theta \leq \pi/2$ radians. We can visualize this very concretely. Imagine a light source as in the left-hand picture below; then a shadow projects onto the vertical y-axis. The shadow locations you would see on the y-axis are precisely the values $\sin(\theta)$, for $0 \leq \theta \leq \pi/2$ radians. Similarly, imagine a light source as in the right-hand picture below; then a shadow projects onto the horizontal x-axis. The shadow locations you would see on the x-axis are precisely the values $\cos(\theta)$, for $0 \leq \theta \leq \pi/2$ radians.





There are two visual conclusions: First, the function values of $\sin(\theta)$ vary from 0 to 1 as θ varies from 0 to $\pi/2$. Secondly, the function values of $\cos(\theta)$ vary from 1 to 0 as θ varies from 0 to $\pi/2$. Of course, we can go ahead and continue analyzing the motion as the ball moves into the second, third and fourth quadrant, ending up back at the starting position (1,0):

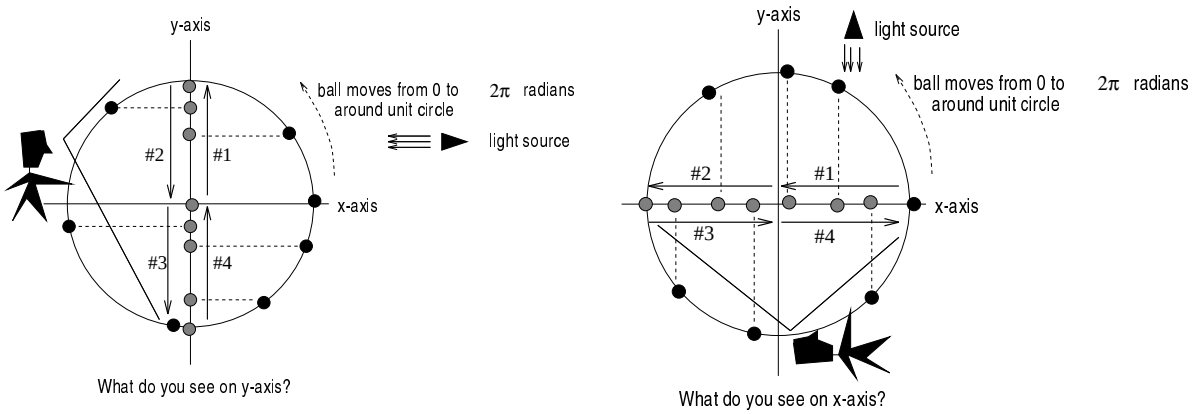


Figure 3.4.1: Analyzing the values of sine and cosine

The conclusion is that after one complete counterclockwise rotation, the values of $\sin(\theta)$ and $\cos(\theta)$ range over the interval $[-1, 1]$. As the ball moves through the four quadrants, we have indicated the "order" in which these function values are assumed by labeling arrows #1-#4: For example, for the sine function, look at the left-hand part of Figure 3.4.1. The values of the sine function vary from 0 up to 1 while the ball moves through the first quadrant (arrow labeled #1), then from 1 down to 0 (arrow labeled #2), then from 0 down to -1 (arrow labeled #3), then from -1 up to 0 (arrow labeled #4).

What about the tangent function? We have seen that the tangent function computes the slope of the hypotenuse of an inscribed triangle. This means we can determine the range of values of $\tan(\theta)$ by investigating the possible slopes for these inscribed triangles. We will maintain the above model of a ball moving around the unit circle.

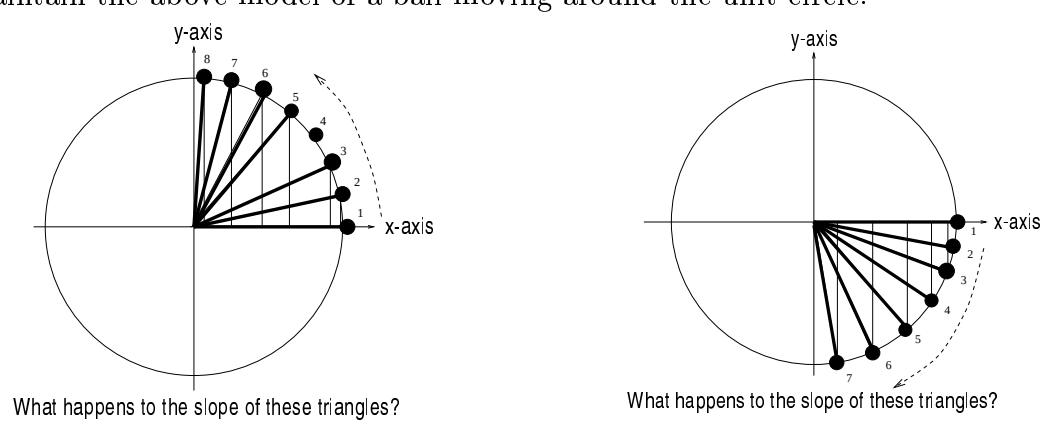


Figure 3.4.2: Analyzing the values of tangent

We look at two cases, each starting at (1,0). In the first quadrant, the ball moves counter-clockwise and in the fourth quadrant it moves clockwise: In the first quadrant, we notice that these hypotenuse slopes are always non-negative, beginning with slope 0 (the degenerate right triangle when $\theta = 0$) then increasing. In fact, as the angle θ approaches $\pi/2$ radians, the ball is getting closer to the position (0,1) and the hypotenuse is approaching a vertical line. This tells us that as θ varies from 0 to $\pi/2$ (but not equal to $\pi/2$), these slopes attain all possible non-negative values. In other words, the range of values for $\tan(\theta)$ on the domain $0 \leq \theta < \pi/2$ will be $0 \leq z < \infty$. Similar reasoning shows that as the ball moves in the fourth quadrant, the slopes of the hypotenuses of the triangles are always non-positive, varying from 0 to ANY negative value. In other words, the range of values for $\tan(\theta)$ on the domain $-\pi/2 < \theta \leq 0$ will be $-\infty < z \leq 0$.

On your calculator, you can varify the visual conclusions we just established by studying the values of $\tan(\theta)$ for θ close (but not equal) to $\frac{\pi}{2}$ radians $= 90^o$:

$$\begin{aligned} \tan(89^o) &= 57.29, \tan(89.9^o) = 572.96, \tan(89.99^o) = 5729.58, ... \\ \tan(-89^o) &= -57.29, \tan(-89.9^o) = -572.96, \tan(-89.99^o) = -5729.58, ... \end{aligned}$$

The fact that the values of the tangent function become arbitrarily large as we get close to $\pm\pi/2$ radians means the function values are *unbounded.*

Circular Function Values 3.4.1: *For any angle θ, we always have $-1 \leq cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$. On domain $0 \leq \theta \leq 2\pi$, the range of both $\cos(\theta)$ and $\sin(\theta)$ is $-1 \leq z \leq 1$. In contrast, on the domain of all θ values for which tangent is defined, the range of $\tan(\theta)$ is all real numbers.*

Note: For the sine and cosine functions, if the domain is not $0 \leq \theta \leq 2\pi$, then we need to consider the "periodic qualities" of the circular functions to determine the range. This is discussed below.

Identities.

There are dozens of formulas that relate the values of two or more circular functions; these are usually lumped under the heading of *Trigonometric Identities.* In this course, we only need a couple frequently used identities. Later on, you might need the more sophisticated ones we have lumped into §2.9.

If we take the point $P = (\cos(\theta), \sin(\theta))$ on the unit circle, corresponding to the standard central position angle θ, then recall the equation of the unit circle tells us $x^2 + y^2 = 1$. But, since the x coordinate is $\cos(\theta)$ and the y coordinate is $\sin(\theta)$, we have $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. It is common notational practice to write $(\cos(\theta))^2 = \cos^2(\theta)$ and $(\sin(\theta))^2 = \sin^2(\theta)$. This leads to the **most important** of all trigonometric identities:

Trigonometric Identity 3.4.2: *For any angle θ, we have the identity* $\cos^2(\theta)+\sin^2(\theta) = 1$.

Adding any multiple of 2π radians (or 360°) to an angle will not change the values of the circular functions. If we focus on radians for a moment, this says that knowing the values of $\cos(\theta)$ and $\sin(\theta)$ on the domain $0 \leq \theta \leq 2\pi$ determines the values for any other possible angle.

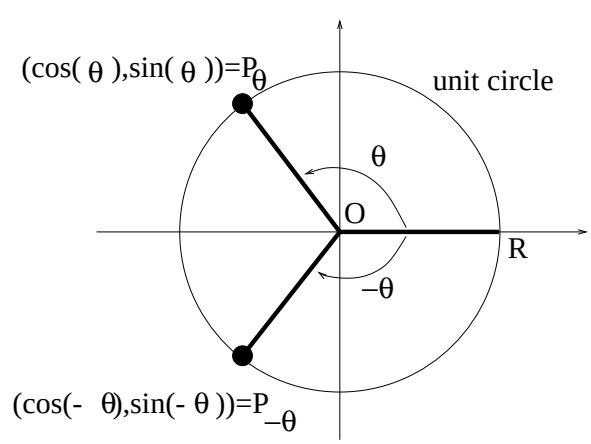
There is something very general going on here, so let's pause a moment to make a definition and then an observation.

Definition of Periodic: *A function $f(\theta)$ is called c-periodic if two things are true:*
(i) $f(\theta + c) = f(\theta)$ *holds for all* θ;
(ii) *There is no smaller* d, $0 < d < c$ *such that* $f(\theta + d) = f(\theta)$ *holds for all* θ.
We usually call c the period of the function.

Using this new terminology, we conclude that the sine and cosine circular functions are 2π-*periodic*. In the case of the tangent circular function, it is also true that $\tan(\theta) = \tan(\theta + 2\pi n)$, for every integer n. However, referring back to the unit circle definitions of the circular functions, we have $\tan(\theta) = \tan(\theta + n\pi)$, for all integers n. If you take $n = 1$, then this tells us that and the tangent circular function is π-periodic. We summarize this information below.

Periodicity Identity 3.4.3: *For any angle θ and any integer $n = 0, \pm 1, \pm 2, \pm 3, \ldots,$ we have* $\cos(\theta) = \cos(\theta + 2\pi n)$, $\sin(\theta) = \sin(\theta + 2\pi n)$ *and* $\tan(\theta) = \tan(\theta + n\pi)$.

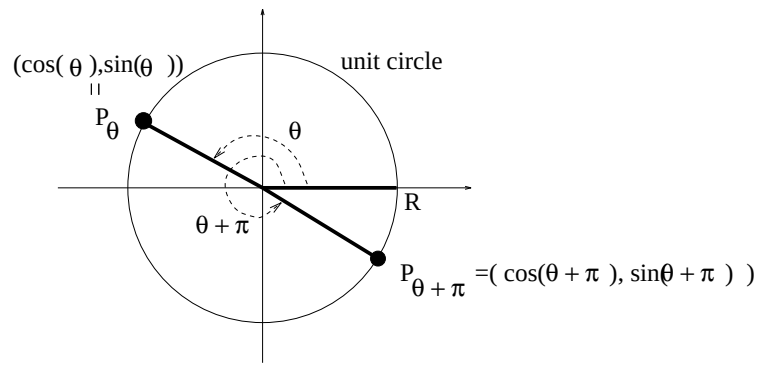
Draw an angle θ and its negative in the same unit circle picture in standard central position. We have indicated the points P_θ and $P_{-\theta}$ used to define the circular functions. It is clear from the picture that P_θ and $P_{-\theta}$ have the same x-coordinate, but the y-coordinates are negatives of one another. This gives the next identity:



Even/Odd Identity 3.4.4: *For any angle θ,* $\sin(-\theta) = -\sin(\theta)$ *and* $\cos(-\theta) = \cos(\theta)$.

If you go back to Exercise 1.7.6, we introduced the terminology of *even and odd functions.* In this language, this result says that the cosine function is an even function and the sine function is an odd function.

Next, draw the angles θ and $\theta + \pi$ in the same unit circle picture in standard central position. We have indicated the corresponding points P_θ and $P_{\theta+\pi}$ on the unit circle and their coordinates in terms of the circular functions:



From the picture, the x-coordinate of P_θ must be the "negative" of the x-coordinate of $P_{\theta+\pi}$ and similarly, the y-coordinate of P_θ must be the "negative" of the y-coordinate of $P_{\theta+\pi}$. This gives us the next identity:

Identity 3.4.5: *For any angle θ, we have* $\sin(\theta + \pi) = -\sin(\theta)$ *and* $\cos(\theta + \pi) = -\cos(\theta)$.

For example, we have $\sin(\frac{5\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$. This calculation leads to a computational observation: Combining Table 3.3.1 with the previous two identities we can compute the EXACT value of $\cos(\theta), \sin(\theta)$ and $\tan(\theta)$ at an angle θ which is a multiple of $30^o = \frac{\pi}{6}$ radians or $45^o = \frac{\pi}{4}$ radians. Here are some sample calculations together with a reference as to "why" each equality is valid:

Examples 3.4.6: (i) $\cos(-45^o) \underset{(3.4.4)}{=} \cos(45^o) \underset{Table}{=} \frac{\sqrt{2}}{2}$.

(ii) $\sin(225^o) \underset{(3.4.3)}{=} \sin(45^o + 180^o) \underset{(3.4.5)}{=} -\sin(45^o) \underset{Table}{=} -\frac{\sqrt{2}}{2}$.

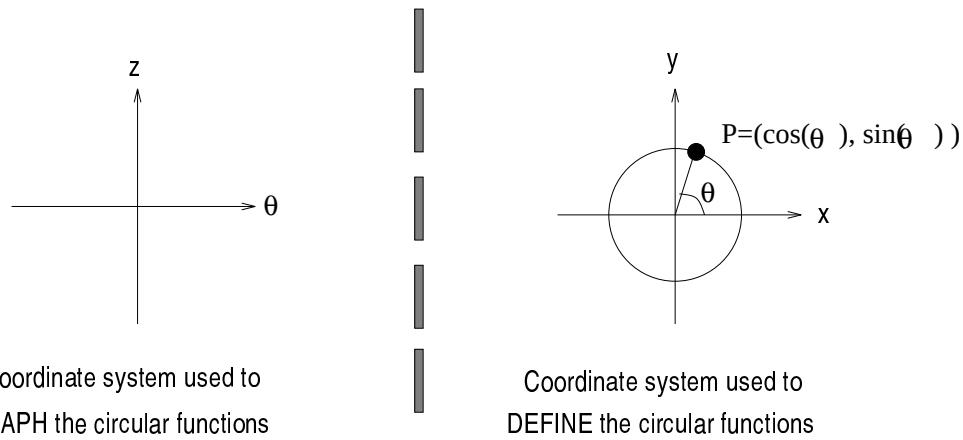
(iii) $\cos(\frac{2\pi}{3}) \underset{(3.4.3)}{=} \cos(-\frac{\pi}{3} + \pi) \underset{(3.4.5)}{=} -\cos(-\frac{\pi}{3}) \underset{(3.4.4)}{=} -\cos(\frac{\pi}{3}) \underset{Table}{=} -\frac{1}{2}$.

Graphs of Circular Functions.

We have introduced three new functions of the variable θ and it is important to understand and interpret the pictures of their graphs. To do this, we need to settle on a coordinate system in which to work. The horizontal axis will correspond to the independent variable, so this should be the θ-axis. We will label the vertical axis, which corresponds to the dependent variable, the z-axis. With these conventions, beginning with any of the circular functions $z = \sin(\theta), z = \cos(\theta)$ or $z = \tan(\theta)$, the graph will be a subset of the θz-coordinate system. Precisely, given a circular function $z = f(\theta)$, the graph consists of all pairs $(\theta, f(\theta))$, where

θ varies over a domain of allowed values. We will record and discuss these graphs below; a graphing device will painlessly produce these for us!

There is a point of possible confusion that needs attention. We purposely did not use the letter "y" for the dependent variable of the circular functions. This is to avoid possible confusion with our construction of the sine and cosine functions using the unit circle. Since we viewed the unit circle inside the xy-coordinate system, the x-coordinates (resp. y-coordinates) of points on the unit circle are computed by $\cos(\theta)$ (resp. $\sin(\theta)$).



A matter of scaling.

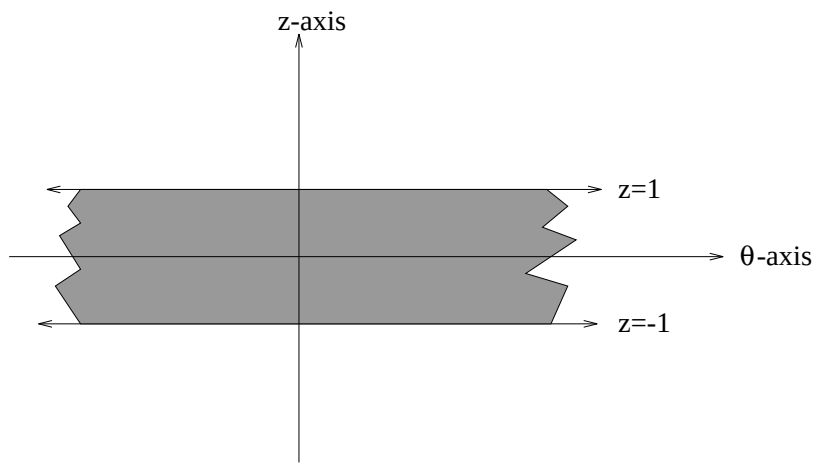
The first issue concerns scaling of the axes used in graphing the circular functions. As we know, the definition of radian measure is directly tied to the lengths of arcs subtended by angles in the unit circle:

... an angle of measure 1 radian inside the unit circle will subtend an arc of length 1 ...

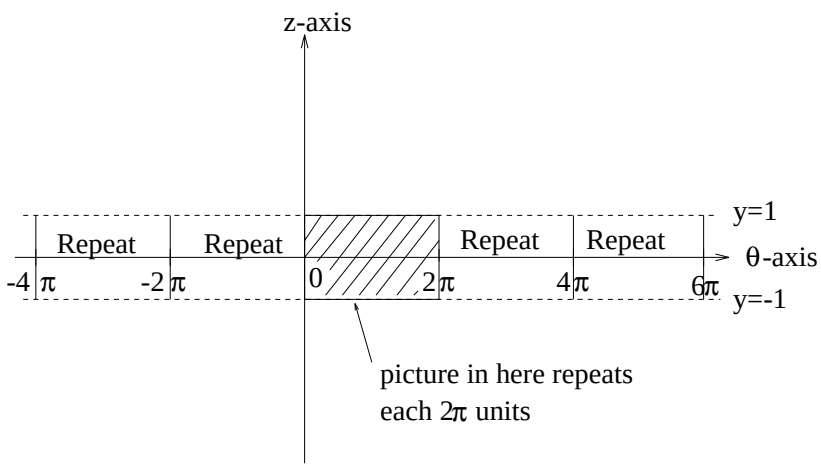
Since length is a good intuitive scaling quantity, it is natural to scale the θ-axis so that the length of 1 radian on the θ-axis (horizontal axis) is the same length as 1 unit on the vertical axis. For this reason, we will work primarily with radian measure when sketching the graphs of circular functions. If we need to work explicitly with degree measure for angles, then we can always convert radians to degrees using the fact: $360^o = 2\pi$ radians.

The sine and cosine graphs.

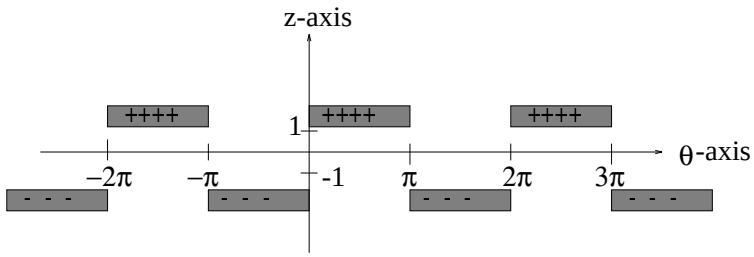
Using (3.4.1), we know that $-1 \leq \sin(\theta) \leq 1$ and $-1 \leq \cos(\theta) \leq 1$. Pictorially, this tells us that the graphs of $z = \sin(\theta)$ and $z = \cos(\theta)$ lie between the horizontal lines $z = 1$ and $z = -1$; i.e. the graphs lie inside the darkened band pictured below:



By (3.4.3), we know that the values of the sine and cosine repeat themselves every 2π radians. Consequently, if we know the graphs of the sine and cosine on the domain $0 \leq \theta \leq 2\pi$, then the picture will repeat for the interval $2\pi \leq \theta \leq 4\pi$, $-2\pi \leq \theta \leq 0$, etc.



Sketching the graph of $z = \sin(\theta)$ for $0 \leq \theta \leq 2\pi$ can be roughly achieved by plotting points. For example, $(\frac{\pi}{6}, \frac{1}{2})$, $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ and $(\frac{\pi}{2}, 1)$ lie on the graph, as do $(\frac{3\pi}{2}, -1)$, $(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2})$, $(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2})$, $(\frac{11\pi}{6}, -\frac{1}{2})$ and $(2\pi, 0)$, etc. If we return to our analysis of the range of values for the sine function in Figure 3.4.1, it is easy to see where $\sin(\theta)$ is positive or negative; combined with §1.4.5, this tells us where the graph is above and below the horizontal axis:



We now include a software plot of the graph of sine function, observing the three qualitative features just isolated: bounding, periodicity and sign properties:

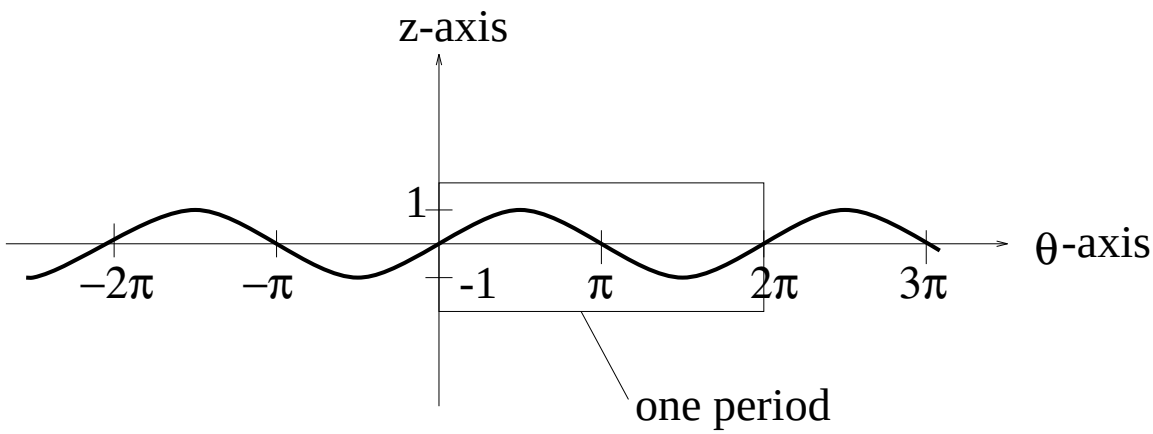
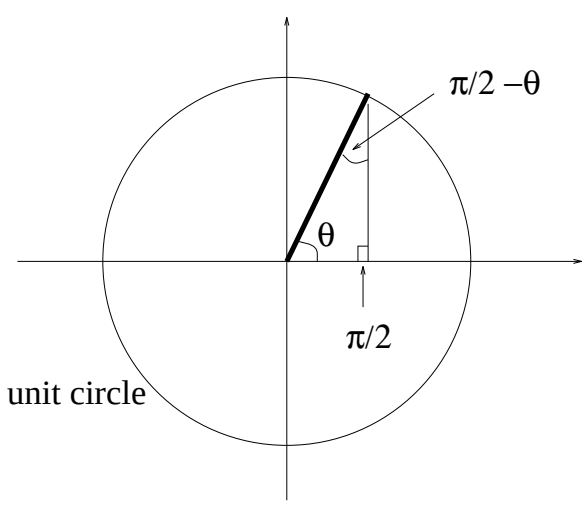


Figure 3.4.4: Graph of $z = \sin(\theta)$

We could repeat this analysis to arrive at the graph of the cosine. Instead, we will utilize an identity. Given an angle θ, place it in central standard position in the unit circle, as one of the four cases of Figure 3.3.3. For example, we have pictured Case I in this figure. Since the sum of the angles in a triangle is $180^o = \pi$ radians, we know that θ, $\pi/2$ and $\frac{\pi}{2} - \theta$ are the three angles of the inscribed right triangle. From the picture, it then follows that



$$\begin{aligned}\cos(\theta) &= \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} \\ &= \frac{\text{side opposite to } \pi/2-\theta}{\text{hypotenuse}} \\ &= \sin(\frac{\pi}{2}-\theta).\end{aligned}$$

Using the same reasoning this identity is valid for all θ. This gives us another useful identity:

Conversion Identity 3.4.7: *For any angle* θ, $\cos(\theta) = \sin(\frac{\pi}{2}-\theta)$ *and* $\sin(\theta) = \cos(\frac{\pi}{2}-\theta)$.

This identity can be used to sketch the graph of the cosine function. First, we do a calculation using our new identity:

$$\cos(\theta) \underset{(3.4.4)}{=} \cos(-\theta) \underset{(3.4.7)}{=} \sin(\frac{\pi}{2}-(-\theta)) = \sin(\theta-(-\frac{\pi}{2})).$$

By the horizontal shifting principle (1.7.3), the graph of $z = \cos(\theta)$ is obtained by horizontally shifting the graph of $z = \sin(\theta)$ by $\frac{\pi}{2}$ units to the LEFT. Here is a plot of the graph of the cosine function:

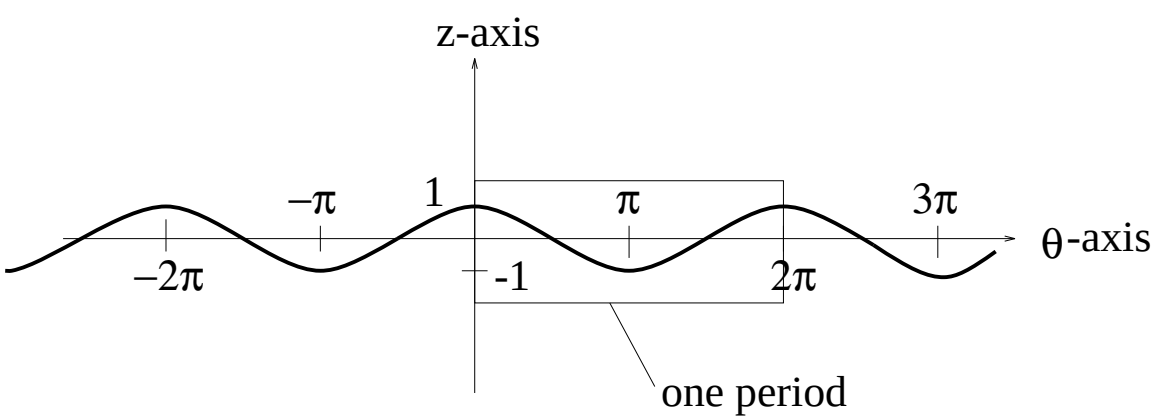


Figure 3.4.5: Graph of $z = \cos(\theta)$

The tangent graph. As we have already seen, unlike the sine and cosine circular functions, the tangent function is NOT defined for all values of θ. Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, here are some properties we can immediately deduce:

- The function $z = \tan(\theta)$ is undefined if and only if $\theta = \frac{\pi}{2}+k\pi$, where $k = 0, \pm1, \pm2, \pm3, \ldots$.
- The function $z = \tan(\theta) = 0$ if and only if $\theta = k\pi$, where $k = 0, \pm1, \pm2, \pm3, \ldots$.
- By (3.4.3), the tangent function is π-periodic, so the picture of the graph will repeat itself every π-units and it is enough to understand the graph when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- On the domain $0 \le \theta < \frac{\pi}{2}$, $\tan(\theta) \ge 0$; on the domain $-\frac{\pi}{2} < \theta \le 0$, $\tan(\theta) \le 0$.

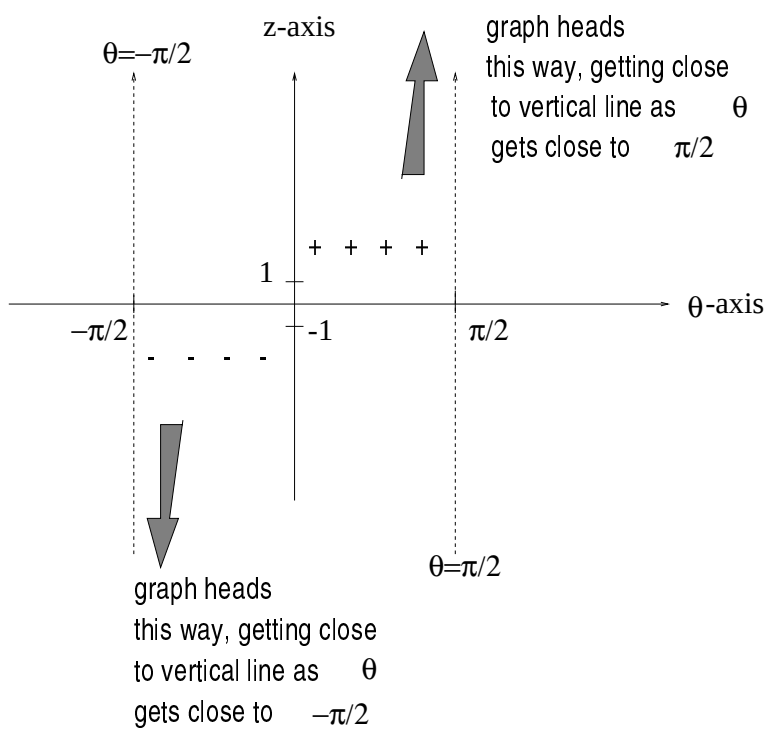
In the θz-coordinate system, the vertical lines $\theta = \frac{\pi}{2} + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ will be *vertical asymptotes* for the graph of the tangent function. Using our slope interpretation in Figure 3.4.2, what becomes clear is this: As the values of θ get close to $\frac{\pi}{2}$, the graph is getting close to the vertical line $\theta = \frac{\pi}{2}$ AND becoming farther and farther away from the horizontal axis: To understand this numerically, first suppose θ is slightly smaller than $\frac{\pi}{2}$, say $\theta = \frac{\pi}{2} - 0.1, \frac{\pi}{2} - 0.01, \frac{\pi}{2} - 0.001$. Then the calculation of $\tan(\theta)$ involves dividing a number very close to 1 by a very small positive number:

$$\tan(\frac{\pi}{2} - 0.1) = 9.9666, \quad \tan(\frac{\pi}{2} - 0.01) = 99.9967, \quad \tan(\frac{\pi}{2} - 0.001) = 1000.$$

Conclude that as θ "approaches $\frac{\pi}{2}$ from below", the values of $\tan(\theta)$ are becoming larger and larger. This says that the function values become "unbounded". Likewise, imagine the case when θ is slightly bigger than $-\frac{\pi}{2}$, say $\theta = -\frac{\pi}{2}+0.1, -\frac{\pi}{2}+0.01, -\frac{\pi}{2}+0.001$. Then the calculation of $\tan(\theta)$ involves dividing a number very close to -1 by a very small positive number:

$$\tan(-\frac{\pi}{2} + 0.1) \quad = \quad -9.9666, \quad \tan(-\frac{\pi}{2} + 0.01) = -99.9967, \quad \tan(-\frac{\pi}{2} + 0.001) = -1000.$$

Conclude that as θ "approaches $-\frac{\pi}{2}$ from above", the values of $\tan(\theta)$ are becoming negative numbers of increasingly larger magnitude:



Again, this tells us the function values are becoming "unbounded". The graph of $z = \tan(\theta)$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ can be roughly achieved by combining the calculations as in (3.4.6) and the qualitative features highlighted. Here is a software plot:

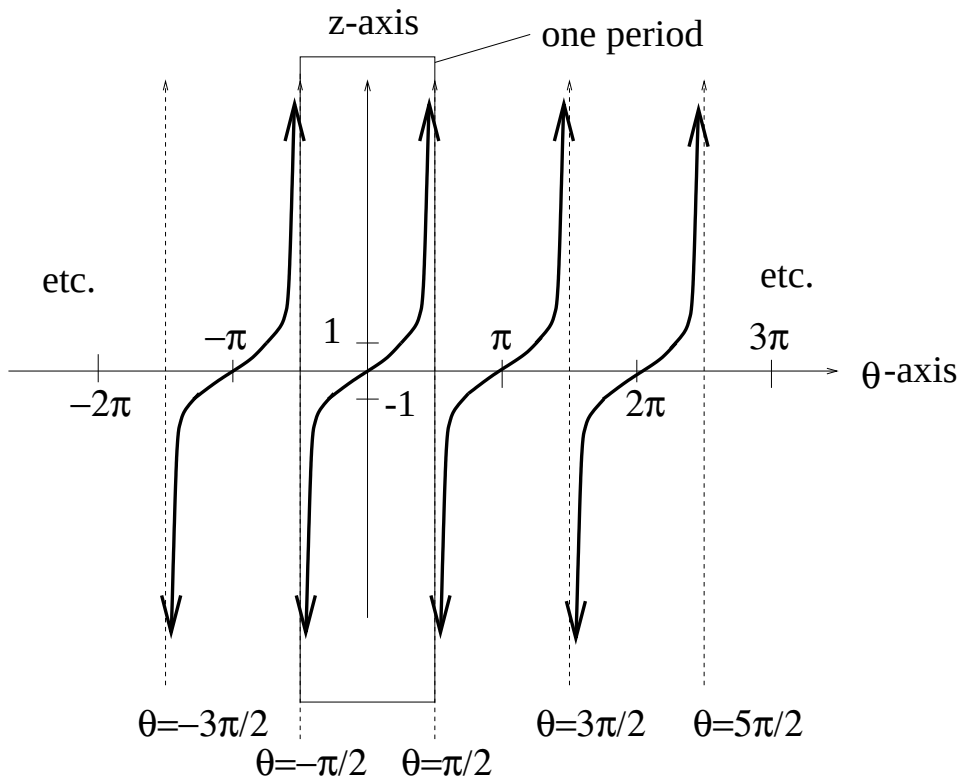
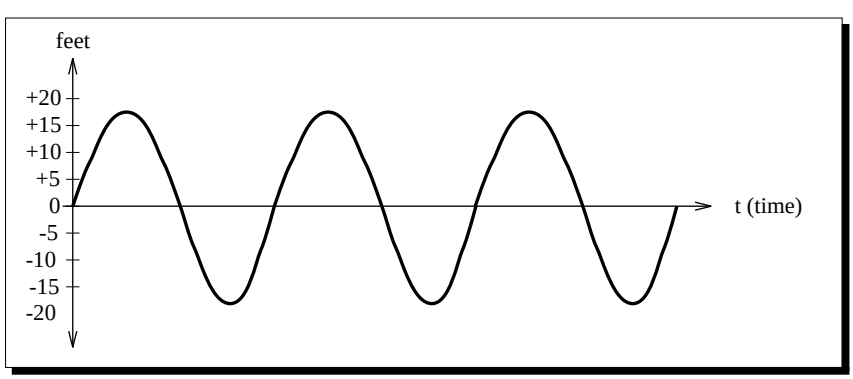


Figure 3.4.6: Graph of $z = tan(\theta)$

Trigonometric Functions.

To become successful mathematical modelers, we must have wide variety of functions in our toolkit. As an illustration, the graph below might represent the height of the tide above some reference level over the course of several days. The curve drawn is clearly illustrating that the height of the tide is "periodic" as a function of time t; in other words, the behavior of the tide repeats itself as time goes by. However, if we try to model this periodic behavior, the only weapon at our disposal would be the circular functions and these require an angle variable, not a time variable such as t; we are stuck!

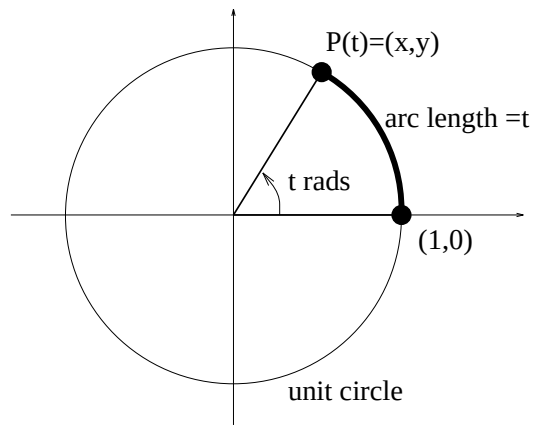


Modeling the tide graph requires the *trigonometric functions*, which lie at the heart of studying all kinds of periodic behavior. We have no desire to "reinvent the wheel", so let's use our previous work on the circular functions to define the trigonometric functions.

A Transition.

Given a real number t, is there a sensible way to define $\cos(t)$ *and* $\sin(t)$*?*

The answer is yes and depends on the ideas surrounding radian measure of angles. Given the positive real number t, we can certainly imagine an angle of measure t radians inside the unit circle (in standard position) and we know the arc subtended by this angle has length t (this is why we use the unit circle).



We already know that $\cos(t \text{ rad})$ and $\sin(t \text{ rad})$ compute the x and y coordinates of the point $P(t)$. In effect, we are just using the measure of the angle t to help us locate the point $P(t)$. An alternate way to locate $P(t)$ is to move along the circumference counterclockwise, beginning at (1,0), until we have an arc of length t; that again puts us at the point $P(t)$. In the case of an angle of measure $-t$ radians, the point $P(-t)$ can be located by moving along the circumference clockwise, beginning at (1,0), until we have an arc of length t.

Trigonometric Functions 3.4.8: *Let t be a real number. We DEFINE the sine function $y = \sin(t)$, the cosine function $y = \cos(t)$ and the tangent function $y = \tan(t)$ by the rules*
$\sin(t) = y$*-coordinate of* $P(t) = \sin(t \text{ rad})$
$\cos(t) = x$*-coordinate of* $P(t) = \cos(t \text{ rad})$.
$\tan(t) = \frac{\sin(t)}{\cos(t)} = \tan(t \text{ rad})$.

We refer to these as the basic *trigonometric functions*. If we are working with radian measure and t is a real number, then there is no difference between evaluating a trigonometric function at the real number t and evaluating the corresponding circular function at the angle of measure t radians.

!!! CAUTION !!!

As a rule, whenever you see an expression involving $\sin(\ldots)$,$\cos(\ldots)$ or $\tan(\ldots)$, we assume "..." is in units of RADIANS, unless otherwise noted. When computing values on your calculator, MAKE SURE YOU ARE USING RADIAN MODE!

!!! CAUTION !!!

Example 3.4.9: *Assume that the number of hours of daylight in Seattle during 1994 is given by the function* $d(t) = 3.7 sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12$, *where* t *represents the day of the year and* $t = 0$ *corresponds to January 1. How many hours of daylight will there be on May 11?*

Solution. To solve the problem, you need to consult a calendar, finding every month has 31 days, except: February has 28 days and April, June, September and November have 30 days. May 11 is the 31+28+31+30+11=131st day of the year. So, there will be $d(131) = 3.7\sin(2(50.5)\pi/366) + 12 = 14.82$ hours of daylight on May 11. □

Graphs of trigonometric functions.

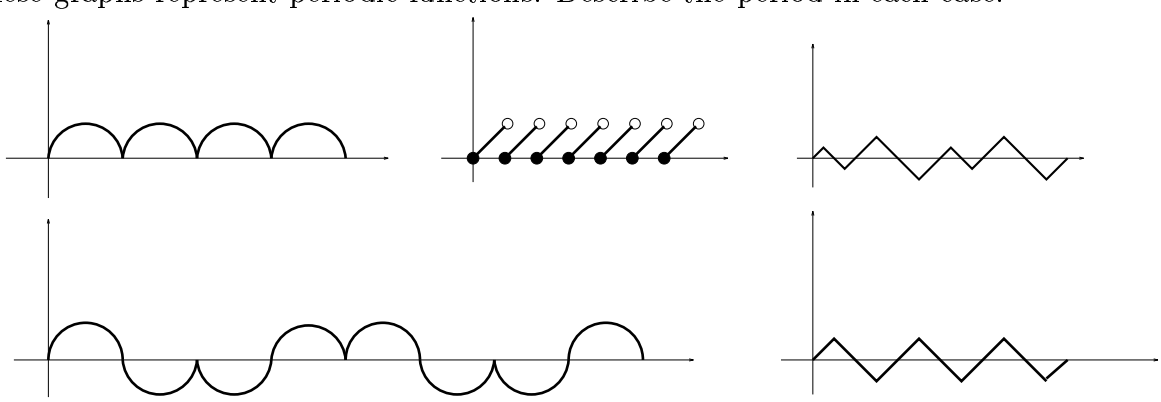
The graphs of the trigonometric functions $y = \sin(t), y = \cos(t)$ and $y = \tan(t)$ will look just like Figures 3.4.4, 3.4.5 and 3.4.6, except that the horizontal axis becomes the t-axis and the vertical axis becomes the y-axis.

Notation for trigonometric functions.

In many texts, you will find the sine function written as $y = \sin t$; i.e. the parenthesis around the "t" are omitted. A similar comment applies to all of the trigonometric functions. We will **never** do this and the reasoning is simply this: Maintaining the parenthesis, as in $y = \sin(t)$, emphasizes the fact that we are dealing with a function and the "input values" are located between the parenthesis. For example, if we write the function $y = \sin(t^2 + 2t + 1)$, it is crystal clear that the sine function is applied to the expression "$t^2 + 2t + 1$"; using the alternate notation yields the expression $y = \sin t^2 + 2t + 1$, which is interpreted to mean $y = (\sin t^2) + (2t + 1)$.

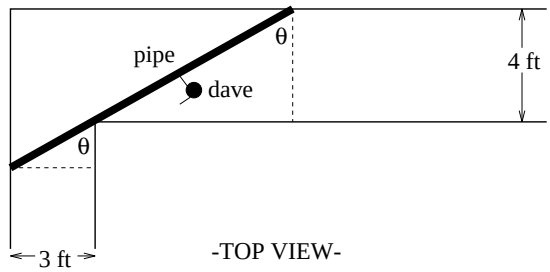
Problems

1. (a) If $\cos(\theta) = \frac{24}{25}$, what are the two possible values of $sin(\theta)$?
   (b) If $\sin(\theta) = -0.8$ and θ is in the third quadrant of the xy plane, what is $\cos(\theta)$?
   (c) If $\sin(\theta) = \frac{3}{7}$, what is $\sin(\frac{\pi}{2} - \theta)$?
2. Start with the equation $\sin(\theta) = \cos(\theta)$.
   (a) Use the unit circle interpretation of the circular functions to find the solutions of this equation; make sure to describe your reasoning.
   (b) Use the tangent function to find the solutions of this equation; make sure to describe your reasoning.
3. (a) If $f(x) = \sin(x)$ and $g(x) = 3x$, what is $f(g(x))$? What is $f(g(5))$? What is $g(f(x))$? What is $g(f(5))$?
   (b) If $f(x) = \sin(x)$ and $g(x) = x^2 + 3x - 5$, what is $f(g(x))$? What is $f(g(1))$? What is $g(f(x))$? What is $g(f(1))$?
   (c) If $f(x) = g(x) = \sin(x)$, what is $f(g(x))$? What is $f(g(1))$? Is $f(g(x)) = [\sin(x)]^2$?
4. These graphs represent periodic functions. Describe the period in each case.



5. For each situation described, which could *possibly* be described using a periodic function?
   (a) The population of rabbits in a large wooded area as a function of time.
   (b) As you pedal a bike, the height of your big toe above the ground as a function of time.
   (c) As you pedal a bike, the distance you have traveled as a function of time.
   (d) The score for the UW womens Bball team as a function of time elapsed during a game.

6.* Dave is replumbing his house and needs to carry a copper pipe around the corner of a hallway. As he cheerfully walks down the hall and rounds the corner, the pipe becomes stuck, as pictured. Assume Dave must always hold the pipe level; i.e. he can't tilt it up or down.



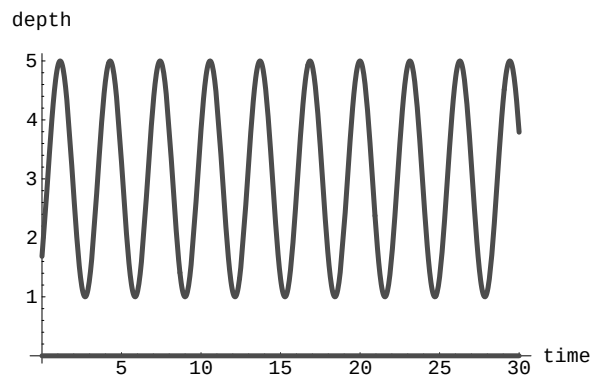
(a) Find a formula for the function $\ell(\theta)$ which computes the length of the longest pipe that will fit with the pictured angle θ.

(b) Describe how you could use the function in a. to find the LONGEST pipe Dave can carry around the corner.

(c) Use a graphing device to approximate the longest pipe Dave can carry around the corner.

3.5 Sinusoidal Functions

A migrating salmon is heading up a portion of the Columbia River. It's depth $d(t)$ (in feet) below the water surface is measured and plotted over a 30 minute period, as a function of time t (minutes). What is the formula for $d(t)$?

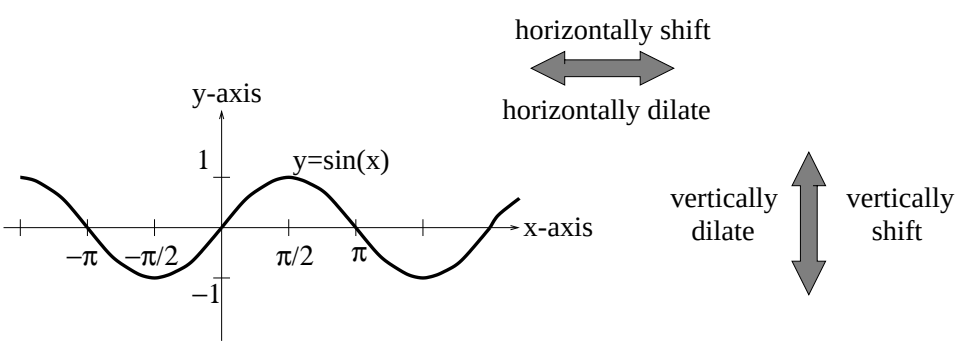


In order to answer the question, we need to introduce an important new family of functions called the *sinusoidal functions*. These functions will play a central role in modeling any kind of periodic phenomena. The amazing fact is that almost any function you will encounter can be approximated by a sum of sinusoidal functions; a result that has far-reaching implications in all of our lives.

A special class of functions.

Beginning with the trigonometric function $y = \sin(x)$, what is the most general function we can build using the graphical techniques of shifting and stretching?

The graph of $y = \sin(x)$ can be manipulated in four basic ways: horizontally shift, vertically shift, horizontally dilate or vertically dilate. Each of these "geometric operations" corresponds to a simple change in the "symbolic formula" for the function, as discussed in §2.5.



If we vertically shift the graph by D units upward, the resulting curve would be the graph of the function $y = \sin(x) + D$; see (2.5.3). Recall, the effect of the sign of D: If D is negative, the effect of shifting D units upward is the same as shifting $|D|$ units downward. Notice, the function $y = \sin(x) + D$ is still a periodic function, having the same period 2π as $y = \sin(x)$.

Notice, whereas the graph of the function $y = \sin(x)$ oscillates between the horizontal lines $y = \pm 1$, the graph of $y = \sin(x) + D$ oscillates between $y = D \pm 1$. For this reason, we sometimes refer to the constant D as the *mean* of the function $y = \sin(x) + D$. In Figure 3.5.1, notice that the graph of $y = \sin(x) + D$ is symmetrically split by the horizontal "mean" line $y = D$.

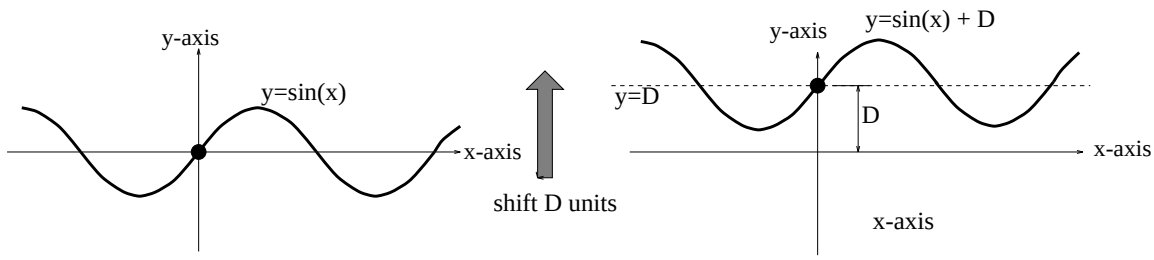


Figure 3.5.1: Interpreting the mean

Next, consider the effect of horizontally shifting the graph of $y = \sin(x)$ by C units to the right. By (2.5.3), the new curve is the graph of the function $y = \sin(x - C)$. Also, recall the effect of the sign of C: If C is negative, the effect of shifting C units right is the same as shifting $|C|$ units left. If the domain of $\sin(x)$ is $0 \leq x \leq 2\pi$, then the domain of $\sin(x - C)$ is $0 \leq x - C \leq 2\pi$, again by (2.5.3). Rewriting this, the domain of $\sin(x - C)$ is $C \leq x \leq 2\pi + C$ and the graph will go through precisely one period on this domain. In other words, the new function $\sin(x - C)$ is still 2π-periodic. The constant C is usually called the *phase shift* of $y = \sin(x - C)$. Looking at Figure 3.5.2, it is possible to interpret C graphically: C will be a point where the graph crosses the horizontal axis *on its way up* from a minimum to a maximum.

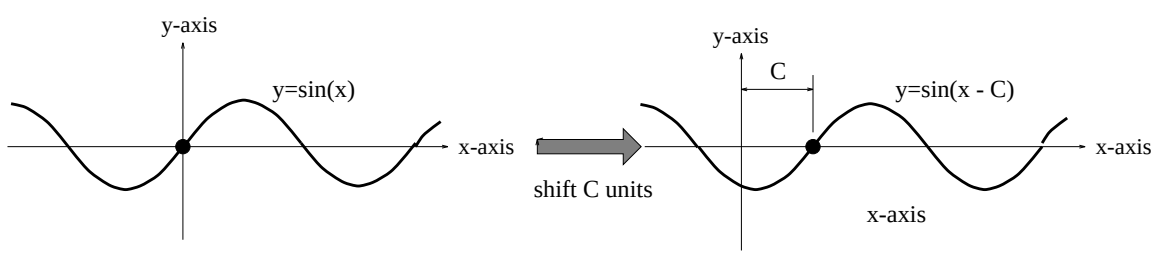


Figure 3.5.2: Interpreting the phase shift

Vertically dilating the graph, either by vertical expansion or compression, leads to a new curve. The graph of this vertically dilated curve is $y = A\sin(x)$, for some positive constant A. Furthermore, if $A > 1$, the graph of $y = A\sin(x)$ is a vertically expanded version of $y = \sin(x)$, whereas, if $0 < A < 1$, then the graph of $y = A\sin(x)$ is a vertically compressed version of $y = \sin(x)$. Notice, the function $y = A\sin(x)$ is still 2π-periodic. What has changed is the band of oscillation: whereas the graph of the function $y = \sin(x)$ stays between the horizontal lines $y = \pm 1$, the graph of $y = A\sin(x)$ oscillates between the horizontal lines $y = \pm A$. We usually refer to A as the *amplitude* of the function $y = A\sin(x)$.

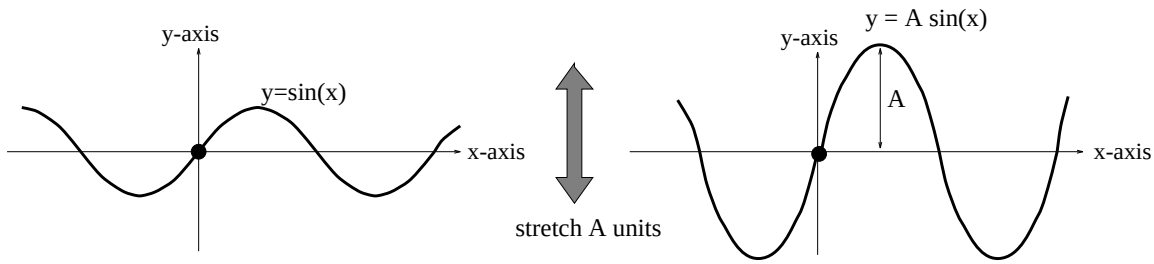


Figure 3.5.3: Interpreting the amplitude

Finally, horizontally dilating the graph, either by horizontal expansion or compression, leads to a new curve. The equation of this horizontally dilated curve is $y = \sin(cx)$, for some constant $c > 0$. We know that $y = \sin(x)$ is a 2π-periodic function and observe that horizontally dilation still results in a periodic function, but the period will typically NOT be 2π. For future purposes, it is useful to rewrite the equation for the horizontally stretched curve in a way more directly highlighting the period. To begin with, once the horizontal stretching factor c is known, we could rewrite

$$c = \frac{2\pi}{B}, \text{ for some } B \neq 0.$$

Here is the point of this yoga with the horizontal dilating constant: If we let the values of x range over the interval $[0,B]$, then $\frac{2\pi}{B}x$ will range over the interval $[0, 2\pi]$. In other words, the function $y = \sin(\frac{2\pi}{B}x)$ is B-periodic and we can read off the *period* of $y = \sin(\frac{2\pi}{B}x)$ by viewing the constant in this mysterious way.

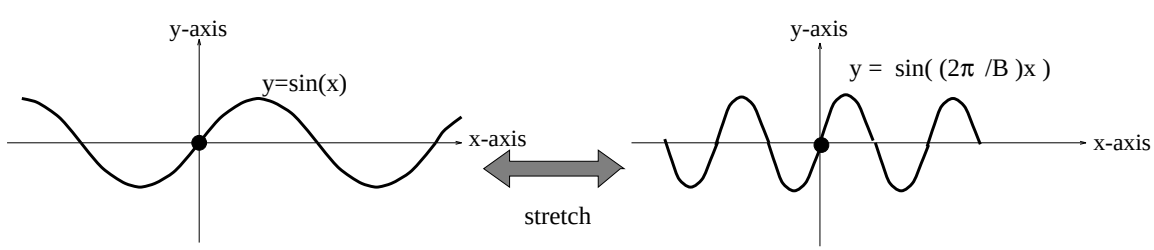


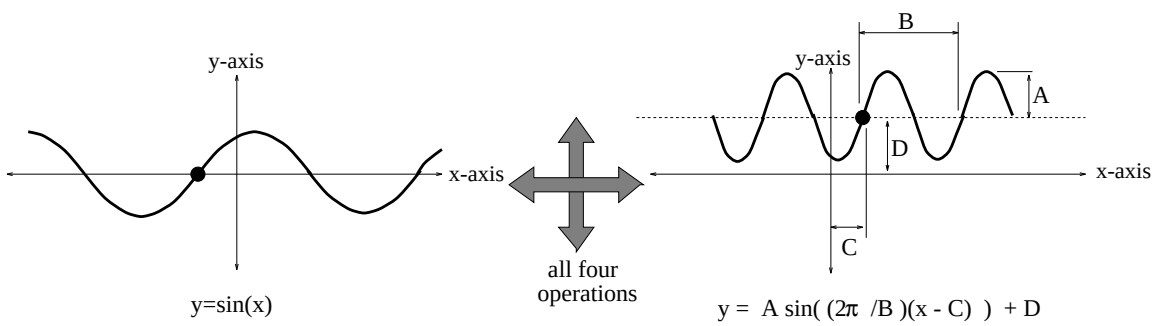
Figure 3.5.4: Interpreting the period

The four constructions outlined lead to a new family of functions.

Sinusoidal Functions 3.5.1: *Let A, B, C and D be fixed constants, where A and B are both positive. Then we can form the new function*

$$y = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

which is called a sinusoidal function. The four constants can be interpreted graphically as indicated:



How to roughly sketch a sinusoidal graph.

Given a sinusoidal function in the standard form

$$y = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

once the constants A, B, C, D are specified, any graphing device can produce an accurate graph. However, it is pretty straightforward to sketch a rough graph by hand and the process will help reinforce the graphical meaning of the constants A, B, C, D. Here is a "five step procedure" one can follow, assuming we are given A, B, C, D. It is a good idea to follow Example (3.5.2) as you read this procedure; that way it will seem a lot less abstract.

1. Draw the horizontal line given by the equation $y = D$; this line will "split" the graph of $y = A\sin(\frac{2\pi}{B}(x - C)) + D$ into symmetrical upper and lower halves.
2. Draw the two horizontal lines given by the equations $y = D \pm A$. These two lines determine a horizontal strip inside which the graph of the sinusoidal function will oscillate. Notice, the points where the sinusoidal function has a maximum value lie on the line $y = D + A$. Likewise, the points where the sinusoidal function has a minimum value lie on the line $y = D - A$. Of course, we do not yet have a prescription that tells us where these maxima (peaks) and minima (valleys) are located; that will come out of the next steps.
3. Since we are given the period B, we know these important facts: (1) The period B is the horizontal distance between two successive maxima (peaks) in the graph. Likewise, the period B is the horizontal distance between two successive minima (valleys) in the graph. (2) The horizontal distance between a maxima (peak) and the successive minima (valley) is $\frac{1}{2}B$.

4. Plot the point (C, D). This will be a place where the graph of the sinusoidal function will cross the mean line $y = D$ *on its way up* from a minima to a maxima. This is not the only place where the graph crosses the mean line; it will also cross at the points obtained from (C, D) by horizontally shifting by any integer multiple of $\frac{1}{2}B$. For example, here are three places the graph crosses the mean line: $(C, D), (C + \frac{1}{2}B, D), (C + B, D)$
5. Finally, midway between (C, D) and $(C + \frac{1}{2}B, D)$ there will be a maxima (peak); i.e. at the point $(C + \frac{1}{4}B, D + A)$. Likewise, midway between $(C + \frac{1}{2}B, D)$ and $(C + B, D)$ there will be a minima (valley); i.e. at the point $(C + \frac{3}{4}B, D - A)$. It is now possible to roughly sketch the graph on the domain $C \leq x \leq C + B$ by connecting the points described. Once this portion of the graph is known, the fact that the function is periodic tells us to simply repeat the picture in the intervals $C + B \leq x \leq C + 2B$, $C - B \leq x \leq C$, etc.

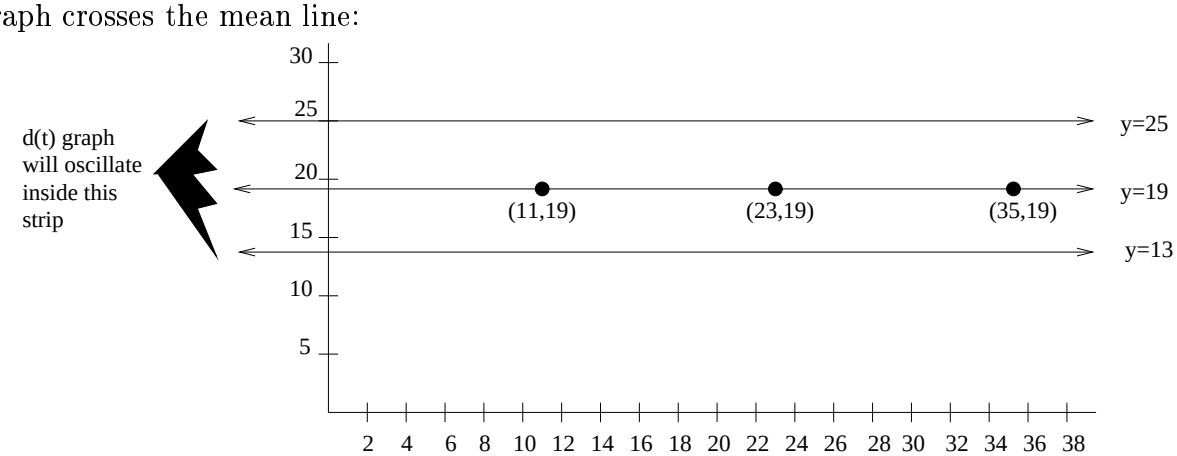
To make sense of this procedure, let's do an explicit example to see how these five steps produce a rough sketch.

Example 3.5.2: *The temperature (in ^oC) of Adri-N's dorm room varies during the day according to the sinusoidal function $d(t) = 6\sin(\frac{\pi}{12}(t-11)) + 19$, where t represents hours after midnight. Roughly sketch the graph of $d(t)$ over a 24 hour period.. What is the temperature of the room at 2:00 pm? What is the maximum and minimum temperature of the room?*

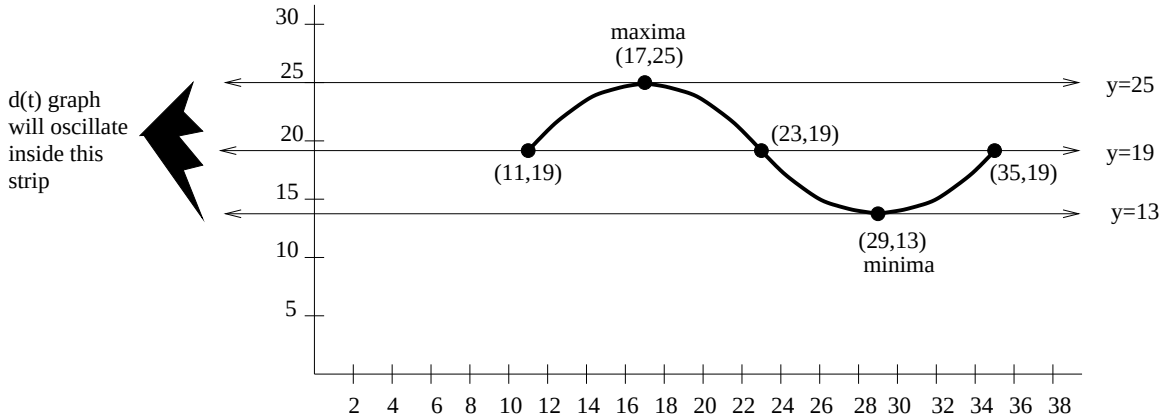
Solution. We begin with the rough sketch. Start by taking an inventory of the constants in this sinusoidal function:

$$d(t) = 6\sin\left(\frac{\pi}{12}(t - 11)\right) + 19 = A\sin\left(\frac{2\pi}{B}(t - C)\right) + D.$$

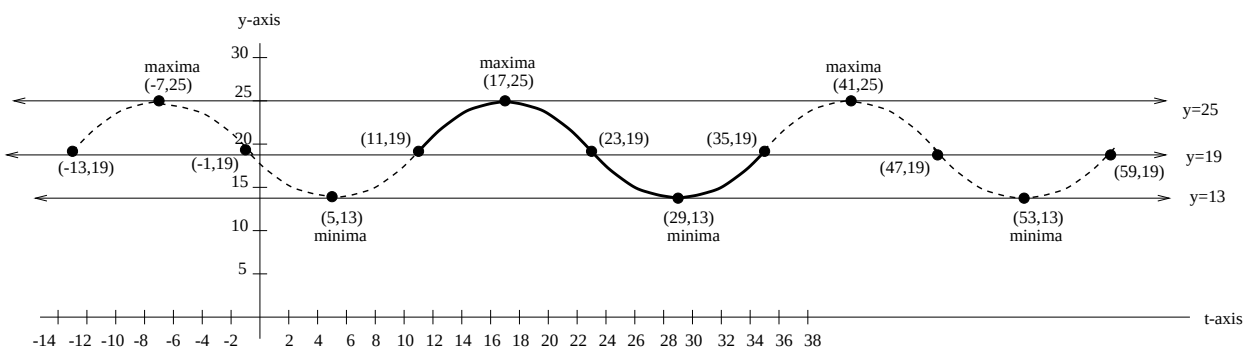
Conclude that $A = 6, B = 24, C = 11, D = 19$. Following the first four steps of the procedure outlined, we can sketch the lines $y = D = 19, y = D \pm A = 19 \pm 6$ and three points where the graph crosses the mean line:



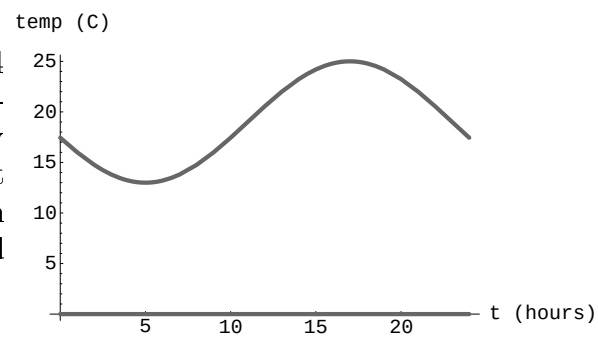
According to the fifth step in the sketching procedure, we can plot the maxima $(C + \frac{1}{4}B, D + A) = (17, 25)$ and the minima $(C + \frac{3}{4}B, D - A) = (29, 13)$. We then "connect the dots" to get a rough sketch on the domain $11 \leq t \leq 35$.



Finally, we can use the fact the function has period 24 to sketch the graph to the right and left by simply repeating the picture every 24 horizontal units.



We restrict the picture to the domain $0 \leq t \leq 24$ and obtain the computer generated graph pictured; as you can see, our rough graph is very accurate. The temperature at 2:00pm is just $d(14) = 23.24\ ^o$C. From the graph, the maximum value of the function will be $D + A = 25\ ^o$C and the minimum value will be $D - A = 13\ ^o$C.



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Functions not in standard sinusoidal form.

Anytime we are given a trigonometric function written in the standard form

$$y = A\sin\left(\frac{2\pi}{B}(x-C)\right)+D,$$

for constants A, B, C, D (with A and B positive), the summary in (3.5.1) tells us everything we could possibly want to know about the graph. But, there are two ways in which we might encounter a trigonometric type function that is not in this standard form:

- The constants A or B might be negative. For example, $y = -2\sin(2x-7)-3$ and $y = 3\sin(-\frac{1}{2}x+1)+4$ are examples that fail to be in standard form.
- We might use the cosine function in place of the sine function. For example, something like $y = 2\cos(3x+1)-2$ fails to be in standard sinusoidal form.

Now what do we do? Does this mean we need to repeat the analysis that led to (3.5.1)? It turns out that if we use our trig identities just right, then we can move any such equation into standard form and read off the amplitude, period, phase shift and mean. In other words, equations that fail to be in standard sinusoidal form for either of these two reasons will still define sinusoidal functions. We illustrate how this is done by way some examples:

Examples 3.5.3: (i) Start with $y = -2\sin(2x-7)-3$, then here are the steps with reference to the required identities to put the equation in standard form:

$$\begin{aligned} y &= -2\sin(2x-7)-3 \\ &= 2\left(-\sin(2x-7)\right)-3 \\ &\underset{(3.4.5)}{=} 2\sin(2x-7+\pi)+(-3) \\ &= 2\sin\left(\frac{2\pi}{\pi}(x-[\frac{7-\pi}{2}])\right)+(-3). \end{aligned}$$

This function is now in the standard form of (3.5.1), so it is a sinusoidal function with phase shift $C = \frac{7-\pi}{2} = 1.93$, mean $D = -3$, amplitude $A = 2$ and period $B = \pi$.

(ii) Start with $y = 3\sin(-\frac{1}{2}x+1)+4$, then here are the steps with reference to the required identities to put the equation in standard form:

$$\begin{aligned} y &= 3\sin(-\frac{1}{2}x+1)+4 \\ &= 3\sin(-(\frac{1}{2}x-1))+4 \\ &\underset{(3.4.4)}{=} 3(-\sin(\frac{1}{2}x-1))+4 \\ &\underset{(3.4.5)}{=} 3\sin(\frac{1}{2}x-1+\pi)+4 \\ &= 3\sin\left(\frac{2\pi}{4\pi}(x-[2-2\pi])\right)+4 \end{aligned}$$

This function is now in the standard form of (3.5.1), so it is a sinusoidal function with phase shift $C = 2-2\pi$, mean $D = 4$, amplitude $A = 3$ and period $B = 4\pi$.

(iii) Start with $y = 2\cos(3x+1)-2$, then here are the steps to put the equation in standard form. A key simplifying step is to use the identity established in Exercise 12, §3.4: $\cos(t) = \sin(\frac{\pi}{2}+t)$.

$$\begin{aligned} y &= 2\cos(3x+1)-2 \\ &= 2\sin(\frac{\pi}{2}+3x+1)-2 \\ &= 2\sin(3x-[-1-\frac{\pi}{2}])+(-2) \\ &= 2\sin\left(\frac{2\pi}{\frac{2\pi}{3}}\left(x-\frac{1}{3}\left[-1-\frac{\pi}{2}\right]\right)\right)+(-2) \end{aligned}$$

This function is now in the standard form of (3.5.1), so it is a sinusoidal function with phase shift $C = \frac{1}{3}[-1-\frac{\pi}{2}]$, mean $D = -2$, amplitude $A = 2$ and period $B = \frac{2\pi}{3}$.

Examples of sinusoidal behavior.

Problems involving sinusoidal behavior come in two basic flavors. On the one hand, we could be handed an explicit sinusoidal function $y = A\sin(\frac{2\pi}{B}(x-C))+D$ and asked various questions. The answers typically require either direct calculation or interpretation of the constants, as in (3.5.1). Example 3.5.2 is typical of this kind of problem. On the other hand, we might be told a particular situation is described by a sinusoidal function and provided some data or a graph. In order to further analyze the problem, we need a "formula", which means finding the constants A, B, C and D. This is a typical scenario in a "mathematical modeling problem": the process of observing data, THEN obtaining a mathematical formula. To find A, take half the difference between the largest and smallest values of $f(x)$. The period B is most easily

found by measuring the distance between to successive maxima (peaks) or minima (valleys) in the graph. The mean D is the average of the largest and smallest values of $f(x)$. The phase shift C (which is usually the most tricky quantity to get your hands on) is found by locating a "reference point". This "reference point" is a location where the graph crosses the mean line $y = D$ *on its way up* from a minimum to a maximum. The funny thing is that the phase shift C is NOT unique; there are an infinite number of correct choices. One choice that will work is C=(x-coordinate of a maximum) - $\frac{B}{4}$. Any other choice of C will differ from this one by a multiple of the period B.

$$\begin{aligned} A &= \frac{\text{max value } - \text{min value}}{2} \\ B &= \text{distance between two successive peaks (or valleys)} \\ C &= (x\text{-coordinate of a maximum})\text{-}\ \frac{B}{4} \\ D &= \frac{\text{max value } + \text{min value}}{2}. \end{aligned}$$

Example 3.5.4: Assume that the number of hours of daylight in Seattle is given by a sinusoidal function $d(t)$ of time. During 1994, assume the longest day of the year is June 21 with 15.7 hours of daylight and the shortest day is December 21 with 8.3 hours of daylight. Find a formula $d(t)$ for the number of hours of daylight on the tth day of the year.

Solution. Because the function $d(t)$ is assumed to be sinusoidal, it has the form $y = A\sin(\frac{2\pi}{B}(t-C)) + D$, for constants A, B, C and D. We simply need to use the given information to find these constants. The largest value of the function is 15.7 and the smallest value is 8.3. Knowing this, from the above discussion we can read off :

$$D = \frac{15.7 + 8.3}{2} = 12 \qquad\qquad A = \frac{15.7 - 8.3}{2} = 3.7.$$

To find the period, we need to compute the time between two successive maximum values of $d(t)$. To find this, we can simply double the time length of one-half period, which would be the length of time between successive maximum and minimum values of $d(t)$. Recalling the calendar information in (3.4.8) on the lengths of months, this gives us the equation

$$B = 2(\text{days between June 21 and December 21}) = 2(183) = 366.$$

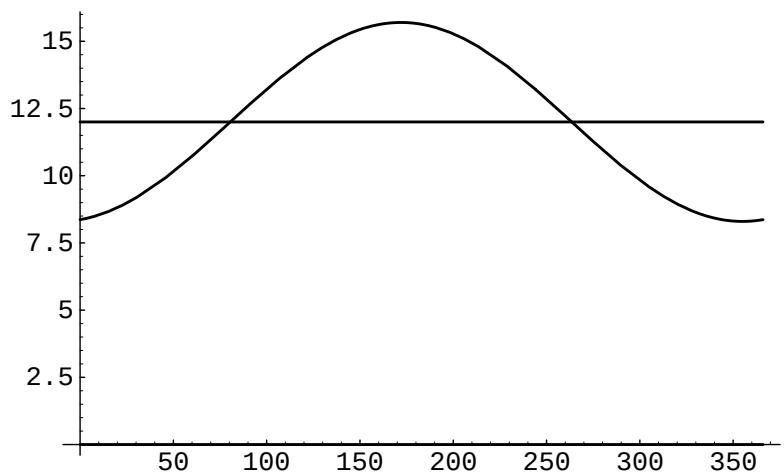
Locating the final constant C requires the most thought. Recall, the longest day of the year is June 21, which is day 172 of the year, so

$$C = (\text{day with max daylight}) - \frac{B}{4} = 172 - \frac{366}{4} = 80.5.$$

In summary, this shows that

$$d(t) = 3.7 \sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12.$$

A rough sketch, following the procedure outlined above, gives this graph on the domain $0 \leq t \leq 366$; we have included the mean line $y = 12$ for reference.



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We close with the example that started this section.

Example 3.5.5: *The depth of a migrating salmon below the water surface changes according to a sinusoidal function of time. The fish varies between 1 and 5 feet below the surface of the water. It takes the fish 1.571 minutes to move from it's minimum depth to it's successive maximum depth. It is located at a maximum depth when $t = 4.285$ minutes. What is the formula for the function $d(t)$ that predicts the depth of the fish after t minutes? What was the depth of the salmon when it was first spotted? During the first 10 minutes, how many times will the salmon be exactly 4 feet below the surface of the water?*

Solution. We know that $d(t) = A\sin(\frac{2\pi}{B}(x - C)) + D$, for appropriate constants A, B, C, D. We need to use the given information to extract these four constants. The amplitude and mean

are easily found using the above formulas:

$$\begin{aligned} A &= \frac{\text{max depth } - \text{min depth}}{2} = \frac{5-1}{2} = 2 \\ D &= \frac{\text{max depth } + \text{min depth}}{2} = \frac{5+1}{2} = 3. \end{aligned}$$

The period can be found by noting that the information about the time between a successive minimum and maximum depth will be half of a period (look at the picture in 3.5.1):

$$B = 2(1.571) = 3.142$$

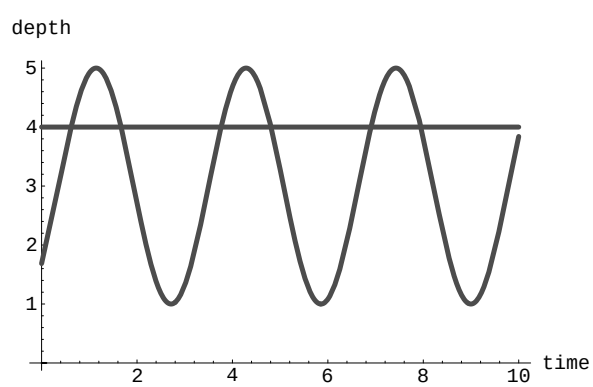
Finally, to find C we

$$C = \text{(time of maximum depth) - } \frac{B}{4} = 4.285 - \frac{3.142}{4} = 3.50.$$

The formula is now

$$d(t) = 2\sin(\frac{2\pi}{3.142}(t-3.5)) + 3 = 2\sin(2t-7) + 3$$

The depth of the salmon when it was first spotted is just $d(0) = 2\sin(-7)+3 = 1.686$ feet. Finally, graphically, the last question amounts to determining how many times the graph of $d(t)$ crosses the line $y = 4$ on the domain [0,10]. This can be done using 3.5.1. A simultaneous picture of the two graphs is given, from which we can see the salmon is exactly 4 feet below the surface of the water six times during the first 10 minutes.



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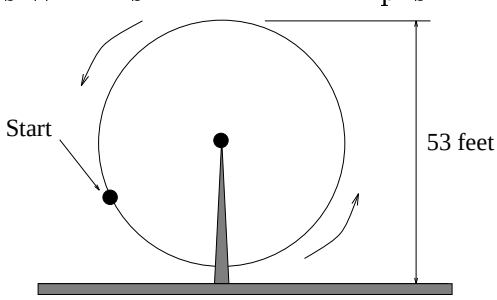
Problems

1. Find the amplitude, period, phase shift and mean of the following sinusoidal functions.
   (a) $y = \sin(2x - \pi) + 1$
   (b) $y = 6\sin(\pi x) - 1$
   (c) $y = -2\sin(2x - \pi) + 3$

2. In this problem, follow the "five step procedure" outlined in the text to roughly sketch the graph of the sinusoidal function $y = 25\sin(1.3\pi + 0.4\pi t) + 28$.

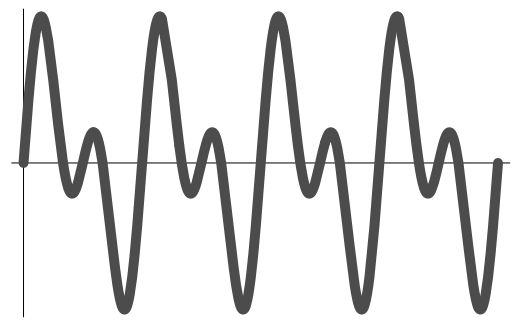
3. Marcel is anxious to learn his Chem 150 grade. He starts out standing in front of Professor Zoller's office door and paces back and forth along a straight line. He begins 20 feet in front of the office door, walks toward the door and in 4 seconds he is 2 feet from the door. He then backs off, moving back and forth sinusoidally as a function of time, all the while moving between 2 feet and 20 feet from the door. After 31 seconds, he works up the courage to knock on the door, so he walks toward the door at a constant speed of 8 ft/sec. Let $m(t)$ be the function which describes the distance from Marcel to the door after t seconds.
   (a) Give a formula for the function $m(t)$ during the first 31 seconds.
   (b) Where is Marcel located after 31 seconds?
   (c) When does he reach the door?
   (d) Give a formula for the multipart function $m(t)$, up until the time he reaches the door.
   (e) Sketch a picture of the graph of $m(t)$.

4. Your seat on a Ferris Wheel is at the indicated position at time $t = 0$.



   Let t be the number of seconds elapsed after the wheel begins rotating counterclockwise. You find it takes 3 seconds to reach the top, which is 53 feet above the ground. The wheel is rotating 12 RPM and the diameter of the wheel is 50 feet. Let $d(t)$ be your height above the ground at time t.
   (a) Argue that $d(t)$ is a sinusoidal function, describing the amplitude, phase shift, period and mean.
   (b) When are the first and second times you are exactly 28 feet above the ground?
   (c) After 29 seconds, how many times will you have been exactly 28 feet above the ground?

5. Here is a graph of the function
   $y = \sin(x) + \sin(2x)$.
   Is this function sinusoidal? Give a reason.



6. A respiratory ailment called "Cheyne-Stokes Respiration" causes the volume per breath to increase and decrease in a sinusoidal manner, as a function of time. For one particular patient with this condition, a machine begins recording a plot of volume per breath versus time (in seconds). Let $b(t)$ be a function of time t that tells us the volume (in liters) of a breath that starts at time t. During the test, the smallest volume per breath is 0.6 liters and this first occurs for a breath that starts 5 seconds into the test. The largest volume per breath is 1.8 liters and this first occurs for a breath beginning 55 seconds into the test.
   (a) Find a formula for the function $b(t)$ whose graph will model the test data for this patient.
   (b) If the patient begins a breath every 5 seconds, what are the breath volumes during the first minute of the test?

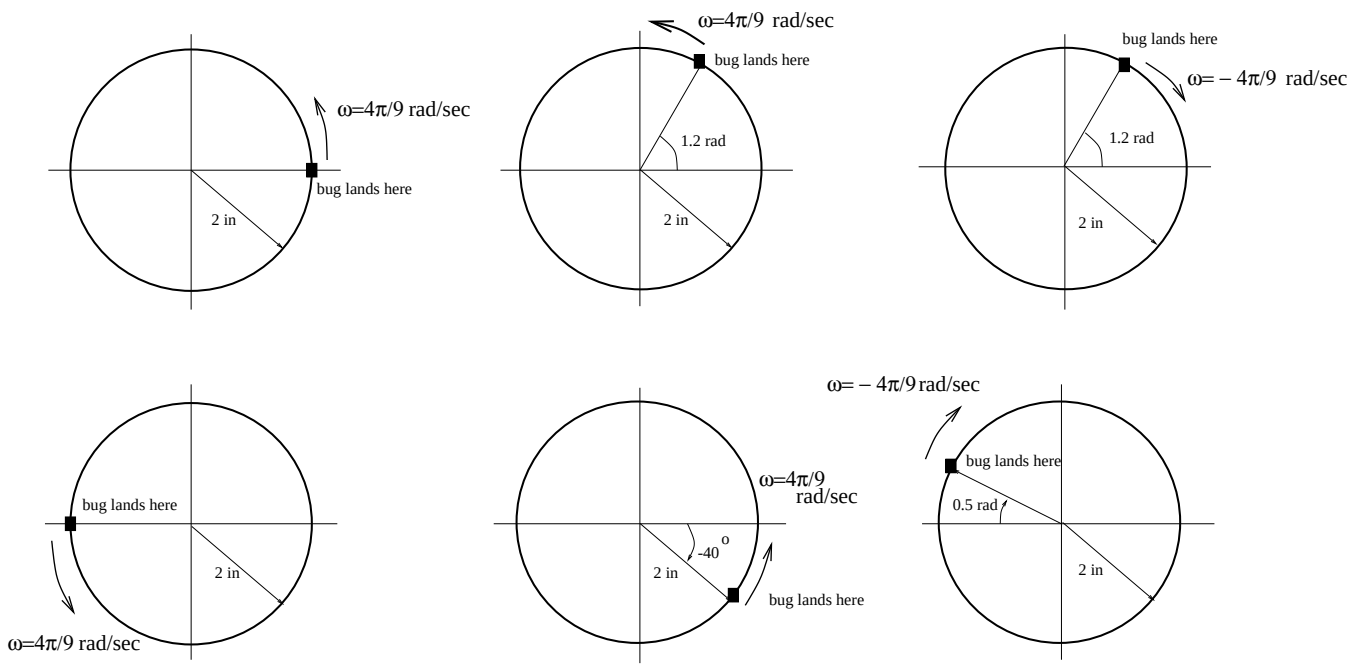
7. The angle of elevation of the sun above the horizon at noon is 18° on December 21, the 355th day of the year. The elevation is 72° at noon on June 21, the 172nd day of the year. These elevations are the minimum and maximum elevations at noon for the year. Assume this particular year has 366 days and that the elevation on day t is given by a sinusoidal function $E(t)$.
   (a) Follow the procedure outlined in this section to sketch a rough graph of $E(t)$. Draw at least two complete cycles of the oscillation, indicating where the maxima and minima occur.
   (b) What are the mean, amplitude, phase shift and period for this function?
   (c) Give four different possible values for the phase shift.
   (d) Write down a formula (as in (3.5.1)) for the function $E(t)$. Confirm it is consistent with (a) using a graphing device.
   (e) What is the elevation at noon February 7? (January has 31 days and we're counting January 1 as the first day of the year.)

8. A bug has landed on the rim of a jelly jar and is moving around the rim. The location where the bug initially lands is described and its angular speed is given. Impose a coordinate system with the origin at the center of the circle of motion. In each of the cases, the earlier

exercise found the coordinates $P(t)$ of the bug at time t. Pick three of the scenarios below and answer these two questions:

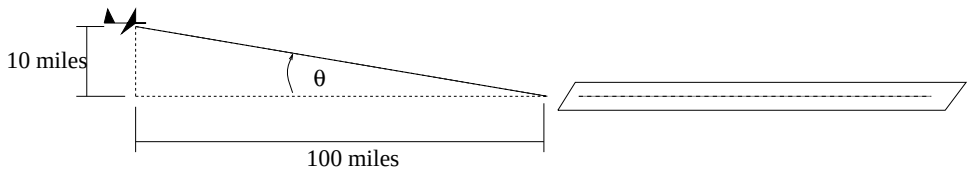
(a) Both coordinates of $P(t)$ are sinusoidal functions in the variable t; i.e. $P(t) = (x(t), y(t))$. Put $x(t)$ and $y(t)$ in standard sinusoidal form. Find the amplitude, mean, period and phase shift for each function.

(b) Sketch a rough graph of the functions $x(t)$ and $y(t)$ in (a) on the domain $0 \leq t \leq 9$.



3.6 Inverse Circular Functions

An aircraft is flying at an altitude 10 *miles above the elevation of an airport. If the airplane begins a steady descent 100 miles from the airport, what is the angle θ of descent?*



The only natural circular function we can use is $z = \tan(\theta)$, leading to the equation:

$$\tan(\theta) = \frac{10}{100} = \frac{1}{10}.$$

The problem is that this equation does not tell us the value of θ. Moreover, none of the equation solving techniques at our disposal (which all amount to algebraic manipulations) will help us solve the equation for θ. What we need is an *inverse function* $\theta = f^{-1}(z)$; then we could use the fact that $\tan^{-1}(\tan(\theta)) = \theta$ (recall (3.6.8)) and obtain:

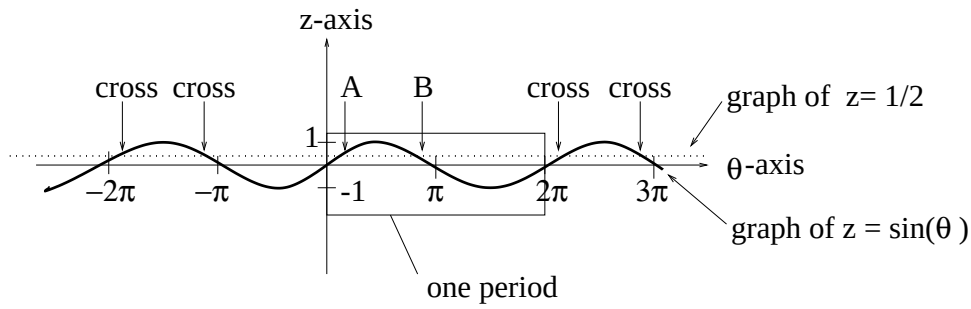
$$\theta \;=\; \tan^{-1}(\tan(\theta)) = \tan^{-1}(\frac{1}{10}).$$

Computationally, without even thinking about what is going on, any scientific calculator will allow us to compute values of an inverse circular function and leads to a solution of our problem. In this example, you will find $\theta = \tan^{-1}(\frac{1}{10}) = 5.71^o$. Punch this into your calculator and verify it!

Solving Three Equations.

Example 3.6.1 *Find all values of θ (an angle) that make this equation true:* $\sin(\theta) = \frac{1}{2}$.

Solution. We begin with a graphical reinterpretation: the solutions correspond to the places where the graphs of $z = \sin(\theta)$ and $z = \frac{1}{2}$ intersect in the θz-coordinate system. Recalling Figure 3.4.4, we can picture these two graphs simultaneously as below:



The first thing to notice is that these two graphs will cross an infinite number of times, so there are infinitely many solutions to (3.6.1)! However, notice there is a predictable spacing of the crossing points, which is just a manifestation of the periodicity of the sine function. In fact, if we can find the two crossing points labeled "A" and "B", then all other crossing points are obtained by adding multiples of 2π to either "A" or "B". By Table 3.3.1, $\theta = \frac{\pi}{6}$ radians is a special angle where we computed $\sin(\frac{\pi}{6}) = \frac{1}{2}$, which tells us that the crossing point labeled "A" is the point $(\frac{\pi}{6}, \frac{1}{2})$. Using the identities (3.4.4) and (3.4.5), notice that

$$\sin(\frac{5\pi}{6}) = \sin(-\frac{\pi}{6} + \pi) = -\sin(-\frac{\pi}{6}) = -(-\sin(\frac{\pi}{6})) = \sin(\frac{\pi}{6}) = \frac{1}{2}.$$

So, $\theta = \frac{5\pi}{6}$ is the only other angle θ between 0 and 2π such that (3.6.1) holds. This corresponds to the crossing point labeled "B", which has coordinates $(\frac{5\pi}{6}, \frac{1}{2})$. In view of the remarks above, the crossing points come in two flavors:

$$\left(\frac{\pi}{6} + 2k\pi, \frac{1}{2}\right), \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots \qquad \text{and} \qquad \left(\frac{5\pi}{6} + 2k\pi, \frac{1}{2}\right), k = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

□

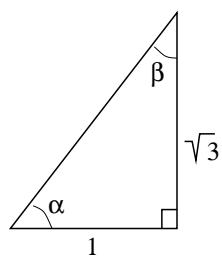
Taking this example as a model, we can tackle the more general problem: For a fixed real number c, describe the solution(s) of the equation $c = f(\theta)$ for each of the circular functions $z = f(\theta)$. Studying solutions of these equations will force us to come to grips with three important issues:

- For what values of c does $f(\theta) = c$ have a solution?
- For a given value of c, how many solutions does $f(\theta) = c$ have?
- Can we restrict the domain so that the resulting function is one-to-one?

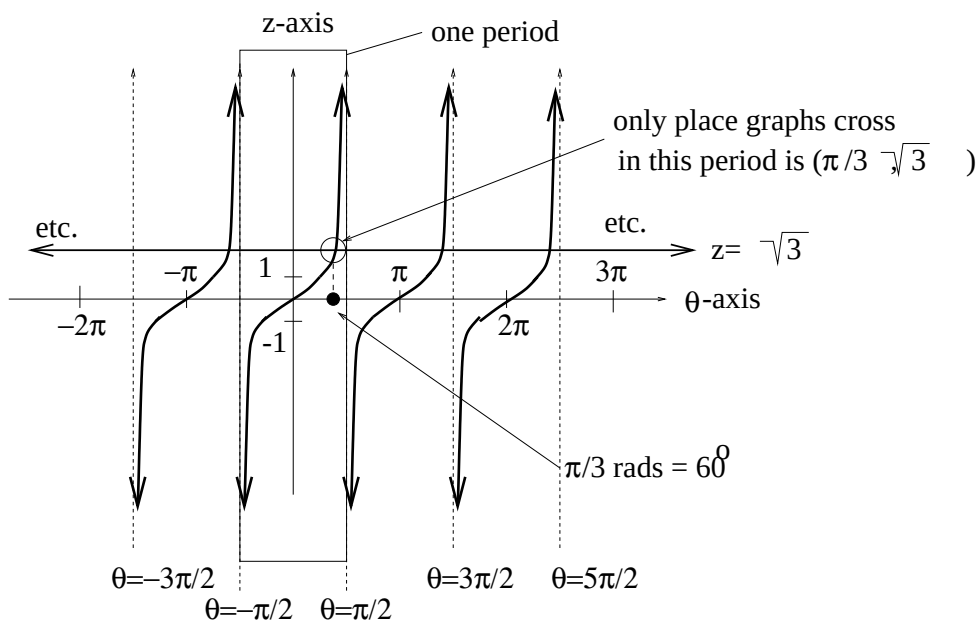
All of these questions must be answered before we can come to grips with any understanding of the inverse functions. Using the graphs of the circular functions, it is an easy matter to arrive at the following qualitative conclusions.

Observation 3.6.2: *None of the circular functions is one-to-one on the domain of all θ values. The equations $c = \sin(\theta)$ and $c = \cos(\theta)$ have a solution if and only if $-1 \le c \le 1$; if c is in this range, there are infinitely many solutions. The equation $c = \tan(\theta)$ has a solution for any value of c and there are infinitely many solutions.*

Example 3.6.3: *If two sides of a right triangle have lengths 1 and $\sqrt{3}$ as pictured below, what are the acute angles α and β?*



Solution. By the Pythagorean Theorem the remaining side has length $\sqrt{1 + (\sqrt{3})^2} = 2$. Since $\tan(\alpha) = \sqrt{3}$, we need to solve this equation for α. Graphically, we need to determine where $z = \sqrt{3}$ crosses the graph of the tangent function:



From (3.6.2), there will be infinitely many solutions to our equation, but notice that there is **exactly one solution** in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and we can find it using Table 3.3.1:

$$\tan(\frac{\pi}{3}) \quad = \quad \tan(60^o) = \frac{\sin(60^o)}{\cos(60^o)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

So, $\alpha = \frac{\pi}{3}$ radians $= 60^o$ is the only acute angle solution and $\beta = 180^o - 60^o - 90^o = 30^o$. $\square$

Inverse Circular Functions.

Except for specially chosen angles, we have not addressed the serious problem of FINDING values of the inverse rules attached to the circular function equations. (Our previous examples were "rigged", so that we could use Table 3.3.1.) To proceed computationally, we need to obtain the *inverse circular functions*. If we were to proceed in a sloppy manner, then a first attempt at defining the inverse circular functions would be to write

- $\sin^{-1}(z) =$ solutions θ of the equation $z = \sin(\theta)$.
- $\cos^{-1}(z) =$ solutions θ of the equation $z = \cos(\theta)$. (3.6.4)
- $\tan^{-1}(z) =$ solutions θ of the equation $z = \tan(\theta)$.

There are two main problems with these rules as they stand. First, to have a solution in the case of $\sin^{-1}$ and $\cos^{-1}$, we need to restrict z so that $-1 \leq z \leq 1$. Secondly, having made this restriction on z in the first two cases, there is no unique solution; rather, there are an infinite number of solutions. This means that the rules $\sin^{-1}, \cos^{-1}$ and $\tan^{-1}$ as they now stand do **not** define functions. Given what we have reviewed about inverse functions, the only way to proceed is to restrict each circular function to a domain of θ values on which it becomes one-to-one, then we can appeal to (2.6.7) and conclude the inverse function makes sense.

At this stage a lot of choice (flexibility) enters into determining the domain on which we should try to invert each circular function. In effect, there are an infinite number of possible choices. If $z = f(\theta)$ denotes one of the three circular functions, there are three natural criteria we use to guide the choice of a restricted domain, which we will call a *principal domain*:

- The domain of $f(\theta)$ should include the angles between 0 and $\frac{\pi}{2}$, since these are the possible acute angles in a right triangle.
- On the restricted domain, the function $f(\theta)$ should take on all possible values in the range of $f(\theta)$. In addition, the function should be one-to-one on this restricted domain.
- The function $f(\theta)$ should be "continuous" on this restricted domain; i.e. the graph on this domain could be traced with a pencil, without lifting it off the paper.

In the case of $z = \sin(\theta)$, the principal domain $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ satisfies our criteria and the picture is given below. Notice, we would **not** want to take the interval $0 \leq \theta \leq \pi$, since $z = \sin(\theta)$ doesn't achieve negative values on this domain; in addition, it's not one-to-one there.

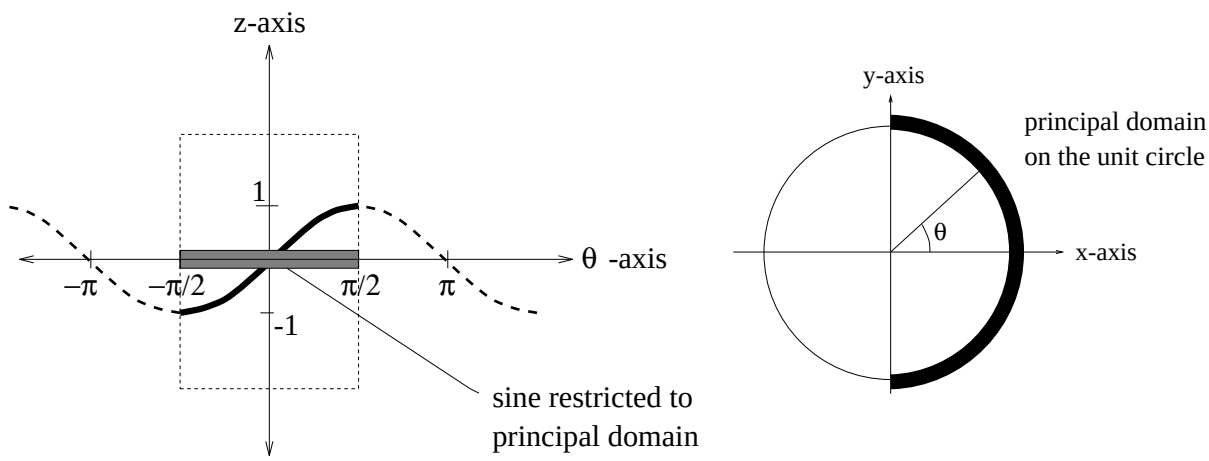


Figure 3.6.1: Principal domain for $\sin(\theta)$

In the case of $z = \cos(\theta)$, the principal domain $0 \leq \theta \leq \pi$ satisfies our criteria and the picture is given below. Notice, we would **not** want to take the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, since $z = \cos(\theta)$ doesn't achieve negative values on this domain; in addition, it's not one-to-one there.

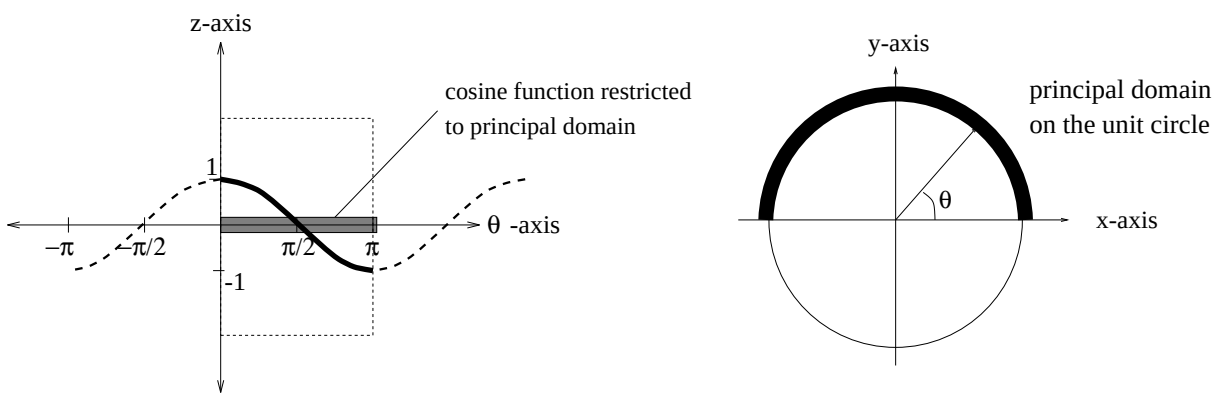


Figure 3.6.2: Principal domain for $\cos(\theta)$

In the case of $z = \tan(\theta)$, the principal domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ satisfies our criteria and the picture is given below. Notice, we would **not** want to take the interval $0 \leq \theta \leq \pi$, since $z = \tan(\theta)$ does not have a continuous graph on this interval; in other words, we do not include the endpoints since $\tan(\theta)$ is undefined there.

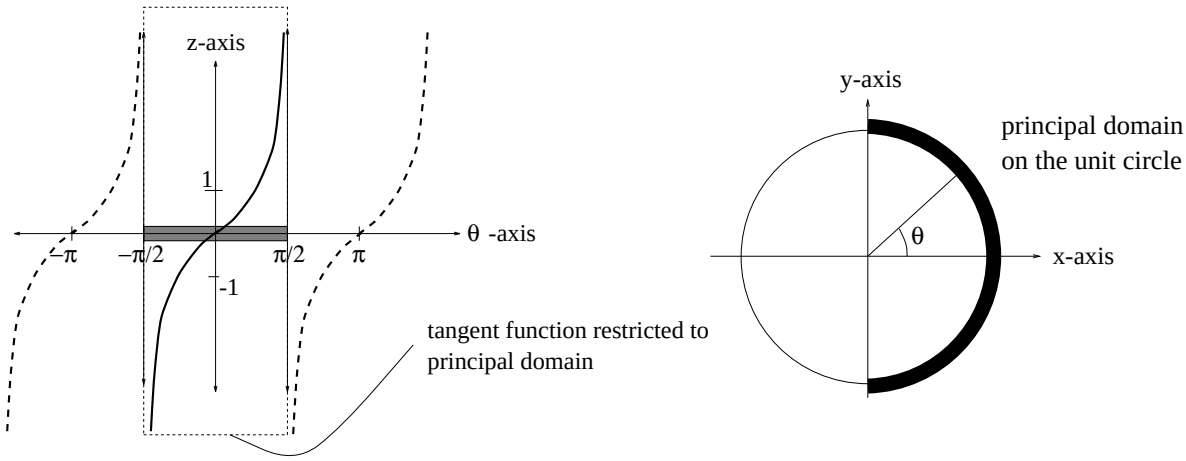


Figure 3.6.3: Principal domain for $\tan(\theta)$

Inverse Circular Functions 3.6.5: *Restricting each circular function to its principal domain, its inverse rule* $f^{-1}(z) = \theta$ *will define a function.*
(i) If $-1 \leq z \leq 1$*, then* $\sin^{-1}(z)$ *is the unique angle* θ *in the principal domain* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ *with the property that* $\sin(\theta) = z$*.*
(ii) If $-1 \leq z \leq 1$*, then* $\cos^{-1}(z)$ *is the unique angle* θ *in the principal domain* $0 \leq \theta \leq \pi$ *with the property that* $\cos(\theta) = z$*.*
(iii) For any real number z*,* $\tan^{-1}(z)$ *is the unique angle* θ *in the principal domain* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ *with the property that* $\tan(\theta) = z$*.*

We refer to the functions defined above as the *inverse circular functions*. These are sometimes referred to as the "arcsine", "arccosine" and "arctangent" functions, though we will not use that terminology. The inverse circular functions give us one solution for each of these equations:

$$\begin{aligned} c &= \cos(\theta) \\ c &= \sin(\theta) \\ c &= \tan(\theta); \end{aligned}$$

these are called the *principal solutions*. We also can refer to these as the *principal values* of the inverse circular function rules $\theta = f^{-1}(z)$.

!!! CAUTION !!!

As usual, be careful with "radian mode" and "degree mode" when making calculations. For example, if your calculator is in "degree" mode and you type in "$\tan^{-1}(18)$", the answer given is "86.82". This means that an angle of measure $\theta = 86.82^o$ has $\tan(86.82) = 18$. If your calculator is in "radian" mode and you type in "$\sin^{-1}(0.9)$", the answer given is "1.12". This means that an angle of measure $\theta = 1.12$ radians has $\sin(1.12) = 0.9$.

!!! CAUTION !!!

There is a key property of the inverse circular functions which is useful in equation solving; it is just a direct translation of (2.6.8) into our current context:

Composition Identity 3.6.6: *We have the following equalities involving compositions of circular functions and their inverses:*

(a) If $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $\sin^{-1}(\sin(\theta)) = \theta$.

(b) If $0 \leq \theta \leq \pi$, then $\cos^{-1}(\cos(\theta)) = \theta$.

(c) If $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\tan^{-1}(\tan(\theta)) = \theta$.

We have been very explicit about the allowed θ values for the equations in (3.6.6). This is important and Exercise 3.6.13 will touch on this issue.

Applications.

As a simple application of (3.6.6), we can return to the beginning of this section and justify the reasoning used to find the angle of descent of the aircraft:

$$\theta = \tan^{-1}((\tan(\theta)) = \tan^{-1}(\frac{1}{10}) = 0.09967\,\text{rad} = 5.71^o.$$

Let's look at some other applications.

Example 3.6.7: *Find two acute angles θ so that the following equation is satisfied:*

$$\frac{9}{4\cos^2(\theta)} = \frac{25^2}{16}(1 - \cos^2(\theta)).$$

Solution. Begin by multiplying each side of the equation by $\cos^2(\theta)$ and rearranging terms:

$$\begin{aligned} \frac{9}{4} &= \frac{25^2}{16}(1 - \cos^2(\theta))\cos^2(\theta) \\ 0 &= \frac{25^2}{16}\cos^4(\theta) - \frac{25^2}{16}\cos^2(\theta) + \frac{9}{4}. \end{aligned}$$

To solve this equation for θ, we the technique of *substitution*. The central idea is to bring the quadratic formula into the picture by making the substitution $z = \cos^2(\theta)$:

$$0 = \frac{25^2}{16}z^2 - \frac{25^2}{16}z + \frac{9}{4}.$$

Applying the quadratic formula, we obtain

$$z = \frac{\frac{25^2}{16} \pm \sqrt{(\frac{25^2}{16})^2 - 4(\frac{25^2}{16})(\frac{9}{4})}}{\frac{2(25^2)}{16}} = 0.9386, 0.06136.$$

We now use the fact that $z = \cos^2(\theta)$ and note the cosine of an acute angle is non-negative to conclude that

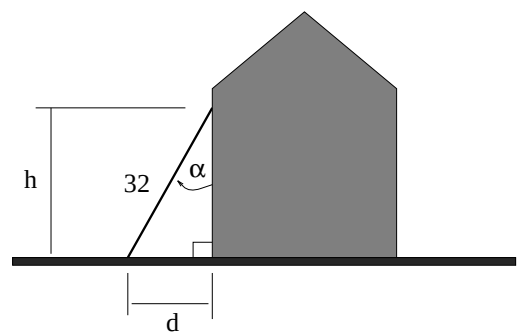
$$\begin{aligned} \cos^2(\theta) &= 0.9386 \text{ or } \cos^2(\theta) = 0.06136 \\ \cos(\theta) &= 0.9688 \text{ or } \cos(\theta) = 0.2477. \end{aligned}$$

Finally, we use the inverse cosine function to arrive at our two acute angle solutions:

$$\begin{aligned} \cos(\theta) &= 0.9688 \Rightarrow \theta = \cos^{-1}(0.9688) = 14.35^o \\ \cos(\theta) &= 0.2477 \Rightarrow \theta = \cos^{-1}(0.2477) = 75.66^o \end{aligned}$$

□

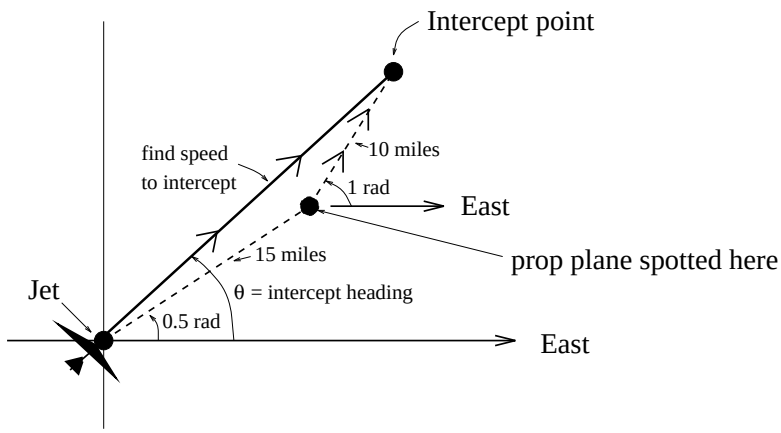
Example 3.6.8: *A 32 ft ladder leans against a building (as shown below) making an angle α with the wall. OSHA (Occupational Safety and Health Administration) specifies a "safety range" for the angle α to be $15^o \leq \alpha \leq 44^o$. If the base of the ladder is $d = 10$ feet from the house, is this a safe placement? Find the highest and lowest points safely accessible.*



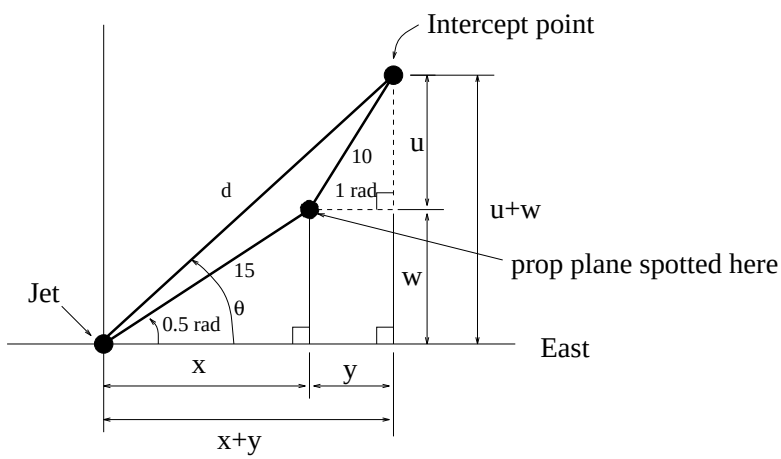
Solution. If $d = 10$, then $\sin(\alpha) = \frac{10}{32}$, so the principal solution is $\alpha = \sin^{-1}(\frac{10}{32}) = 18.21^o$; this lies within the safety zone. From the picture, it is clear that the highest point safely reached will occur precisely when $\alpha = 15^o$ and as this angle increases, the height h decreases until we reach the lowest safe height when $\alpha = 44^o$. We need to solve two right triangles. If $\alpha = 15^o$, then $h = 32\cos(15^o) = 30.9$ ft. If $\alpha = 44^o$, then $h = 32\cos(44^o) = 23.02$ ft. □

Example 3.6.9 *A Coast Guard jet pilot makes contact with a small unidentified propeller plane 15 miles away at the same altitude in a direction 0.5 radians counterclockwise from East. The prop plane flies in the direction 1.0 radians counterclockwise from East. The jet has been instructed to allow the prop plane to fly 10 miles before intercepting. In what direction should the jet fly to intercept the prop plane? If the prop plane is flying 200 mph, how fast should the jet be flying to intercept?*

Solution. Here is a picture of the situation:



After a look at the picture, three right triangles pop out and beg to be exploited. We highlight these in the figure below, labeling the various sides. We will work in radian units and label θ to be the required intercept heading.



We will first determine the sides $x + y$ and $u + w$ of the large right triangle. To do this, we have

$$\begin{aligned} x &= 15\cos(0.5) = 13.164 \text{ miles} \\ y &= 10\cos(1.0) = 5.403 \text{ miles} \\ w &= 15\sin(0.5) = 7.191 \text{ miles} \\ u &= 10\sin(1.0) = 8.415 \text{ miles.} \end{aligned}$$

We now have $\tan(\theta) = \frac{w+u}{x+y} = 0.8405$, so the principal solution is $\theta = 0.699$ radians, which is about 40.05^o. This is the only acute angle solution, so we have found the required intercept heading.

To find the intercept speed, first compute your distance to the intercept point, which is the length of the hypotenuse of the big right triangle: $d = \sqrt{(18.567)^2 + (15.606)^2} = 24.254$

miles. You need to travel this distance in the same amount of time T it takes the prop plane to travel 10 miles at 200 mph; i.e. $T = \frac{10}{200} = 0.05$ hours. Thus, the intercept speed s is

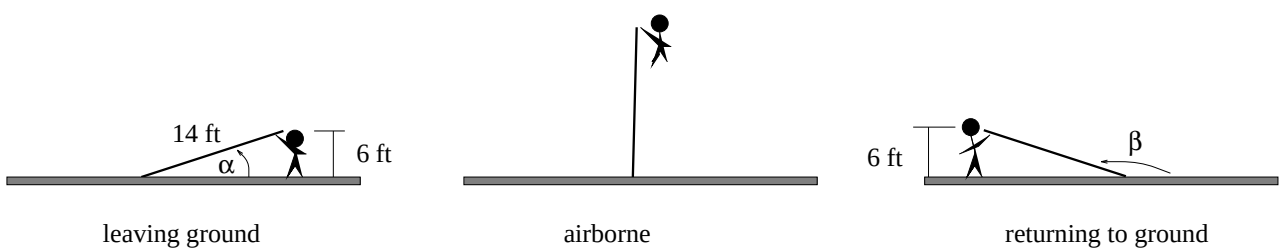
$$s = \frac{\text{distance traveled}}{\text{time } T \text{ elapsed}} = \frac{24.254}{0.05} = 485\,\text{mph}.$$

□

In Chapter 3, we will develop an alternate approach to this problem using *velocity vectors*.

In certain applications, knowledge of the principal solutions for the equations $c = \cos(\theta)$, $c = \sin(\theta)$ and $c = \tan(\theta)$ is not sufficient. Here is a typical example of this, illustrating the reasoning required.

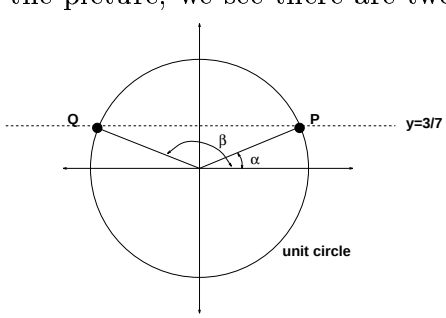
Example 3.6.10 *A rigid 14 ft pole is used to vault. The vaulter leaves and returns to the ground when the tip is 6 feet high, as indicated. What are the angles of the pole with the ground on takeoff and landing?*



Solution. From the obvious right triangles in the picture, we are interested in finding angles θ where $\sin(\theta) = \frac{6}{14} = \frac{3}{7}$. The idea is to proceed in three steps:

- Find the principal solution of the equation $\sin(\theta) = \frac{3}{7}$;
- Find **all** solutions of the equation $\sin(\theta) = \frac{3}{7}$;
- Use the constraints of the problem to find α and β among the set of all solutions.

Solving the equation $\sin(\theta) = \frac{3}{7}$ involves finding the points on the unit circle with y-coordinate equal to $\frac{3}{7}$. From the picture, we see there are two such points, labeled P and Q.



The coordinates of these points will be $P = (\cos(\alpha), \frac{3}{7})$ and $Q = (\cos(\beta), \frac{3}{7})$. Notice, α is the principal solution of our equation $\sin(\theta) = \frac{3}{7}$, since $0 \leq \alpha \leq 90^o$; so $\alpha = \sin^{-1}(\frac{3}{7}) = 25.38^o$. In general, the solutions come in two basic flavors:

$$\theta \;=\; \alpha + 2k(180^o) = 25.38^o + 2k(180^o), \text{ or, } \;\; \theta = \beta + 2k(180^o),$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$. To find the angle β, we can use basic properties of the circular functions:

$$\sin(\beta) = \sin(\alpha) = -\sin(-\alpha) = \sin(180^o - \alpha) = \sin(154.62^o).$$

This tells us $\beta = 154.62^o$. □

Two strategies to solve trigonometric equations.

So far, our serious use of the inverse trigonometric functions has focused on situations that ultimately involve triangles. However, many trigonometric modeling problems have nothing to do with triangles and so we need to free ourselves from the necessity of relying on such a geometric picture. There are two general strategies for finding solutions to the equations $c = \sin(\theta)$, $c = \cos(\theta)$ and $c = \tan(\theta)$:

- The first procedure is summarized in (3.6.11) below. This method has the advantage of offering a "prescription" for solving the equations; the disadvantage is you can lose intuition toward interpreting your answers.
- The second procedure is graphical in nature and is illustrated in (3.6.12) below. This method usually clarifies interpretation of the answers, but it does require more work since an essential step is to roughly sketch the graph of the trigonometric function (following the procedure of §3.5 or using a graphing device).

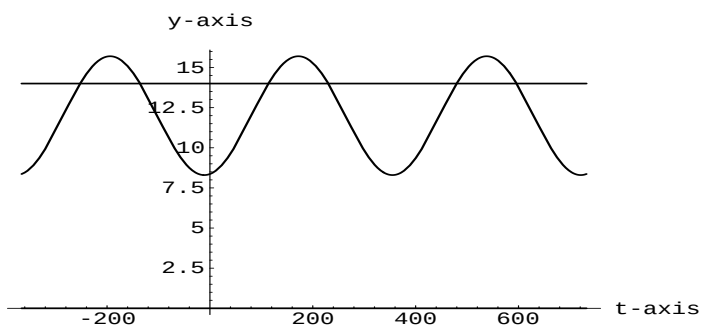
Each approach has its merits as you will see in the exercises.

Strategy 3.6.11: *To find ALL solutions to the equations $c = \sin(\theta)$, $c = \cos(\theta)$ and $c = \tan(\theta)$, we can lay out a foolproof strategy.*

Step	sine case	cosine case	tangent case
1. Find principal solution	$\theta = \sin^{-1}(c)$	$\theta = \cos^{-1}(c)$	$\theta = \tan^{-1}(c)$
2. Find symmetry solution	$\theta = -\sin^{-1}(c) + \pi$	$\theta = -\cos^{-1}(c)$	not applicable
3. Write out multiples of period $k = 0, \pm 1, \pm 2, \ldots$	$2k\pi$	$2k\pi$	$k\pi$
4. Obtain general principal solutions	$\theta = \sin^{-1}(c) + 2k\pi$	$\theta = \cos^{-1}(c) + 2k\pi$	$\theta = \tan^{-1}(c) + k\pi$
5. Obtain general symmetry solutions	$\theta = -\sin^{-1}(c) + \pi + 2k\pi$	$\theta = -\cos^{-1}(c) + 2k\pi$	not applicable

Example 3.6.12: *Assume that the number of hours of daylight in your hometown during 1994 is given by the function* $d(t) = 3.7\sin\left(\frac{2\pi}{366}(t-80.5)\right)+12$, *where* t *represents the day of the year. Find the days of the year during which there will be approximately 14 hours of daylight?*[1]

Solution. To begin, we want to roughly sketch the graph of $y = d(t)$ on the domain $0 \le t \le 366$. If you apply the graphing procedure discussed in §3.5, you obtain the sinusoidal graph below on the larger domain $-366 \le t \le 732$. (The reason we use a larger domain is so that the "pattern" that will arise in the strategy described below is more evident. Ultimately, we will restrict our attention to the smaller domain $0 \le t \le 366$.) To determine when there will be 14 hours of daylight, we need to solve the equation $14 = d(t)$. Graphically, this amounts to finding the places where the line $y = 14$ intersects the graph of $d(t)$. As can be seen in the picture below, there are several such intersection points.



We now outline a "three step strategy" to find all of these intersection points (which amounts to solving the equation $14 = d(t)$):

1. Principal Solution. We will find one solution by using the inverse sine function. If we start with the function $y = \sin(t)$ on its principal domain $\frac{-\pi}{2} \le t \le \frac{\pi}{2}$, then we can compute the domain of $d(t) = 3.7\sin\left(\frac{2\pi}{366}(t-80.5)\right)+12$:

$$\begin{aligned} -\pi/2 &\le \frac{2\pi}{366}(t-80.5) \le \pi/2 \\ -11 &\le t \le 172. \end{aligned}$$

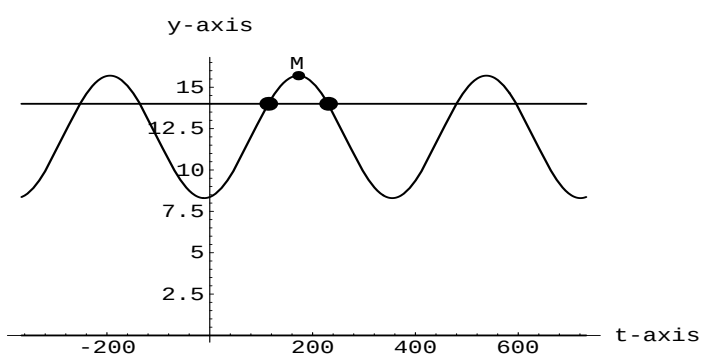
[1] You can get the actual data from the naval observatory at this world wide web address: http://tycho.usno.navy.mil/time.html

Now, using the inverse sine function we can find the principal solution to the equation $14 = d(t)$:

$$\begin{aligned} 14 &= 3.7\sin\left(\frac{2\pi}{366}(t-80.5)\right)+12 \qquad (3.6.13)\\ 0.54054 &= \sin\left(\frac{2\pi}{366}(t-80.5)\right)\\ 0.57108 &= \sin^{-1}(0.54054) = \frac{\pi}{183}(t-80.5)\\ t &= 113.8 \end{aligned}$$

Notice, this answer is in the domain $-11 \le t \le 172$. In effect, we have found THE ONLY SOLUTION on this domain. Conclude that there will be about 14 hours of daylight on the 114th day of the year.

2. Symmetry Solution. To find another solution to the equation $14 = d(t)$, we will use symmetry properties of the graph of $y = d(t)$. This is where having the graph of $y = d(t)$ is most useful. We know the a maxima on the graph occurs at the point $M = (172, 15.7)$; review Example 3.5.4 for a discussion of why this is the case. From the graph, we can see there are two symmetrically located intersection points on either side of M. The principal solution gives the intersection point $(113.8, 14)$. This point is 58.2 horizontal units to the left of M; see the picture below. So a symmetrically positioned intersection point will be $(172 + 58.2, 14) = (230.2, 14)$.



In other words, $t = 230.2$ is a second solution to the equation $14 = d(t)$. We call this the symmetry solution.

3. Other Solutions. To find all other solutions of $14 = d(t)$, we add integer multiples of the period $B = 366$ to the t-coordinates of the principal and symmetric intersection points. On the domain $-366 \le t \le 732$ we get these six intersection points; refer to the picture of the graph:

$$(-252.2, 14), (113.8, 14), (479.8, 14), (-135.8, 14), (230.2, 14), (596.2, 14).$$

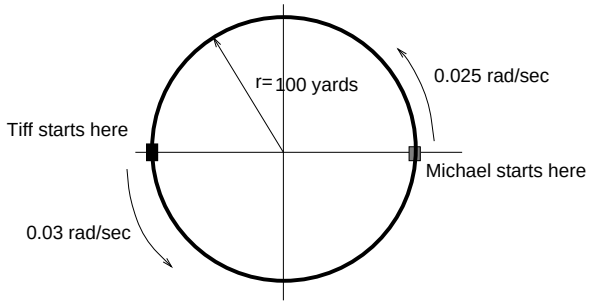
So, on the domain $-366 \leq t \leq 732$ we have these six solutions to the equation $14 = d(t)$:

$$t = -252.2, -135.8, 113.8, 230.2, 479.8, 596.2.$$

To conclude the problem, we only are interested in solutions in the domain $0 \leq t \leq 366$, so the answers are $t = 113.8, 230.2$; i.e. on days 114 and 230 there will be about 14 hours of daylight. □

Problems

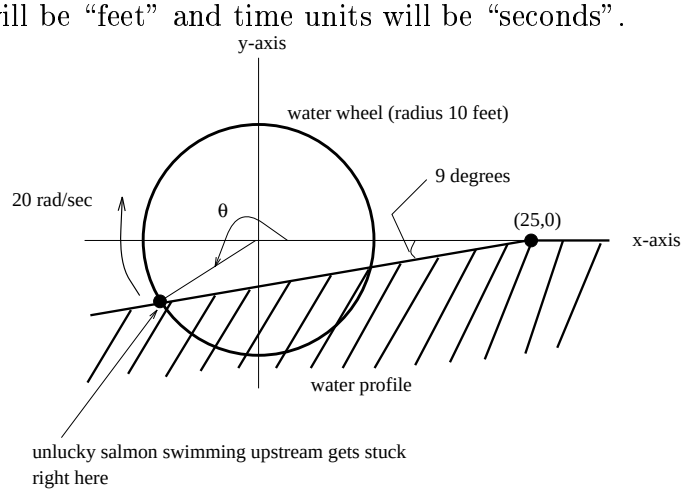
1. Let's make sure we can handle the symbolic and mechanical aspects of working with the inverse trigonometric functions:
   (a) Set your calculator to "radian mode" and compute to four decimal places:
      (i) $\sin^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2$.
      (ii) $\cos^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2$.
      (iii) $\tan^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2$.
   (b) Redo part (a) with your calculator set in "degree mode".
   (c) Find four solutions of $\tan(2x^2 + x - 1) = 5$. Use a graphing device to interpret your four solutions.
   (d) Solve for x: $\tan^{-1}(2x^2 + x - 1) = 0.5$
2. (a) If $y = \sin(x)$ on the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 2\sin(3x - 1) + 3$? How many solutions does the equation $4 = 2\sin(3x - 1) + 3$ have on the domain D and what are they?
   (b) If $y = \sin(t)$ on the domain $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 8\sin(\frac{2\pi}{1.2}(t-0.3))+18$. How many solutions does the equation $22 = 8\sin(\frac{2\pi}{1.2}(t-0.3))+18$ have on the domain D and what are they?
3. For each of the functions below:
   A. Sketch the function and show the horizontal line where $y =$ constant.
   B. Find the principal solution and the symmetry solution.
   C. Indicate the other solutions on your graph and describe their relationship to the principal and symmetry solutions.
   (a) $y = \sin\left(x + \frac{\pi}{6}\right)$, $y = -1$.
   (b) $y = \sin(2x - \pi)$, $y = \frac{1}{4}$.
4. Tiffany and Michael begin running around a circular track of radius 100 yards. They start at the locations pictured. Michael is running 0.025 rad/sec counterclockwise and Tiffany is running 0.03 rad/sec counterclockwise. Impose coordinates as pictured.



(a) Where is each runner located (in xy-coordinates) after 8 seconds?
(b) How far has each runner traveled after 8 seconds?
(c) Find the angle swept out by Michael after t seconds.
(d) Find the angle swept out by Tiffany after t seconds.
(e) Find the xy-coordinates of Michael and Tiffany after t seconds.
(f) Find the first time when Michael's x-coordinate is -50.
(g) Find the first time when Tiffany's x-coordinate is -50.
(h) Find when Tiffany passes Michael the first time.

5. Assume that the number of hours of daylight in New Orleans in 1994 is given by the function $D(x) = \frac{7}{3}\sin\left(\frac{2\pi}{365}x\right) + \frac{35}{3}$, where x represents the number of days after March 21.
   (a) Find the number of hours of daylight on January 1, May 18 and October 5.
   (b) On what days of the year will there be approximately 10 hours of daylight?

6. Flowing water causes a water wheel to turn in a CLOCKWISE direction. The wheel has radius 10 feet and has an angular speed of $\omega = -20$ rad/sec. Impose a coordinate system with the center of the wheel as the origin. The profile of the flowing water is pictured below. At time $t = 0$, an unlucky salmon gets caught in the wheel at the pictured location. Distance units will be "feet" and time units will be "seconds".



(a) How long does it take the salmon to complete one revolution?
(b) What is the salmon's linear speed?
(c) What is the equation of the line modeling the profile of the flowing water?
(d) Where (xy-coordinates) does the salmon get stuck on the wheel?
(e) What is the initial angle θ pictured?

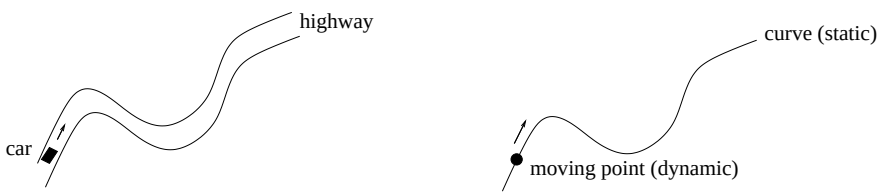
(f) Let $P(t) = (x(t), y(t))$ be the location of the salmon at time t seconds; find the formulas for $x(t)$ and $y(t)$.
(g) Where (xy-coordinates) is the salmon located after 0.12 seconds?
(h) When is the salmon first 4 feet to the right of the y-axis?

7. The air temperature (F°) above a frozen lake during the month of June is given by $L(t) = 20\sin(\frac{\pi}{12}t - \frac{5\pi}{6}) + 45$, where t represents hours elapsed since 12:00 a.m. June 1. The lake ice will begin to break up after 100 hours of 60° thaw time; you can only count time periods when the temperature is at least 60°. When will the ice break up?

Chapter 4

Parametized Curves

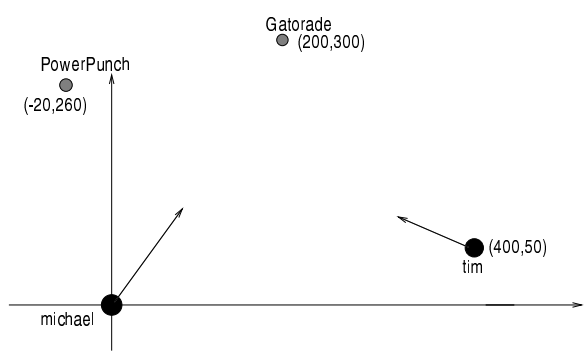
Imagine a car is traveling along the highway and you look down at the situation from high above:



We can adopt at least two different viewpoints: We can focus on the entire highway all at once, which is modeled by a curve in the plane; this is a "static viewpoint". We could study the movement of the car along the highway, which is modeled by a point moving along the curve; this is a "dynamic viewpoint". The ideas in this chapter are "dynamic", involving motion along a curve in the plane; in contrast, our previous work has tended to involve the "static" study of a curve in the plane. We will combine our understanding of linear functions, quadratic functions and circular functions to explore a variety of dynamic problems.

4.1 Parametric Equations

After a vigorous soccer match, Tim and Michael decide to have a glass of their favorite refreshment. They each run in a straight line along the indicated paths at a speed of 10 ft/sec. Will Tim and Michael collide?



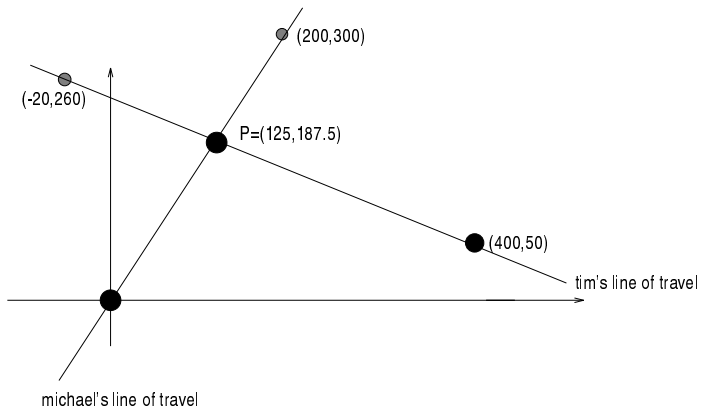
As a first step, we can model the lines along which both Tim and Michael will travel:

$$\text{Michael's line of travel:} \qquad f(x) = \frac{3}{2}x$$

$$\text{Tim's line of travel:} \qquad g(x) = -\frac{1}{2}(x - 400) + 50.$$

It is an easy matter to determine where these two lines cross: Set $f(x) = g(x)$ and solve for x, getting $x = 125$, so the lines intersect at $P = (125, 187.5)$.

Unfortunately, we have NOT yet determined if the runners collide. The difficulty is that we have found where the two lines of travel cross, but we have not worried about the individual locations of Michael and Tim along the lines of travel. In fact, if we compute the distance from the starting point of each person to P, we find:

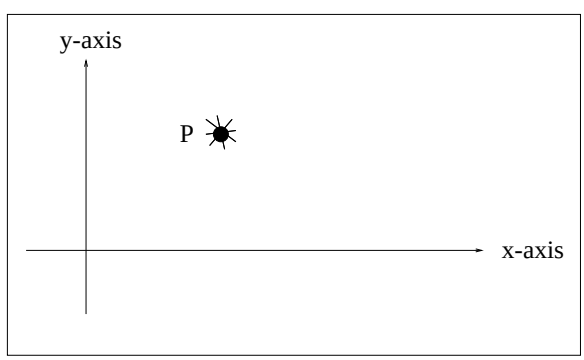


$$\begin{aligned} \text{dist(Mike},P) &= \sqrt{(0-125)^2 + (0-187.5)^2} = 225.35\,\text{feet} \\ \text{dist(Tim},P) &= \sqrt{(400-125)^2 + (50-187.5)^2} = 307.46\,\text{feet} \end{aligned}$$

Since these distances are different and both runners have the same speed, Tim and Michael do not collide!

Motivation: Keeping track of a bug.

Imagine a bug is located on the desktop. How can you best study its motion as time passes?



Let's denote the location of the bug when you first observed it by P. If we let t represent time elapsed since first spotting the bug (say in units of seconds), then we can let $P(t)$ be the *new* location of the bug at time t. When $t = 0$, which is the instant you first spot the bug, the location $P(0) = P$ is the initial location. For example, the path followed by the bug might look something like the dashed path in the next Figure; we have indicated the bug's explicit position at four future times: $t_1 < t_2 < t_3 < t_4$.

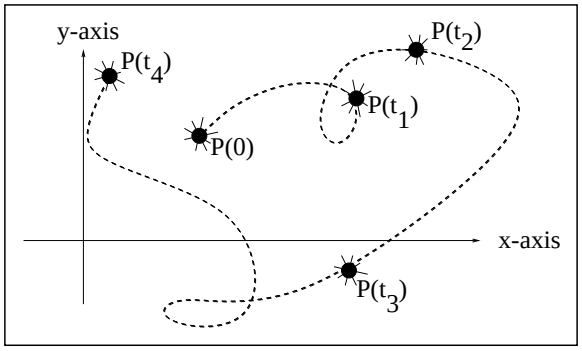


Figure 4.1.1: The path of the bug

How can we describe the curve in Figure 4.1.1 ?

To start, lets define a couple of *new* functions. Given a time t, we have the point $P(t)$ in the plane, so we can define:

$$\begin{aligned} x(t) &= x\text{-coordinate of } P(t) \text{ at time } t, \\ y(t) &= y\text{-coordinate of } P(t) \text{ at time } t. \end{aligned}$$

In other words, the point $P(t)$ is described as

$$P(t) = (x(t), y(t)).$$

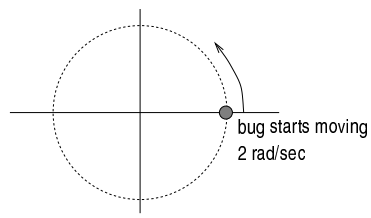
We usually call $x = x(t)$ and $y = y(t)$ the *coordinate functions* of $P(t)$. Also, it is common to call the pair of functions $x = x(t)$ and $y = y(t)$ the *parametric equations* for the curve. Anytime we describe a curve using parametric equations, we usually call it a *parametrized curve.*

Given parametric equations $x = x(t)$ and $y = y(t)$, the *domain* will be the set of t values we are allowed to plug in. Notice, we are using the same set of t-values to plug into both of the equations. Describing the curve in Figure 4.1.1 amounts to finding the parametric equations $x(t)$ and $y(t)$. In other words, we typically want to come up with "formulas" for the functions $x(t)$ and $y(t)$. Depending on the situation, this can be easy or very hard.

Examples of Parametrized Curves.

We have already worked with some interesting examples of parametric equations.

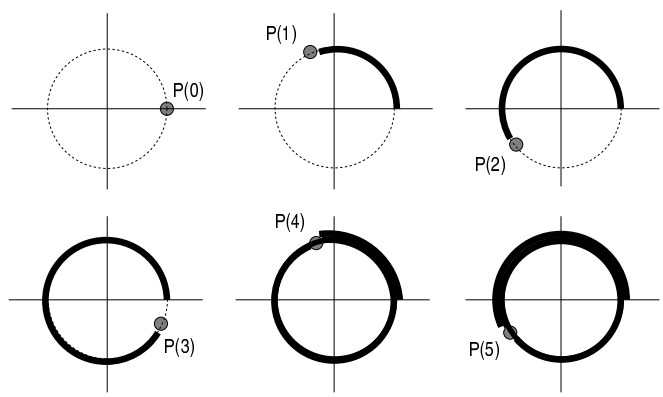
Example 4.1.1: *A bug begins at the location (1,0) on the unit circle and moves counterclockwise with an angular speed of $\omega = 2$ rad/sec. What are the parametric equations for the motion of the bug during the first 5 seconds? Indicate, via "snapshots", the location of the bug at 1 second time intervals.*



Solution. We can use (3.2.2) to find the angle swept out after t seconds: $\theta = \omega t = 2t$ radians. The parametric equations are now easy to describe:

$$\begin{aligned} x = x(t) &= \cos(2t) \\ y = y(t) &= \sin(2t). \end{aligned}$$

If we restrict t to the domain $[0, 5]$, then the location of the bug at time t is given by $P(t) = (\cos(2t), \sin(2t))$. We locate the bug via six one-second snapshots: □

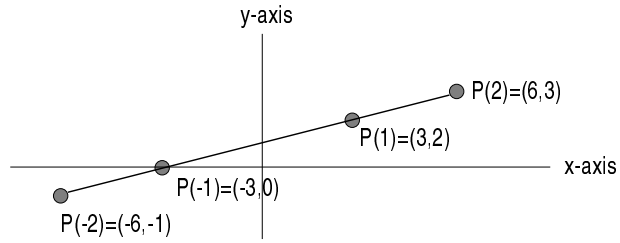


When modeling motion along a curve in the plane, we would typically be given the curve and try to find the parametric equations. We can turn this around: Given a pair of functions $x = x(t)$ and $y = y(t)$, let

$$P(t) = (x(t), y(t)), \tag{4.1.2}$$

which assigns to each input t a point in the xy-plane. As t ranges over a given domain of allowed t values, we will obtain a collection of points in the plane. We refer to this as the *graph* of the parametric equations $\{x(t), y(t)\}$. Thus, we have now described a process which allows us to obtain a picture in the plane given a pair of equations in a common single variable t. Again, we call curves that arise in this way *parametrized curves*. The terminology comes from the fact we are describing the curve using an auxiliary variable t, which is called the describing "parameter". In applications, t often represents time.

Example 4.1.3: *The graph the parametric equations $x(t) = 3t, y(t) = t + 1$ on the domain $-2 \leq t \leq 2$ is pictured; it is a line segment. As we let t increase from -2 to 2, we can observe the motion of the corresponding points on the curve.*



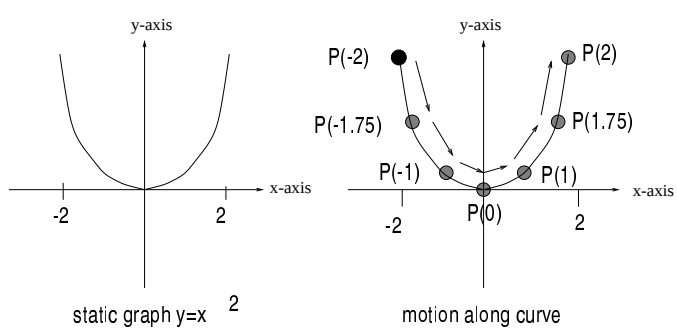
Function graphs.

It is important to realize that the graph of every function can be thought of as a parametrized curve. Here is the reason why: Given a function $y = f(x)$, recall the graph consists of points $(x, f(x))$, where x runs over the allowed domain values. If we define

$$\begin{aligned} x = x(t) &= t \\ y = y(t) &= f(t), \end{aligned}$$

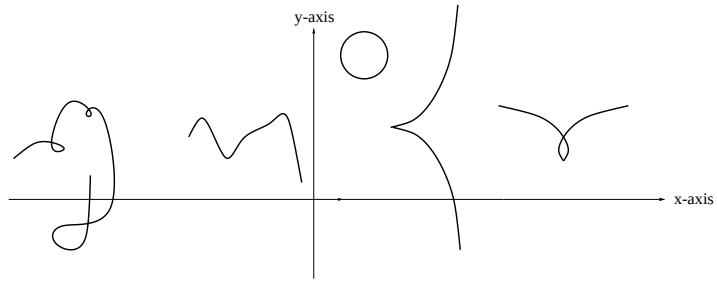
then plotting the points $P(t) = (x(t), y(t)) = (t, f(t))$ gives us the graph of f. We gain one important thing with this new viewpoint: Letting $x = t$ increase in the domain, we now have the ability to dynamically view a point $P(t)$ moving along the function graph. Here is an example of how this all works:

Example 4.1.4: *Consider the function $y = x^2$ on the domain $-2 \leq x \leq 2$. As a parametrized curve, we would view the graph of $y = x^2$ as all points of the form $P(t) = (t, t^2)$, where $-2 \leq t \leq 2$. If t increases from -2 to 2, the corresponding points $P(t)$ move along the curve as pictured:*



For example, $P(-2) = (-2, (-2)^2) = (-2, 4)$, $P(1) = (1, 1^2) = (1, 1)$, etc.

Not every parametrized curve is the graph of a function. For example, consider these possible curves in the plane: The second curve from the left is the graph of a function; the other curves violate the vertical line test.



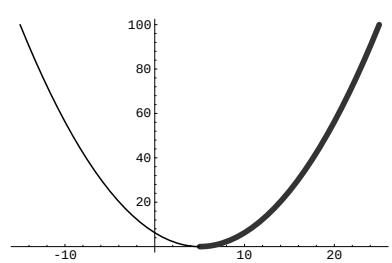
A useful trick.

There is an approach to understanding a parametrized curve which is sometimes useful: Begin with the equation $x = x(t)$. Solve the equation $x = x(t)$ for t in terms of the single variable x; i.e., obtain $t = g(x)$. Then substitute $t = g(x)$ into the other equation $y = y(t)$, leading to an equation involving only the variables x and y. If we were given the allowed t values, we can use the equation $x = x(t)$ to determine the allowed x values, which will be the domain of x values for the function $y = y(g(x))$. This may be a function with which we are familiar or can plot using available software.

Example 4.1.5: *Start with the parametrized curve given by the equations $x = x(t) = 2t + 5$ and $y = y(t) = t^2$, when $0 \le t \le 10$. Find a function $y = f(x)$ whose graph gives this parametrized curve.*

Solution. Following the suggestion, we begin by solving $x = 2t + 5$ for t, giving $t = \frac{1}{2}(x - 5)$. Plugging this into the second equation gives $y = (\frac{1}{2}(x-5))^2 = (\frac{1}{4})(x-5)^2$. Conclude that (x, y) is on the parametrized curve if and only if the equation $y = f(x) = (\frac{1}{2}(x-5))^2 = (\frac{1}{4})(x-5)^2$ is satisfied. This is a quadratic function, so the graph will be an upward opening parabola with vertex (5,0).

Since the t domain is $0 \le t \le 10$, we get a new inequality for the x domain: $0 \le \frac{1}{2}(x - 5) \le 10$. Solving this, we get $0 \le x - 5 \le 2(10)$, so $5 \le x \le 25$. This means the graph of the parametrized curve is the graph of the function $y = (\frac{1}{4})(x-5)^2$, with the domain of x values $[5, 25]$. Here is a plot of the graph of $y = f(x)$; the thick portion is the parametrized curve we are studying. □



<u>Circular motion</u>.

We can describe the motion of an object around a circle using parametric equations. This will involve the trigonometric functions.

The general setup to imagine is pictured: An object moving around a circle of radius r centered at a point (x_c, y_c) in the xy-plane. The path traced out is the circle. However, the location of the object at time t will depend on a number of things:

- The starting location P of the object;
- The angular speed ω of the object;
- The radius r and the center (x_c, y_c).

We will build up to the general solution by considering two cases, the first being a special case of the second.

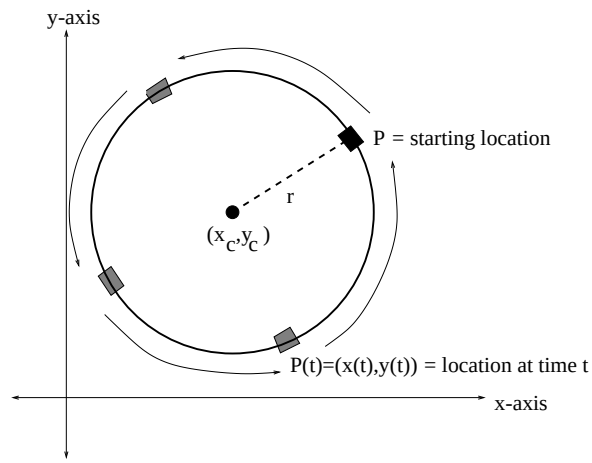
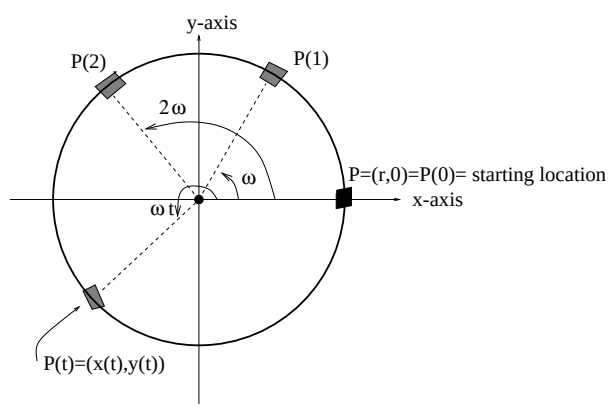


Figure 4.1.2: Circular motion

Standard circular motion.

As a first case to consider, assume that the center of the circle is $(0, 0)$ and the starting location $P = (r, 0)$, as pictured below. If the angular speed is ω, then the angle θ swept out in time t will be $\theta = \omega t$; this requires that the time units in ω agree with the time units of t! We denote by $P(t) = (x(t), y(t))$ the xy-coordinates of the object at time t. At time t, we can compute the coordinates of $P(t) = (x(t), y(t))$ using the circular functions:

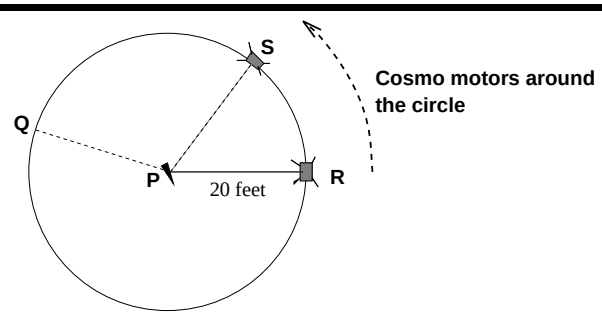
$$\begin{aligned} x &= x(t) = r\cos(\omega t) \\ y &= y(t) = r\sin(\omega t). \end{aligned}$$



This parametrizes motion starting at $P = (r, 0)$. Using the shifting technology of §2.5, we are led to a general description of this type of circular motion, which involves a circle of radius r centered at a point (x_c, y_c); we refer to this situation as *standard circular motion*.

Standard Circular Motion 4.1.6: *Assume an object is moving around a circle of radius r centered at (x_c, y_c) with a constant angular speed of ω. Assume the object begins at $P = (x_c + r, y_c)$. Then the location of the object at time t is given by the parametric equations: $x = x(t) = x_c + r\cos(\omega t)$ and $y = y(t) = y_c + r\sin(\omega t)$.*

Example 4.1.7: *Cosmo the dog is tied to a 20 foot long tether, as in Figure 3.2.1. Assume Cosmo starts at the location "R" in the Figure and maintains a tight tether, moving around the circle at a constant angular speed $\omega = \frac{\pi}{5}$ radians/second. Parametrize Cosmos motion and determine where the dog is located after 3 seconds and after 3 minutes.*



-TOP VIEW-

Solution. Impose a coordinate system so that the pivot point of the tether is $(x_o, y_o) = (0, 0)$. Since $\omega > 0$, Cosmo is walking counterclockwise around the circle. By (4.1.6), the location of Cosmo after t seconds is $P(t) = (x(t), y(t)) = (20\cos(\frac{\pi}{5}t), 20\sin(\frac{\pi}{5}t))$. After 3 seconds, Cosmo is located at $P(3) = (20\cos(\frac{3\pi}{5}), 20\sin(\frac{3\pi}{5})) = (-6.18, 19.02)$. After 3 minutes = 180 seconds, the location of the dog will be $P(180) = (20\cos(\frac{180\pi}{5}), 20\sin(\frac{180\pi}{5})) = (20, 0)$, which is the original starting point. □

General circular motion.

The circular motion of an object can begin at any location P on the circle. To handle the general case, we follow an earlier idea and introduce an auxiliary *relative coordinate system*: The $x_{rel}y_{rel}$-coordinates are obtained by drawing lines parallel to the xy-axis and passing through (x_c, y_c). We are using the subscript "rel" to stand for "relative". This new relative coordinate system has origin (x_c, y_c) and allows us to define the *initial angle* θ_o, which indexes the starting location P, as pictured below:

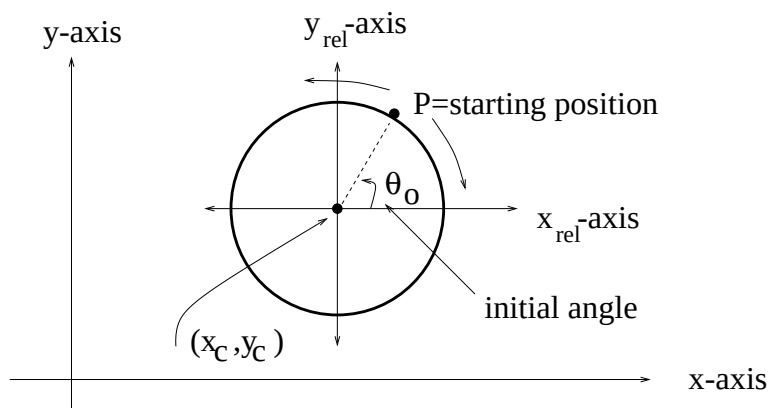


Figure 4.1.3: Initial angle and auxiliary axis

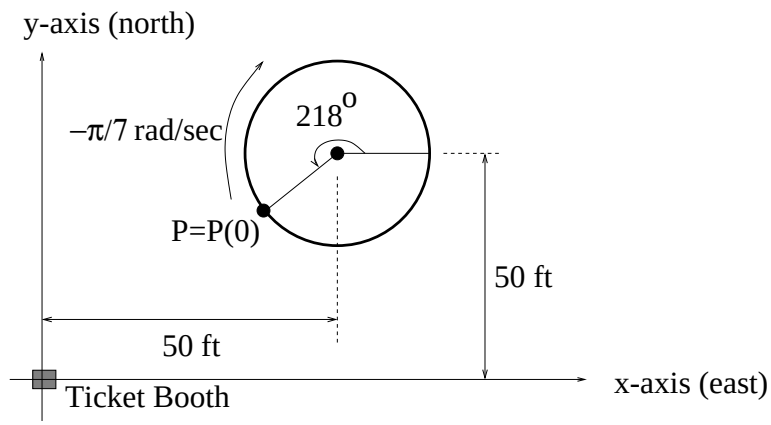
Assume the object starts at P and is moving at a constant angular speed ω around the pictured circle of radius r. Then after time t has elapsed, the location of the object is indexed by sweeping out an angle ωt, **starting from** θ_o. In other words, the location after t units of time is going to be determined by the central standard angle $\theta_o + \omega t$ with initial side the positive x_{rel}-axis. This means that if $P(t) = (x(t), y(t))$ is the location of the object at time t,

$$\begin{aligned} x &= x(t) = x_c + r\cos(\theta_o + \omega t) \\ y &= y(t) = y_c + r\sin(\theta_o + \omega t). \end{aligned}$$

Notice, the case of standard circular motion is just the scenario when $\theta_o = 0$ and these parametric equations collapse to those of (4.1.6).

General Circular Motion 4.1.8: *Assume an object is moving around a circle of radius r centered at (x_c, y_c) with a constant angular speed of ω. Assume the object begins at the location P with initial angle θ_o, as in Figure 4.1.3. The object location at time t is given by: $x = x(t) = x_c + r\cos(\theta_o + \omega t)$ and $y = y(t) = y_c + r\sin(\theta_o + \omega t)$.*

Example 4.1.9: *A rider jumps on a merry-go-round of radius 20 feet at the pictured location. The ride rotates at the constant angular speed of* $\omega = -\frac{\pi}{7}$ *radians/second. The center of the platform is located 50 feet East and 50 feet North of the ticket booth for the ride. What are the parametric equations describing the location of the rider? Where is the rider after 18 seconds have elapsed? How far from the ticket booth is the rider after 18 seconds have elapsed?*



Solution. In this example, since $\omega < 0$, the rotation is clockwise. Since the angular speed is given in radians, we need to convert the initial angle to radians as well: $218^o = 3.805$ radians. Impose a coordinate system so that the center of the ride is (50,50) and its radius is 20 feet. By (4.1.8), the parametric equations for the rider are given by $x = x(t) = 50 + 20\cos(3.805 - \frac{\pi}{7}t)$ and $y = y(t) = 50 + 20\sin(3.805 - \frac{\pi}{7}t)$. The location after 18 seconds will be

$$P(18) \quad = \quad (x(18), y(18)) = (50 + 20\cos(-4.273), 50 + 20\sin(-4.273)) = (41.49, 68.10).$$

The distance from $P(18)$ to the origin is

$$d \quad = \quad \sqrt{(41.49)^2 + (68.10)^2} = 79.74 \text{ feet}.$$

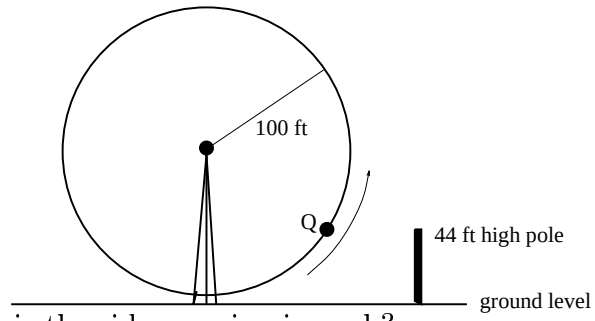
□

Problems

1. Return to Example 4.1.1. What are the coordinates of the bug in the six "snapshots" in the solution? When will the bug first cross into the second quadrant? When will the bug first have x-coordinate=-0.2?.
2. Sketch the curve represented by the parametric equations. Choose four specific t values and indicate the corresponding points on the curve. As t moves from left to right in the domain, indicate how the corresponding points on the curve are moving.
   (a) $x(t) = 1, y(t) = 4 - 2t, 0 \leq t \leq 5$.

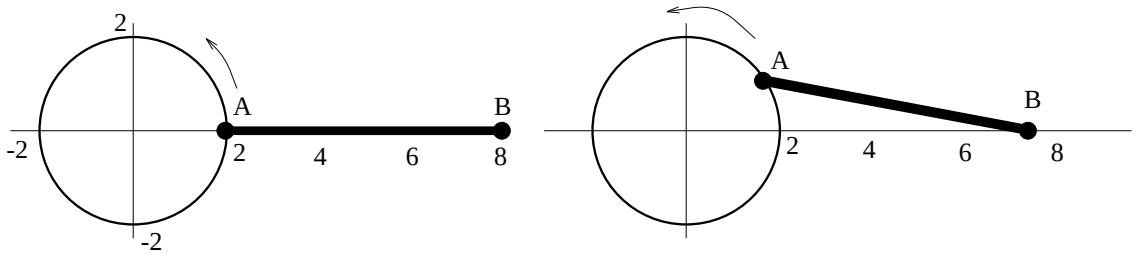
(b) $x(t) = t - 1, y(t) = 4, 0 \leq t \leq 5$.
(c) $x(t) = t - 1, y(t) = 4 - 2t, 0 \leq t \leq 5$.
(d) $x(t) = 4 - t, y(t) = 2t - 6, 0 \leq t \leq 5$.

3. A ferris wheel of radius 100 feet is rotating at a constant angular speed ω counterclockwise. Using a stopwatch, the rider finds it takes 3.4 seconds to go from the lowest point on the ride to a point Q, which is level with the top of a 44 ft pole. Assume the lowest point of the ride is 3 feet above ground level.



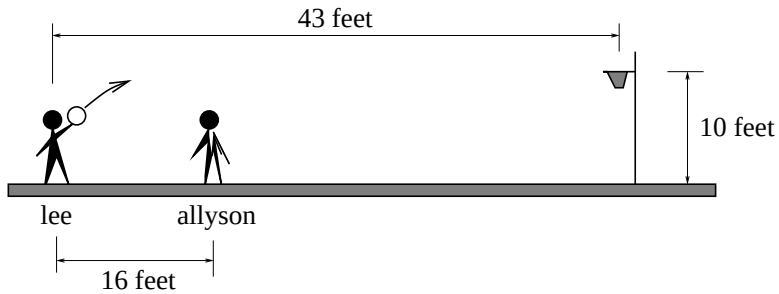
(a) What is the angular speed ω? How fast is the rider moving in mph?
(b) Find parametric equations for the motion of a rider on the wheel, assuming the rider begins at the lowest point on the wheel.
(c) Due to a malfunction, the ride abruptly stops 35 seconds after it began. To the nearest foot, how high above the ground is the rider?

4.* A six foot long rod is attached at one end A to a point on a wheel of radius 2 feet, centered at the origin. The other end B is free to move back and forth along the x-axis. The point A is at $(2, 0)$ at time $t = 0$, and the wheel rotates counterclockwise at constant speed with a period of 3 seconds.



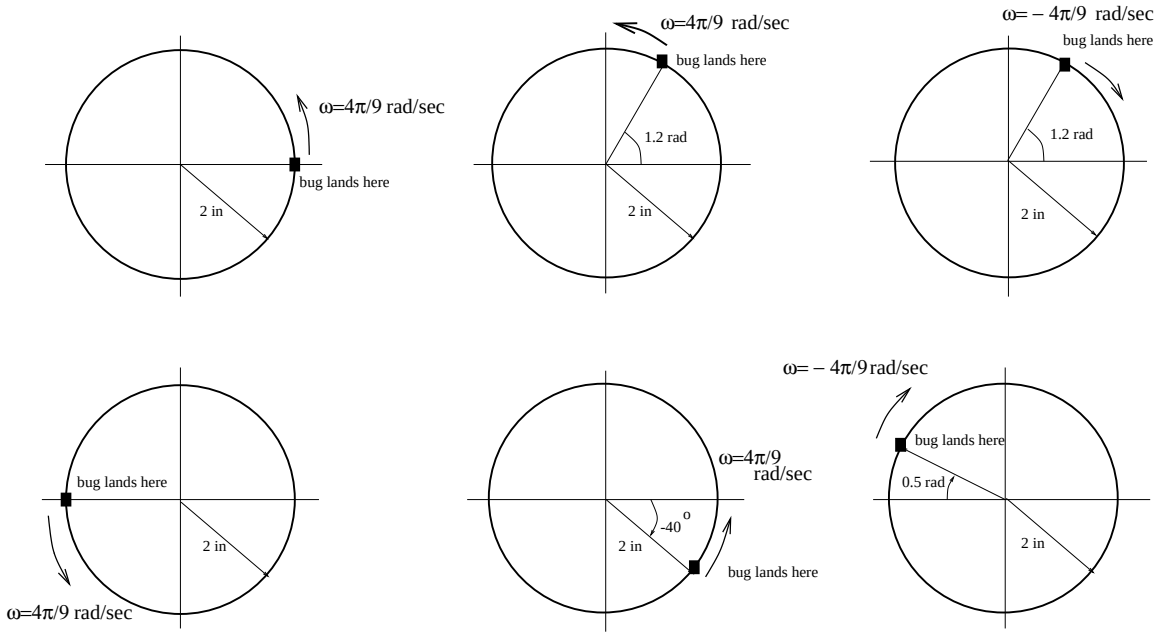
(a) As the point A makes one complete revolution, indicate in the picture the direction and range of motion of the point B.
(b) Find the coordinates of the point A as a function of time t.
(c) Find the coordinates of the point B as a function of time t.
(d) What is the x-coordinate of the point B when $t = 1$?
(e) Find the first two times when the x coordinate of the point B is 5.

5. Lee has been boasting all year about his basketball skill and Allyson can't stand it anymore. So, she has challenged Lee to a game of one-on-one. With 2 seconds to go, Allyson is leading 14-12 and Lee fires a desperation three. Impose coordinates with Lee's feet at the origin and use units of feet on each axis. The path of the ball is described by the parametric equations: $x(t) = 28.925t, y(t) = -16t^2 + 34.472t + 6$.



(a) How high is the ball when Lee fires his shot?
(b) When is the ball directly over Allyson?
(c) Allyson can jump and block a ball 9.5 feet above the floor. Can Allyson block Lee's shot?
(d) Where and when is the ball 22 feet above the floor?
(e) Where and when does the ball reach it's highest point above the floor?
(f) Who wins the game?
(g) Solve the equation $x = x(t)$ for t in terms of x, then plug this into the equation for $y = y(t)$. Sketch the graph of the resulting function $y = f(x)$; this is the path of Lee's shot.

6. In the pictures below, a bug has landed on the rim of a jelly jar and is moving around the rim. The location where the bug initially lands is described and its angular speed is given. Impose a coordinate system with the origin at the center of the circle of motion. In each of the cases, find parametric equations describing the location of the bug at time t seconds.

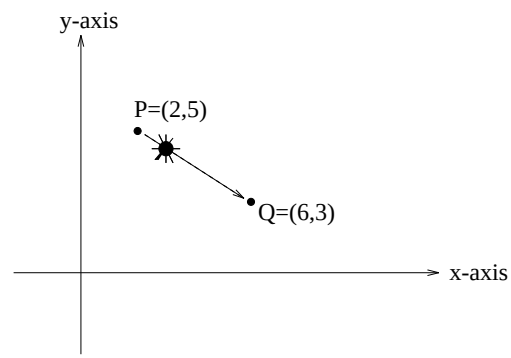


4.2 Linear Motion

The simplest example of a parametrized curve arises when studying the motion of an object along a straight line in the plane. We will start by studying this kind of motion when the starting and ending locations are known.

Motion of a Bug.

Example 4.2.1: *A bug is spotted at $P = (2, 5)$ in the xy-plane. The bug walks in a straight line from P to $Q = (6, 3)$ at a constant speed s. It takes the bug 5 seconds to reach Q. Assume the units of our coordinate system are feet. What is the speed s of the bug along the line connecting P and Q? Compute the horizontal and vertical speeds of the bug and show they are both constant.*



Solution. A standard technique in motion problems is to analyze the x and y-motion separately. This means we look at the projection of the bug location onto the x and y-axis separately, studying how each projection moves. We can think of these projections as "shadows" cast by a flashlight onto the two axes:

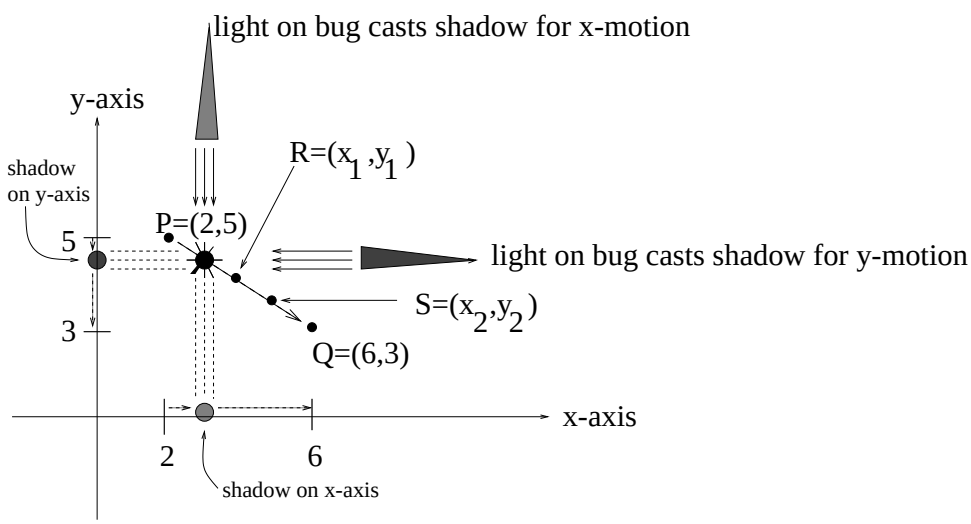


Figure 4.2.1: How to view motion in the coordinates

For the x-motion, we study the "shadow" on the x-axis which starts at "2" and moves toward "6" on the x-axis. For the y-motion, we study the "shadow" on the y-axis which starts at "5" and moves toward "3" along the y-axis.

In general, speed is computed by dividing distance by time elapsed, so

$$s = \frac{\text{dist}(P,Q)}{5} \text{ ft/sec} = \frac{\sqrt{(5-3)^2+(2-6)^2}}{5} \text{ ft/sec} = \frac{2\sqrt{5}}{5} \text{ feet/sec.} \tag{4.2.2}$$

This is the speed of the bug along the line connecting P and Q

The hard part of this problem is to show that the speed of the horizontal and vertical shadows are also constant. This might seem obvious when you first think about it. In order to actually show it, let's take two intermediary positions $R = (x_1, y_1)$ and $S = (x_2, y_2)$ along the bugs path. We are going to relate the *horizontal speed* v_x of the bug between x_1 and x_2, the *vertical speed* v_y of the bug between y_1 and y_2 and the speed s of the bug from R to S. Actually, because there are $\pm$ directions for the x and y axes, we will allow horizontal and vertical "speed" to be a $\pm$ quantity, with the obvious meaning. If it takes T seconds for the bug to travel from R to S, then T is the elapsed time for the horizontal motion from x_1 to x_2 and also the elapsed time for the vertical motion from y_1 to y_2. The horizontal speed v_x is the *directed distance* $\Delta x = (x_2 - x_1)$ divided by the time elapsed T, whereas the vertical speed v_y is the *directed distance* $\Delta y = (y_2 - y_1)$ divided by the time elapsed T.

We want to show that v_x and v_y are both constants!

To do this, we have these three equations:

$$\begin{aligned} v_x &= \frac{\Delta x}{T} = \frac{x_2 - x_1}{T} \\ v_y &= \frac{\Delta y}{T} = \frac{y_2 - y_1}{T} \\ s &= \frac{\textit{distance}(R,S)}{T} = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{T} = \frac{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}{T}. \end{aligned}$$

Now, square each side of the three equations and combine them to conclude: $s^2T^2 = v_x^2T^2 + v_y^2T^2$. We can multiply through by T^2 and that gives us the key equation:

$$s^2 = v_x^2 + v_y^2. \tag{4.2.3}$$

On the other hand, the ratio of the vertical and horizontal speed gives

$$\begin{aligned} \frac{v_y}{v_x} &= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{"slope of line connecting } P \text{ and } Q\text{"} = -\frac{1}{2} \qquad (4.2.4) \\ v_y &= -\frac{1}{2}v_x. \end{aligned}$$

Since v_x is positive,

$$\begin{aligned} s &= \sqrt{v_x^2 + v_y^2} = \sqrt{v_x^2 + (-\frac{1}{2}v_x)^2} = \frac{\sqrt{5}}{2}|v_x| = \frac{\sqrt{5}}{2}v_x; \text{i.e.,} \\ v_x &= \frac{2}{\sqrt{5}}s = \frac{2}{\sqrt{5}}(\frac{2\sqrt{5}}{5}) = \frac{4}{5} = 0.8 \ \text{ feet/sec.} \end{aligned}$$

By (4.2.4), $v_y = (-\frac{1}{2})\frac{4}{5} = -0.4$ feet/sec. This shows the speed in both the horizontal and vertical directions is constant as the bug moves from R to S. Since R and S were *any* two intermediary points between P and Q, the horizontal and vertical bug speeds are constant. □

In Problem 4.2(a), we will look at a very simple example of motion in the plane where the speed along a curve is constant, but the speed in the horizontal and vertical directions is NOT constant. This will highlight the very special nature of linear motion, where things work out so nicely.

Example 4.2.5: *Parametrize the motion in the previous problem.*

Solution. We have shown $v_x = 0.8$ feet/sec and $v_y = -0.4$ feet/sec, so

$$\begin{aligned} x(t) &= (x\text{-coordinate of the bug at time } t) \\ &= (\text{beginning } x\text{-coordinate}) + (\text{distance traveled in } x\text{-direction in } t \text{ seconds}) \\ &= 2 + 0.8t. \end{aligned}$$

$$\begin{aligned} y(t) &= (y\text{-coordinate of the bug at time } t) \\ &= (\text{beginning } y\text{-coordinate}) + (\text{distance traveled in } y\text{-direction in } t \text{ seconds}) \\ &= 5 - 0.4t. \end{aligned}$$

As a check, notice that $P(0) = (2 + (0.8)0, 5 - (0.4)0) = (2, 5)$ is the starting location and $P(5) = (2 + 5(0.8), 5 - 5(0.4)) = (6, 3)$ is the ending location. □

General Setup.

Given two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane, we can study motion of an object along the line connecting P and Q. In so doing, you need to first specify the starting location and the ending location of the object; lets say we start at P and proceed to Q. Fix the distance units used in the coordinate system (feet, inches, miles, meters, etc.) and the time units used (seconds, hours, years, etc.). As highlighted in the solution of (4.2.5), the key is to analyze the x-motion and y-motion separately.

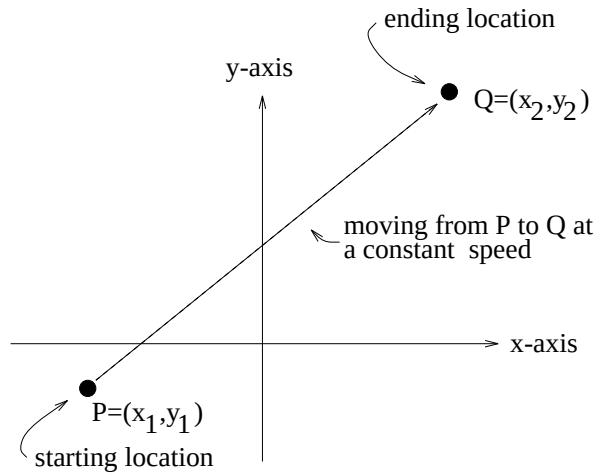


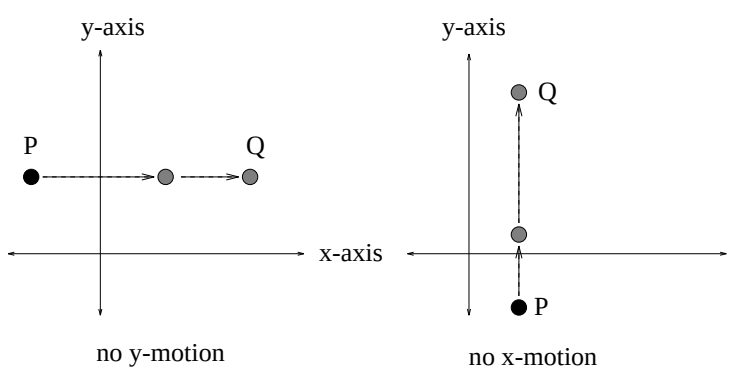
Figure 4.2.2: Typical scenario

We will be imposing an assumption that the speed along the line connecting P and Q in Figure 4.2.1 is a constant. There is a crucial observation we need to make, a special case of which was the content of (4.2.1).

Linear Motion 4.2.6: *Assume that an object moves from P to Q along a straight line at a constant speed s, as in Figure 4.2.2. Then the speed v_x in the x-direction and the speed v_y in the y-direction are both constant. We also have two useful formulas:*

$$\begin{aligned} s^2 &= v_x^2 + v_y^2, \\ \frac{v_y}{v_x} &= \textit{slope of line of travel, when the line is non-vertical.} \end{aligned}$$

This fact is established using the same reasoning as in Example 4.2.1. Let's make a few comments. To begin with, if the line of travel is either vertical or horizontal, then either $v_x = 0$ or $v_y = 0$ and (4.2.6) isn't really saying anything of interest.



For any other line of travel, we can use the reasoning used in Example 4.2.1. Pay attention that the *horizontal speed* v_x and the *vertical speed* v_y are both *directed quantities*; i.e. these can be positive or negative. The sign of v_x will indicate the direction of motion: If v_x is positive, then the horizontal motion is to the right and if v_x is negative, then the horizontal motion is to the left. Similarly, the sign of v_y tells us if the vertical motion is upward or downward.

!!! CAUTION !!! The formulas in (4.2.6) only work if we are talking about linear motion. A problem at the end of this section illustrates how the conclusion can fail for nonlinear motion problems. !!! CAUTION !!!

Returning to Figure 4.2.2, to describe the x-motion, two pieces of information are needed: the starting location (in the x-direction) and the constant speed v_x in the x-direction. So,

$$\begin{aligned} x(t) &= (x\text{-coordinate of the object at time } t) \\ &= (\text{beginning } x\text{-coordinate}) + (\text{distance traveled in } x\text{-direction in } t \text{ time units}) \\ &= x_1 + v_x \cdot t. \end{aligned}$$

If we are not given the horizontal velocity directly, rather the time T required to travel from P to Q, then we could compute v_x using the fact that the object starts at $x = x_1$ and travels to $x = x_2$:

$$\begin{aligned} v_x &= \frac{\text{directed horizontal distance traveled}}{\text{time required to travel this distance}} \\ &= \frac{\text{ending } x\text{-coordinate - starting } x\text{-coordinate}}{\text{time required to travel this distance}} = \frac{\Delta x}{T} = \frac{x_2 - x_1}{T}. \end{aligned}$$

To describe the y-motion in Figure 4.2.2, we proceed similarly. We will denote by v_y the constant vertical speed of the object, then after t time units the object has traveled $v_y \cdot t$ units. So,

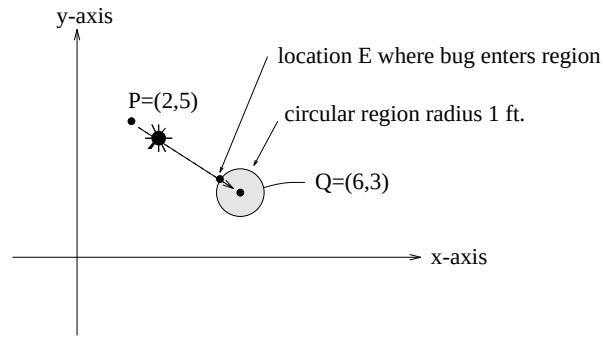
$$\begin{aligned} y(t) &= (y\text{-coordinate of object at time } t) \\ &= (\text{beginning } y\text{-coordinate}) + (\text{distance traveled in } y\text{-direction in } t \text{ time units}) \\ &= y_1 + v_y \cdot t. \end{aligned}$$

In summary,

Linear Motion 4.2.7: *Suppose an object begins at a point $P = (x_1, y_1)$ and moves at a constant speed s along a line connecting P to another point $Q = (x_2, y_2)$. Then the motion of this object will trace out a line segment which is parametrized by the equations:*

$$\begin{aligned} x &= x(t) = x_1 + v_x \cdot t, \\ y &= y(t) = y_1 + v_y \cdot t. \end{aligned}$$

Example 4.2.8: *Return to the linear motion problem studied in Example (4.2.1) and (4.2.5). However, now assume that the point $Q = (6, 3)$ is located at the center of a circular region of radius 1 ft. When and where does the bug enter this circular region?*



Solution. The parametric equations for the linear motion of the bug are given by:

$$x = x(t) = 2 + 0.8t, \quad y = y(t) = 5 - 0.4t.$$

The equation of the boundary of the circular region centered at Q is given by

$$(x - 6)^2 + (y - 3)^2 = 1.$$

To find where and when the bug crosses into the circular region, we determine where and when the linear path and the circle equation have a simultaneous solution. To find such a location, we simply plug $x = x(t)$ and $y = y(t)$ into the circle equation:

$$\begin{aligned} (x(t) - 6)^2 + (y(t) - 3)^2 &= 1 \\ (2 + 0.8t - 6)^2 + (5 - 0.4t - 3)^2 &= 1 \\ 16 - 6.4t + 0.64t^2 + 4 - 1.6t + .16t^2 &= 1 \\ 0.8t^2 - 8t + 19 &= 0. \end{aligned}$$

Notice, this is an equation in the single variable t. Finding the solution of this equation will tell us *when* the bug crosses into the region. Once we know *when* the bug crosses into the region, we can determine the location by plugging this time value into our parametric equations. By the quadratic formula, we find the solutions are

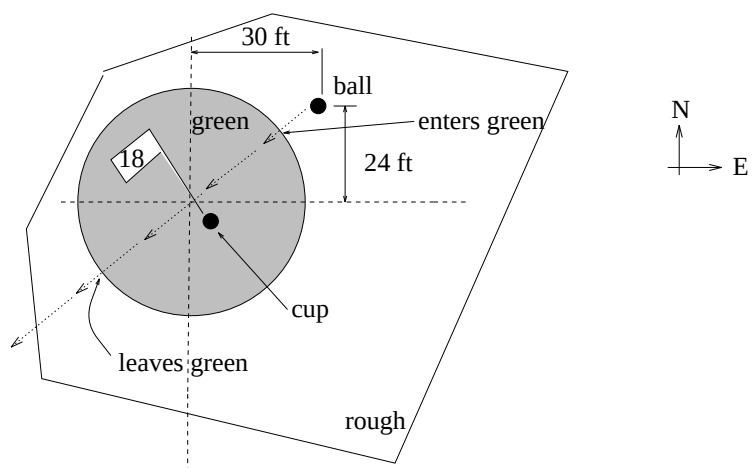
$$t = \frac{8 \pm \sqrt{64 - 4(19)(0.8)}}{1.6} = 6.12,\ 3.88.$$

We know that the bug reaches the point Q in 5 seconds, so the second solution $t = 3.88$ seconds is the time when the bug crosses into the circular region. (If the bug had continued walking in a straight line directly through Q, then the time when the bug leaves the circular region would correspond to the other solution $t = 6.12$ seconds.) Finally, the location E of the bug when it crosses into the region is $E = (x(3.88), y(3.88)) = (5.10, 3.45)$. □

Problems

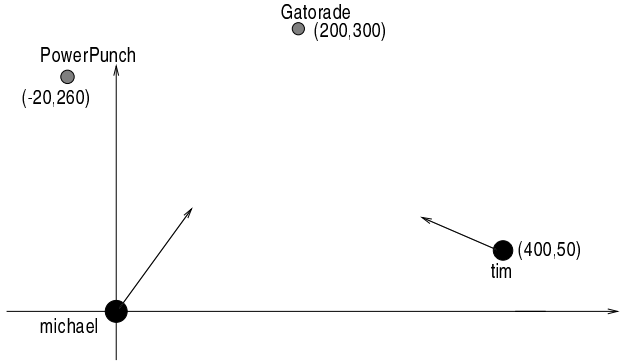
1. Return to Example 4.2.1, but assume the bug begins at Q and walks toward P. Again assume the bug walks at a constant speed, requiring 5 seconds to traverse the distance.
   (a) Describe the motion of the bug pictorially and via parametric equations.
   (b) Find the speed and direction of the bug along the line connecting Q to P? How about the horizontal and vertical speeds?
   (c) How far is the bug from the origin after 2 seconds?

2. The cup on the 18th hole of a golf course is located 4 feet south and 4 feet east of the center of a circular green which is 54 feet in DIAMETER. Your ball is located as in the picture below. Assume that you mistakenly putt the ball which travels at a constant speed in a straight line toward the center of the green. It takes 7 seconds to reach the center of the green. Use distance units of FEET and time units of SECONDS in this problem.



   (a) Find parametric equations for the motion of the ball.
   (b) How far is the ball from the cup after 5 seconds?
   (c) Where and when does the ball cross into the green and out of the green?
   (d) Find when and where the ball is closest to the cup. (Hint: Construct a line thru the cup perpendicular to the path of the ball, then find where these two lines intersect.)
   (e) Find a function of t that computes the distance from the ball to the cup at any time t.
   (f) When and where is the ball 8 feet from the cup?

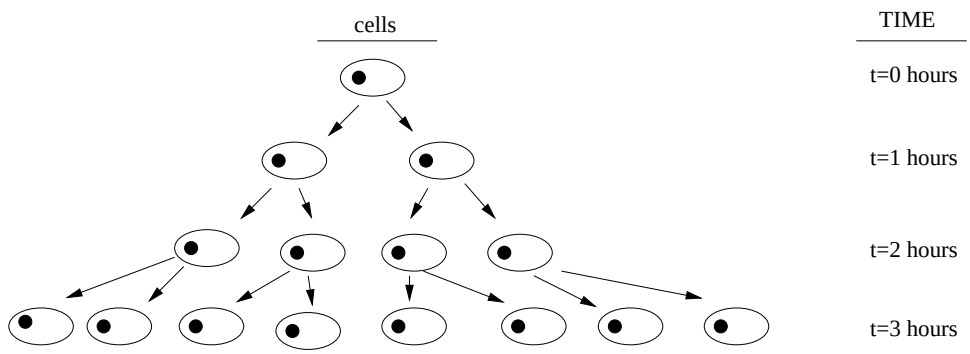
3. After a vigorous soccer match, Tim and Michael decide to have a glass of their favorite refreshment. They each run in a straight line along the indicated paths at a speed of 10 ft/sec. Parametrize the motion of Tim and Michael individually. Find when and where Tim and Michael are closest to one another; also compute this minimum distance.



Chapter 5

Exponential Functions

If we start with a single yeast cell under favorable growth conditions, then it will divide in one hour to form two identical "daughter cells". In turn, after another hour, each of these daughter cells will divide to produce two identical cells; we now have four identical "granddaughter cells" of the original parent cell. Under ideal conditions, we can imagine how this "doubling effect" will continue:



The question is this:

Can we find a function of t that will predict (i.e. model) the number of yeast cells after t hours?

If we tabulate some data (as at right), the conclusion is that the formula

$$N(t) = 2^t$$

predicts the number of yeast cells after t hours.

Total hours	Number of yeast cells
0	$1=2^0$
1	$2=2^1$
2	$4=2^2$
3	$8=2^3$
4	$16=2^4$
5	$32=2^5$
6	$64=2^6$

Now, lets make a very slight change. Suppose that instead of starting with a single cell, we begin with a population of 3×10^6 cells; a more realistic situation. If we assume that the population of cells will double every hour, then reasoning as above will lead us to conclude that the formula

$$N(t) = (3 \times 10^6)2^t$$

gives the population of cells after t hours. Now, as long as t represents a non-negative integer, we know how to calculate $N(t)$. For example, if $t = 6$, then

$$N(t) = (3 \times 10^6)2^6 = (3 \times 10^6)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = (3 \times 10^6)64 = 192 \times 10^6.$$

The key point is that computing $N(t)$ only involves simple arithmetic. But what happens if we want to know the population of cells after 6.37 hours? That would require that we work with the formula

$$N(t) = (3 \times 10^6)2^{6.37}$$

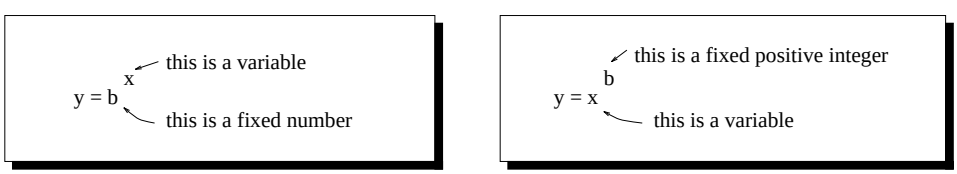
and the rules of arithmetic do not suffice to calculate $N(t)$. We are stuck, since we must understand the meaning of an expression like $2^{6.37}$. In order to proceed, we will need to review the algebra required to make sense of raising a number (such as 2) to a non-integer power. We need to understand the precise meaning of expressions like: $2^{6.37}$, $2^{\sqrt{5}}$, $2^{-\pi}$, etc.

5.1 Functions of Exponential Type

On a symbolic level, the class of functions we are trying to motivate is easily introduced. We have already studied the monomials $y = x^b$, where x was our input <u>variable</u> and b was a fixed positive integer <u>exponent</u>. What happens if we turn this around, interchanging x and b, defining a new rule:

$$y = f(x) = b^x. \tag{5.1.1}$$

We refer to x as the *power* and b the *base*. An expression of this sort is called a function of *exponential type*. Actually, if your algebra is a bit rusty, it is easy to initially confuse functions of exponential type and monomials:

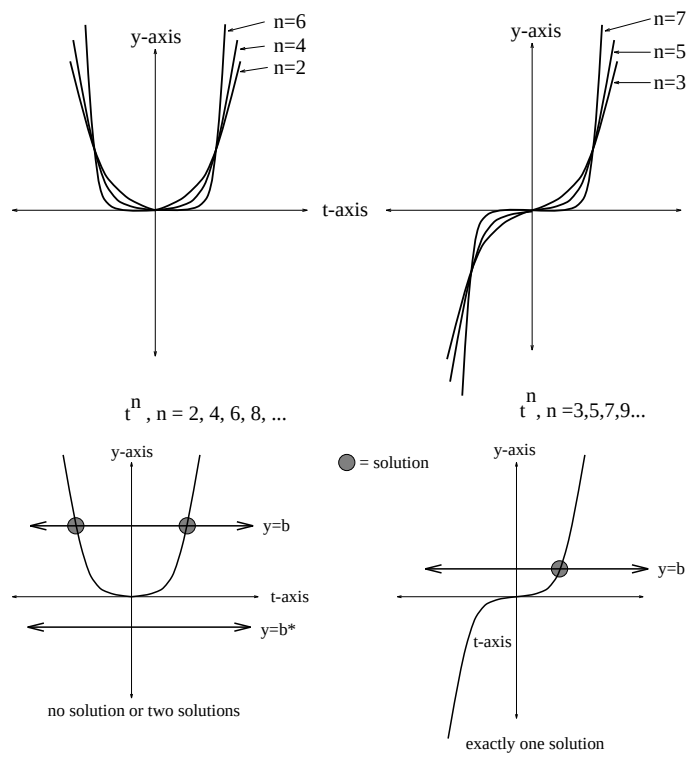


Exponential Picture Monomial Picture

Reviewing the Rules of Exponents.

To be completely honest, making sense of the expression $y = b^x$ for *all* numbers x requires the tools of *Calculus*, but it is possible to establish a reasonable comfort level by handling the case when x is a rational number. If $b \geq 0$ and n is a positive integer (i.e. $n = 1, 2, 3, 4, \ldots$), then we can try to solve the equation

$$t^n = b. \tag{5.1.2}$$



A solution t to this equation is called an nth root of b. Return to §1.9 and recall the graph of the function $y = t^n$ in the ty-coordinate system. This leads to complications, depending on whether n is even or odd. In the odd case, for any real number b, notice that the graph of $y = b$ will always cross the graph of $y = t^n$ exactly once, leading to one solution of (5.1.2). On the other hand, if n is even and $b < 0$, then the graph of $y = t^n$ will miss the graph of $y = b$, implying there are no solutions to the equation in (5.1.2). (There will be *complex* solutions to equations such as $t^2 = -1$, involving the imaginary complex numbers $\pm i = \pm\sqrt{-1}$, but we are only working with real numbers in this course.) Also, again in the case when n is even, it can happen that there are two solutions to (5.1.2). We do not want to constantly worry about this even/odd distinction, so we will henceforth **assume** $b > 0$. To eliminate possible ambiguity, we will single out a particular n^{th}-root; we define the symbols:

$$\sqrt[n]{b} = b^{\frac{1}{n}} = \text{ the largest real } n^{th} \text{ root of } b. \tag{5.1.3}$$

Thus, whereas ± 1 are both 4th-roots of 1, we have defined $\sqrt[4]{1} = 1$.

In order to manipulate $y = b^x$ for rational x, we need to recall some basic facts from algebra.

Rational Exponent Facts 5.1.4: *For all positive integers p and q, and any real number base $b > 0$, we have*

$$b^{\frac{p}{q}} = \left(\sqrt[q]{b}\right)^p = \sqrt[q]{b^p}.$$

For any rational numbers r and s, and for all positive bases a and b:
(1) Product of power rule: $b^r b^s = b^{r+s}$
(2) Power of power rule: $(b^r)^s = b^{rs}$
(3) Power of product rule: $(ab)^r = a^r b^r$
(4) Zero exponent rule: $b^0 = 1$
(5) Negative power rule: $b^{-r} = \frac{1}{b^r}$

These rules have two important consequences, one theoretical and the other more practical. On the first count, recall that any rational number r can be written in the form $r = \frac{p}{q}$, where p and q are integers. Consequently, using these rules, we see that the expression $y = b^x$ defines a function of x, whenever x is a rational number. On the more practical side of things, using the rules we can calculate and manipulate certain expressions. For example,

$$\begin{aligned} 27^{\frac{2}{3}} &= \left(\sqrt[3]{27}\right)^2 = 3^2 = 9; \\ 8^{-\frac{5}{3}} &= \left(\sqrt[3]{8}\right)^{-5} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}. \end{aligned}$$

As another example, go back to the Chapter introduction and calculate:

$$2^{19.75} = 2^{19}2^{3/4} = (524,288)\sqrt[4]{2^3} = (524288)(1.6818) = \$8817.48; \tag{5.1.5}$$

this means that the painters make an extra \$3574.60 for an extra 45 minutes work!

The sticky point which remains is knowing that $f(x) = b^x$ actually defines a function for *all real values of* x. This is **not** easy to verify and we are simply going to accept it as a fact. The difficulty is that we need the fundamentally new concept of a *limit*, which is the starting point of a *Calculus* course. Once we know the expression does define a function, we can also verify that the rules of (5.1.4) carry through for all real exponent powers. Your calculator should have a "y to the x key", allowing you to calculate expressions such as $\pi^{\sqrt{2}}$ involving non-rational powers.

Here are the key modeling functions we will work with in this Chapter.

Definition 5.1.6: *A function of exponential type has the form*

$$A(x) = A_o b^x,$$

for some $b > 0$, $b \neq 1$ and $A_o \neq 0$.

We will refer to the formula in (5.1.6) as the *standard exponential form.* Just as with standard forms for quadratic and sinusoidal functions, we sometimes need to do a little calculation to put an equation in standard form. The constant A_o is called the *initial value* of the exponential function; this is because if x represents time, then $A(0) = A_o b^0 = A_o$ is the value of the function at time $x = 0$; i.e. the initial value of the function.

Example 5.1.7: *Write the equations $y = 8^{3x}$ and $y = 7(\frac{1}{2})^{2x-1}$ in standard exponential form.* In both cases, we just use the rules of exponents to manuever the given equation into standard form:

$$\begin{aligned} y &= 8^{3x} = (8^3)^x = 512^x \\ y &= 7\left(\frac{1}{2}\right)^{2x-1} = 7\left(\frac{1}{2}\right)^{2x}\left(\frac{1}{2}\right)^{-1} = 7\left(\left(\frac{1}{2}\right)^2\right)^x 2 = 14\left(\frac{1}{4}\right)^x \end{aligned}$$

The Functions $y = A_o b^x$.

We know $f(x) = 2^x$ defines a function of x, so we can study basic qualitative features of its graph. The data assembled in the solution of the "Contractor Scam" beginning this Chapter, plus the rules of exponents, produce a number of points on the graph.

x	2^x	Point on the graph of $y = 2^x$
⋮	⋮	⋮
-2	1/4	(-2,1/4)
-1	1/2	(-1,1/2)
0	1	(0,1)
1	2	(1,2)
2	4	(2,4)
3	8	(3,8)
⋮	⋮	⋮

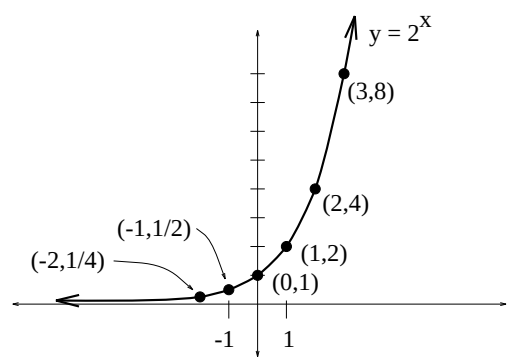


Figure 5.1.1: Graph of $y = 2^x$

This graph exhibits four key qualitative features that deserve mention:

- The graph is always above the horizontal axis; i.e. the function values are always positive.
- The graph has y-intercept 1 and is increasing.
- The graph becomes closer and closer to the horizontal axis as we move left; i.e. the x-axis is a horizontal asymptote for the left-hand portion of the graph.
- The graph becomes higher and higher above the horizontal axis as we move to the right; i.e., the graph is *unbounded* as we move to the right.

We can use these features to argue that $y = 2^x$ is not the graph of any function we have studied thus far. For example, the graph in Figure 5.1.1 is not the graph of any polynomial, since a

polynomial graph never becomes asymptotic to a horizontal axis. It cannot be a sinusoidal function, since the values are not bounded, etc.

The special case of $y = 2^x$ is representative of the function $y = b^x$, but there are a few subtle points that need to be addressed. First, recall we are always assuming that our base $b > 0$. We will consider three separate cases: $b = 1, b > 1$ and $0 < b < 1$.

The case $b = 1$.

In the case $b = 1$, we are working with the function $y = 1^x = 1$; this is not too exciting, since the graph is just a horizontal line. We will ignore this case.

The case $b > 1$.

If $b > 1$, the graph of the function $y = b^x$ is qualitatively similar to the situation for $b = 2$, which we just considered. The only difference is the exact amount of "concavity" in the graph, but the four features highlighted above are still valid. The left-hand figure below indicates how these graphs compare for three different values of b. Functions of this type exhibit what is typically referred to as *exponential growth*; this codifies the fact that the function values grow rapidly as we move to the right along the x-axis.

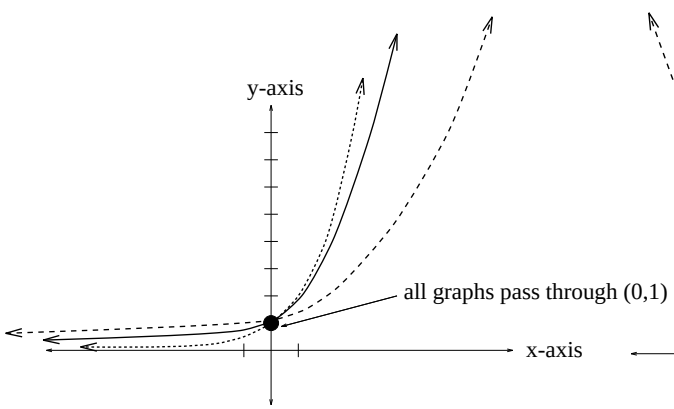


Figure 5.1.2: Graph of $y = b^x$, $b > 1$.

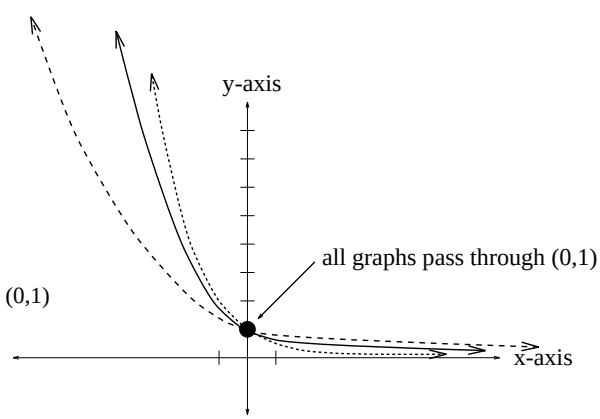


Figure 5.1.3: Graph of $y = b^x$, $0 < b < 1$.

The case $0 < b < 1$.

We can understand the remaining case $0 < b < 1$, by using the remarks above and our work in §2.5. First, with this condition on b, notice that $\frac{1}{b} > 1$, so the graph of $y = (\frac{1}{b})^x$ is of the type in Figure 5.1.2. Now, using the rules of exponents:

$$y = (\frac{1}{b})^{-x} = ((\frac{1}{b})^{-1})^x = b^x.$$

By the reflection principle, the graph of $y = (\frac{1}{b})^{-x}$ is obtained by reflecting the graph of $y = (\frac{1}{b})^x$ about the y-axis. Putting these remarks together, if $0 < b < 1$, we conclude that the graph of $y = b^x$ will look like the right-hand figure above. Notice, the graphs in Figure 5.1.3

share qualitative features, mirroring the features outlined previously, with the "asymptote" and "unbounded" portions of the graph interchanged. Graphs of this sort are often said to exhibit *exponential decay*, in the sense that the function values rapidly approach zero as we move to the right along the x-axis.

Features of Exponential Type Functions 5.1.8: *Let b be a positive real number, not equal to 1. The graph of $y = b^x$ has these four properties:*
(1) The graph is always above the horizontal axis.
(2) The graph has y-intercept 1.
(3) If $b > 1$ (resp. $0 < b < 1$), the graph becomes closer and closer to the horizontal axis as we move to the left (resp. move to the right); this says the x-axis is a horizontal asymptote for the left-hand portion of the graph (resp. right-hand portion of the graph).
(4) If $b > 1$ (resp. $0 < b < 1$), the graph becomes higher and higher above the horizontal axis as we move to the right (resp. move to the left); this says that the graph is unbounded as we move to the right (resp. move to the left).
If $A_o > 0$, the graph of the function $y = A_o b^x$ is a vertically expanded or compressed version of the graph of $y = b^x$. If $A_o < 0$, we additionally reflect about the x-axis.

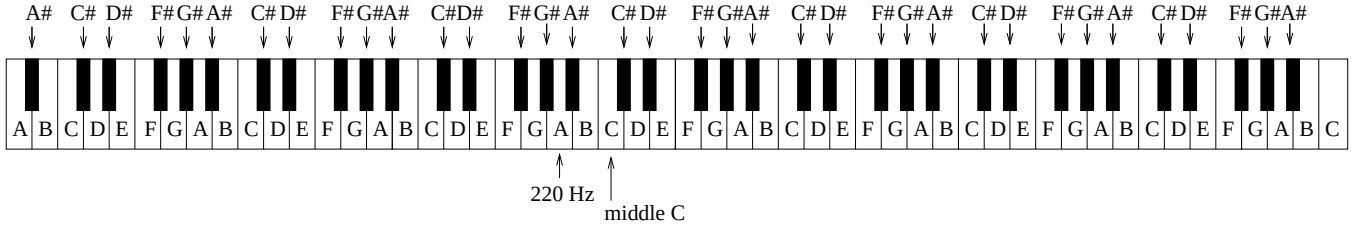
Piano Frequency Range.

A sound wave will cause your eardrum to move back and forth and this can be modeled using sinusoidal functions. In the case of a so-called *pure tone*, this motion is modeled by a single sinusoidal function of the form

$$d(t) \quad = \quad A\sin(2\pi f t),$$

where f is called the *frequency*, in units of "periods/unit time", called "Hertz" and abbreviated "Hz". The coefficient A is related to the actual displacement of the eardrum, which is, in turn, related to the loudness of the sound. A person can typically perceive sounds ranging from 20 Hz to 20,000 Hz.

A piano keyboard is commonly tuned according to a rule requiring that each key (white and black) has a frequency $2^{1/12}$ times that of the previous key. Assuming that the key A below middle C has a frequency of 220 Hz, we can determine the frequency of every key on the keyboard. We indicate the layout of the keyboard below. Recall, the "white" keys are labeled "A,B,C,D,E,F,G", then the sequence repeats itself. The black keys fit into this sequence as "sharps", so that an octave on the keyboard involves 12 keys: "A,A$^\#$,B,C,C$^\#$,D,D$^\#$,E,F,F$^\#$,G,G$^\#$". The layout is given below (and one of the exercises is based on this information):



Problems

1. Let's brush up on the required calculator skills. Use a calculator to approximate:
   (a) 3^{π}
   (b) $4^{2+\sqrt{5}}$
   (c) π^{π}
   (d) $5^{-\sqrt{3}}$
   e. 3^{π^2}
   f. $\sqrt{11^{\pi-7}}$
   g. $(5^{\sqrt[3]{23}})\sin(\frac{11}{3}\pi)$
   h. $\sin^{-1}(2^{-\pi})$

2. Using graphical reasoning (and no logarithms) solve these equations:
   (a) $3^x = 3^5$
   (b) $2^x = 0$
   (c) $2^x = 1$
   d. $5^x = 25$
   e. $(3^x - 1)(2x - 5) = 0$
   f. $\pi^x = \sqrt{\pi}$

3. Put each equation in standard exponential form:
   (a) $y = 3(2^{-x})$
   (b) $y = 4^{-x/2}$
   (c) $y = \pi^{\pi x}$
   (d) $y = \frac{5}{0.345^{2x-7}}$

4. (a) Begin with the function $y = f(x) = 2^x$.
   (i) Rewrite each of the following functions in standard exponential form: $f(2x)$, $f(x-1)$, $f(2x-1)$, $f(2(x-1))$, $3f(x)$, $3f(2(x-1))$.
   (ii) Is the function $3f(2(x-1)) + 1$ a function of exponential type?
   (iii) Sketch the graphs of $f(x)$, $f(2x)$, $f(2(x-1))$, $3f(2(x-1))$ and $3f(2(x-1)) + 1$ in the same coordinate system and explain which graphical operation(s) (vertical shifting, vertical dilation, horizontal shifting, horizontal dilation) have been carried out.
   (b) In general, explain what happens when you apply the four construction tools of §2.5 (vertical shifting, vertical dilation, horizontal shifting, horizontal dilation) to the standard exponential model $y = A_o b^x$. For which of the four operations is the resulting function still a standard exponential model?

5. According to the *Merck Manual*, the relationship between the height h (in inches), weight w (in pounds) and surface area S (in square meters) for a human is approximated by the equation:
$$S = 0.0104h^{0.425}w^{0.725}\,\text{meter}^2$$
   Fix h to be your height, in inches. Let $S = S(w)$ be the resulting function computing your surface area as a function of your weight w. What is your surface area? How much weight must you gain for your surface area to increase by 5%? Decrease by 2%?

5.2 Exponential Modeling

A computer industry spokesperson has predicted that the number of subscribers to Microsoft Network, a new internet provider, will grow exponentially for the first 5 years. Assume this person is correct. If Microsoft has 100,000 subscribers after 6 months and 750,000 subscribers after 12 months, how many subscribers will there be after 5 years?

The solution to this problem offers a template for many similar exponential modeling applications. First, we are assuming that the number of subscribers $N(x)$, where x represents years, is computed using a function of exponential type. That tells us

$$N(x) = N_o b^x,$$

for some N_o and $b > 1$. Next, we are given two pieces of information about the future growth, which gives the equations:

$$\begin{aligned} N(0.5) &= 100000; \quad \text{i.e.,} \quad N_o b^{0.5} = 100000 \\ N(1) &= 750000; \quad \text{i.e.,} \quad N_o b = 750000. \end{aligned}$$

If we divide the second equation by the first, we get

$$\begin{aligned} \frac{b^1}{b^{0.5}} &= 7.5 \\ b^{0.5} &= b^{1/2} = \sqrt{b} = 7.5 \\ b &= 56.25. \end{aligned}$$

If we plug this value of b into either equation (say the first one), we can solve for N_o: $N_o = \frac{100000}{(56.25)^{0.5}} = 13,333$. We now know that the number of *Network* subscribers will be predicted by

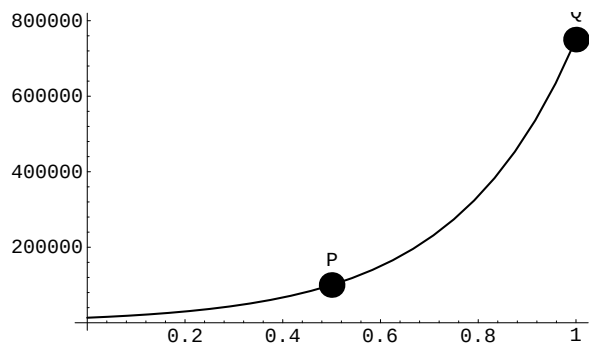
$$N(x) = 13,333(56.25)^x.$$

In five years, we obtain $N(5) = 7,508,300,000,000$. This exceeds the population of the Earth (which is between 5 and 6 billion), so mathematics, in this case, shows that all the hype can't be taken at face value. □

There are two important conclusions we can draw from this problem. First, the given information provides us with two points on the graph of the function $N(x)$:

$$\begin{aligned} P &= (0.5, 100000) \\ Q &= (1, 750000). \end{aligned}$$

More importantly, this example illustrates a very important principal we can use when modeling with functions of exponential type.



Key Fact 5.2.1: *A function of exponential type can be determined if we are given two data points on its graph.*

!!! CAUTION !!!

When you use the above strategy to find the base b of the exponential model, make sure to write down a lengthy decimal approximation. As a rule of thumb, go for twice as many significant digits as you are otherwise using in the problem.

!!! CAUTION !!!

The Method of Compound Interest.

You walk into a Bank with P_o dollars (usually called principal), wishing to invest the money in a savings account. You expect to be rewarded by the Bank and paid interest, so how do you compute the total value of the account after t years?

The future value of the account is really a function of the number of years t elapsed, so we can write this as a function $P(t)$. Our goal is to see that $P(t)$ is a function of exponential type. In order to compute the future value of the account, the Bank provides any savings account investor with two important pieces of information:

$$\begin{aligned} r &= \text{annual (decimal) interest rate} \\ n &= \text{the number of compounding periods per year} \end{aligned}$$

The number n tells us how many times each year the Bank will compute the total value $P(t)$ of the account. For example, if $n = 1$, the calculation is done at one year intervals; if $n = 12$, the calculation is done each month, etc. The bank will compute the value of your account after a typical compounding period by using the *periodic rate of return* $\frac{r}{n}$. For example, if the

interest rate percentage is 12% and the compounding period is monthly (i.e., $n = 12$), then the annual (decimal) interest rate is 0.12 and the periodic rate is $\frac{0.12}{12} = 0.01$.

!!! CAUTION !!!

The number r always represents the decimal interest rate, which is a decimal between 0 and 1. If you are given the interest rate percentage (which is a positive number between 0 and 100), you need to convert to a decimal by dividing by 100.

!!! CAUTION !!!

Two Examples.

Let's consider an example: $P_o = \$1000$ invested at the annual interest percentage of 8% compounded yearly, so $n = 1$ and $r = 0.08$. To compute the value $P(1)$ after one year, we will have

$$\begin{aligned} P(1) &= P_o + (\text{periodic rate})P_o \\ &= P_o + rP_o = P_o(1+r) \\ &= \$1000(1+0.08) = \$1080. \end{aligned}$$

To compute the value after two years, we need to apply the periodic rate to the value of the account after one year:

$$\begin{aligned} P(2) &= P(1) + (\text{periodic rate})P(1) \\ &= P_o(1+r) + rP_o(1+r) = P_o(1+r)^2 \\ &= \$1000(1+0.08)^2 = \$1166.40. \end{aligned}$$

Notice, the amount the Bank has paid after two years is \$166.40, which is slightly bigger than twice the \$80 paid after one year. To compute the value after three years, we need to apply the periodic rate to the value of the account after two years:

$$\begin{aligned} P(3) &= P(2) + (\text{periodic rate})P(2) \\ &= P_o(1+r)^2 + rP_o(1+r)^2 = P_o(1+r)^3 \\ &= \$1000(1+0.08)^3 = \$1259.71. \end{aligned}$$

Again, notice the amount the Bank is paying after three years is \$259.71, which is slightly larger than three times the \$80 paid after one year. Continuing on in this way, to find the value after t years, we arrive at the formula

$$P(t) = P_o(1+r)^t = \$1000(1.08)^t.$$

In particular, after 5 and 10 years, the value of the account (to the nearest dollar) will be \$1469 and \$2159, respectively.

As a second example, suppose we begin with the same \$1000 and the same annual interest percentage 8%, but now compound monthly, so $n = 12$ and $r = 0.08$. The value of the account

after one compounding period is $P(1/12)$, since a month is one-twelfth of a year. Arguing as before, paying special attention that the periodic rate is now $\frac{r}{n} = \frac{0.08}{12}$, we have

$$\begin{aligned} P(1/12) &= P_o + (\text{periodic rate})P_o \\ &= P_o(1 + \frac{.08}{12}) \\ &= \$1000(1 + 0.006667) = \$1006.67. \end{aligned}$$

After two compounding periods, the value is $P(2/12)$,

$$\begin{aligned} P(2/12) &= P(1/12) + (\text{periodic rate})P(1/12) \\ &= P_o(1 + \frac{.08}{12}) + (\frac{0.08}{12})P_o(1 + \frac{.08}{12}) \\ &= P_o(1 + \frac{.08}{12})^2 \\ &= \$1000(1 + 0.006667)^2 = \$1013.38. \end{aligned}$$

Continuing on in this way, after k compounding periods have elapsed, the value will be $P(\frac{k}{12})$, which is computed as

$$P(k/12) = P_o(1 + \frac{.08}{12})^k.$$

It is possible to rewrite this formula to give us the value after t years, noting that t years will lead to $12t$ compounding periods; i.e., set $k = 12t$ in the previous formula:

$$P(t) = P_o(1 + \frac{.08}{12})^{12t}$$

For example, after 1, 5 and 10 years, the value of the account, to the nearest dollar, would be \$1083, \$1490 and \$2220.

Discrete Compounding.

The two examples above highlight a general formula for computing the future value of an account.

Discrete Compounding 5.2.2: *Suppose an account is opened with P_o principal. If the decimal interest rate is r and the number of compounding periods per year is n, then the value $P(t)$ of the account after t years will be*

$$P(t) = P_o(1 + \frac{r}{n})^{nt}.$$

Notice, the future value $P(t)$ is a function of exponential type; the base is the number $(1 + \frac{r}{n})$, which will be greater than one. Since $P_o > 0$, the graph will be qualitatively similar to the ones pictured in Figure 5.1.2.

Example 5.2.3: *At birth, your Uncle Hans secretly purchased a \$5000 U.S. Savings Bond for \$2500. The conditions of the bond state that the U.S. Government will pay a minimum annual interest rate of $r = 8.75\%$, compounded quarterly. Your Uncle has given you the bond as a gift, subject to the condition that you cash the bond at age 35 and buy a red Porsche. On your way to the Dealer, you receive a call from your tax accountant informing you of a 28% tax on the capital gain you realize through cashing in the bond; the capital gain is the selling price of the bond minus the purchase price. Before stepping onto the showroom floor, compute how much cash will you have on hand, after the U.S. Government shares in your profits.*

Solution. The value of your bond after 35 years is computed by the formula in (5.1.10), using $P_o = \$2500$, $r = 0.0875$, $n = 4$ and $t = 35$. Plugging this all in, we find that the selling price of the bond is

$$P(35) \quad = \quad \$2500(1 + \frac{0.0875}{4})^{4(35)} = \$51,716.42.$$

The capital gain will be \$51,716.42 - \$2500 = \$49,216.42 and the tax due is \$(49,216.42)(0.28) = \$13,780.60. You are left with \$51,716.42 - \$13,780.60 = \$37,935.82. Better make that a used Porsche! □

The Number e and the Exponential Function.

What happens to the future value of an investment of $\$P_o$ as the number of compounding periods is increased?

For example, return to our earlier example: P_o=\$1000 and an annual interest percentage of 8%. After 1 year, the table below indicates the value of the investment for various compounding periods: yearly, quarterly, monthly, weekly, daily and hourly.

n	**Value after 1 year (to nearest dollar)**
1 (yearly compounding)	$\$1000(1 + 0.08)^1 = \$1080$
4 (quarterly compounding)	$\$1000(1 + \frac{0.08}{4})^4 = \$1082.43$
12 (monthly compounding)	$\$1000(1 + \frac{0.08}{12})^{12} = \$1083.00$
52 (weekly compounding)	$\$1000(1 + \frac{0.08}{52})^{52} = \$1083.22$
365 (daily compounding)	$\$1000(1 + \frac{0.08}{365})^{365} = \$1083.28$
8760 (hourly compounding)	$\$1000(1 + \frac{0.08}{8760})^{8760} = \$1083.29$

We could continue on, considering "minute" and "second" compounding and what we will find is that the value will be at most \$1083.29. This illustrates a general principal:

> Initially increasing the number of compounding periods makes a significant difference in the future value; however, eventually there appears to be a limiting value.

Let's see if we can understand mathematically why this is happening. The first step is to recall the discrete compounding formula:

$$P(t) = P_o\left(1+\frac{r}{n}\right)^{nt}.$$

If our desire is to study the effect of increasing the number of compounding periods, this means we want to see what happens to this formula as n gets BIG. To analyze this, it is best to rewrite the expression using a substitution trick: Set $z = \frac{n}{r}$, so that $n = rz$ and $\frac{r}{n} = \frac{1}{z}$. Plugging in, we have

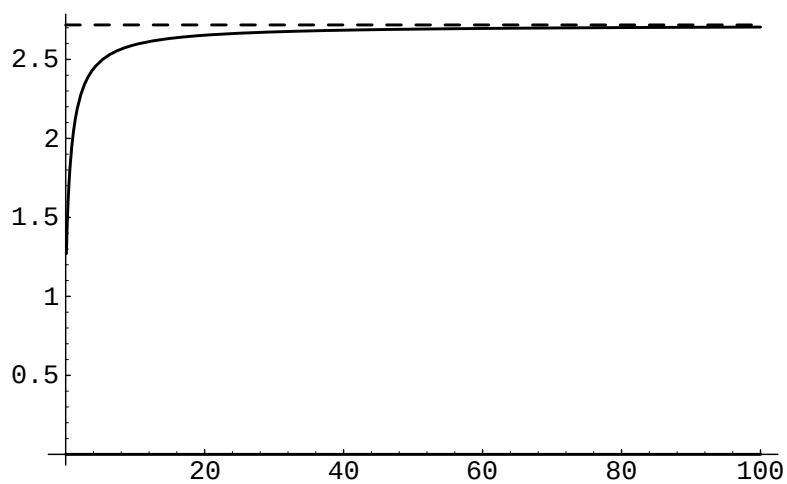
$$P(t) = P_o\left(1+\frac{r}{n}\right)^{nt} = P_o\left(1+\frac{1}{z}\right)^{rzt} = P_o\left(\left(1+\frac{1}{z}\right)^{z}\right)^{rt}. \tag{5.2.4}$$

So, since r is a fixed number and $z = \frac{n}{r}$, letting n get BIG is the same as letting z become BIG in (5.2.4). This all means we need to answer this new question:

What happens to the expression $(1+\frac{1}{z})^z$ *as* z *becomes large?*

On the one hand, the power in the expression is getting large; at the same time, the base is getting close to 1. This makes it very tricky to make quick predictions about the outcome. It is best to first tabulate some numerical data for the values of $y = g(z) = \left(1+\frac{1}{z}\right)^z$ and look at a plot of this function graph on the domain $0.01 \le z \le 100$:

z	$(1+\frac{1}{z})^z$
1	2
2	2.25
3	2.37037
4	2.4414
20	2.65329
100	2.70481
1000	2.71692
10^9	2.71828



You can see from this plot, the graph of $y = \left(1+\frac{1}{z}\right)^z$ approaches the "dashed" horizontal asymptote, as z becomes BIG. We will let the letter "e" represent the spot where this horizontal line crosses the vertical axis and e$\approx$ 2.7182818. This number is only an approximation, since e is known to be an irrational number. What sets this irrational number apart from the ones you are familiar with (e.g. $\sqrt{2}$, π, etc.) is that defining the number e requires a "limiting" process. This will be studied a lot more in your *Calculus* course. The new number e is a

positive number greater than 1, so we can study the function:

$$y = e^x. \tag{5.2.5}$$

Since $e > 1$, the graph will share the properties in Figure 5.1.2. This function is usually referred to as THE *exponential function*. Scientific calculators will have a key of the form "$\exp(x)$" or "e^x".

Calculator drill.

Plugging in $x = 1$, you can compute an approximation to e on your calculator; you should get $e = 2.7183$ to four decimal places. Make sure you can compute expressions like e^3, e^π or $\sqrt{e} = e^{1/2}$; to four decimal places, you should get $20.0855, 23.1407$ and 1.6487.

Back to the original problem...

We can now return to our future value formula in (5.2.4) and conclude that as the number of compounding periods increases, the future value is approaching a limiting value:

$$P(t) \quad = \quad P_o\left((1+\frac{1}{z})^z\right)^{rt} \qquad \Longrightarrow \qquad P_o e^{rt}.$$

The right hand limiting formula $Q(t) = P_o e^{rt}$ computes the future value using what is usually referred to as *continuous compounding*. From the investors viewpoint, this is the best possible scheme for computing future value.

Continuous Compounding 5.2.6: *The future value of P_o dollars principal invested at an annual decimal interest rate of r under continuous compounding after t years is $Q(t) = P_o e^{rt}$; this value is always greater than the value of $P_o(1+\frac{r}{n})^{nt}$, for any discrete compounding scheme. In fact, $P_o e^{rt}$ is the limiting value.*

Problems

1. The town of *Pinedale, Wyo.* is experiencing a population boom. In 1990, the population was 860 and five years later it was 1210.
   (a) Find a linear model $l(x)$ and an exponential model $p(x)$ for the population of Pinedale in the year 1990+x.
   (b) What will be the population of Pinedale in 2000 under these two models?
   (c) Using graphical techniques, discuss when the predicted population of the linear model exceeds that of the exponential model by at least 10.
   (d) Use a graphing device to determine when the predicted population under the linear model exceeds the exponential model population by at least 10.

2. Tiffany is a model rocket enthusiast. She has been working on a pressurized rocket filled with laughing gas. According to her design, if the atmospheric pressure exerted on the rocket is less than 10 pounds/sq.in., the laughing gas chamber inside the rocket will explode. Tiff worked from a formula $p = (14.7)e^{-h/10}$ pounds/sq.in. for the atmospheric pressure h miles above sea level. Assume that the rocket is launched at an angle of α^o above level ground at sea level with an initial speed of 1400 feet/sec.
   (a) If the angle of launch is $\alpha = 12^o$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in mid-air?
   (b) If the angle of launch is $\alpha = 82^o$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in mid-air?
   (c)* Assume that the maximum elevation to avoid premature explosion is 3.8526 miles. Find the largest launch angle α so that the rocket will not prematurely explode.
3. Return to the Earning Power Problem in §1.4. Use the data in Table 1.4.1 to obtain exponential models $M(x)$ and $W(x)$ for Men's and Women's Earning power in the year $1970 + x$, respectively. What will be the earnings in 1997? In 2010? In 2100? From these calculations, what can you say about whether women are gaining on men?

5.3 Logarithmic Functions

If we invest $P_o = \$1000$ at an annual rate of $r = 8\%$ compounded continuously, how long will it take for the account to have a value of \$5000?

The formula $P(t) = 1000e^{0.08t}$ gives the value after t years, so we need to solve the equation:

$$\begin{aligned} 5000 &= 1000e^{0.08t} \\ 5 &= e^{0.08t}. \end{aligned}$$

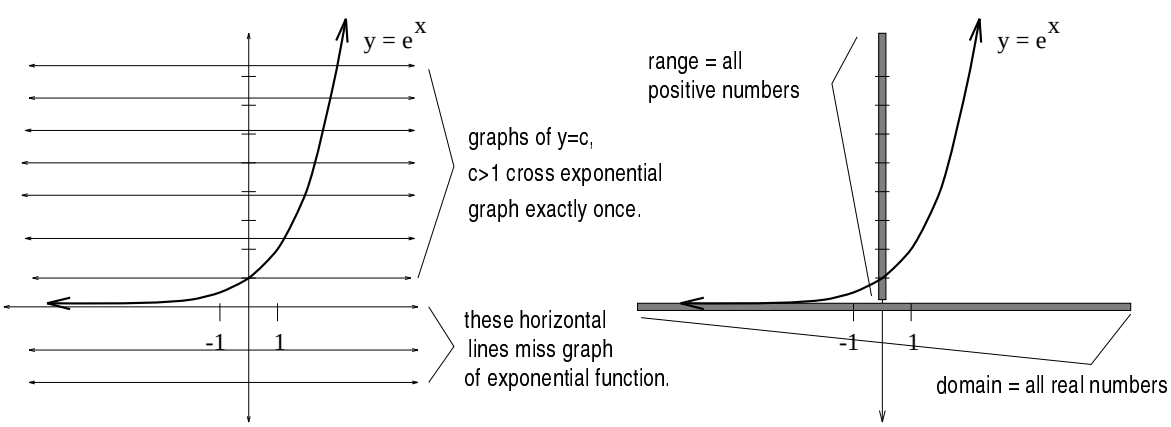
Unfortunately, algebraic manipulation will not lead to a further simplification of this equation; we are stuck! In spirit, the same sort of thing happened to us at the beginning of §2.7. Then, just as now, the required technique involves the theory of inverse functions. Assuming we can find the inverse function of $f(t) = e^t$, we can apply $f^{-1}(t)$ to each side of the equation and solve for t:

$$\begin{aligned} f^{-1}(5) &= f^{-1}(e^{0.08t}) = 0.08t \\ (12.5)f^{-1}(5) &= t \end{aligned}$$

The goal in this section is to describe the function f^{-1}, which is usually denoted by the symbol $f^{-1}(t) = \ln(t)$ and called the *natural logarithm* function. On your calculator, you will find a button dedicated to this function and we can now compute $\ln(5) = 1.60944$. Conclude that the solution is $t = 20.12$ years.

The Inverse Function of $y = e^x$.

If we sketch a picture of the exponential function on the domain of all real numbers and keep in mind the properties in (5.1.7), then every horizontal line above the x-axis intersects the graph of $y = e^x$ exactly once; see left-hand picture below. The range of the exponential function will consist of all possible y-coordinates of points on the graph. Using the graphical techniques of §2.2, we can see that the range of will be all POSITIVE real numbers; see the right-hand picture below.



By the horizontal line test, this means the exponential function is one-to-one and the inverse rule $f^{-1}(c)$ will define a function

$$f^{-1}(c) = \begin{cases} \text{the unique solution of the equation } c = e^x, & \text{if } c > 0 \\ \text{undefined,} & \text{if } c \leq 0. \end{cases} \tag{5.3.1}$$

This inverse function is called the *natural logarithm* function, denoted $\ln(c)$. We can sketch the graph of the the natural logarithm as follows: First, by (2.8.7), the domain of the function $\ln(y) = x$ is just the range of the exponential function, which we noted is all positive numbers. Likewise, the range of the function $\ln(y) = x$ is the domain of the exponential function, which we noted is all real numbers. Interchanging x and y, the graph of the natural logarithm function $y = \ln(x)$ can be obtained by flipping the graph of $y = e^x$ across the line $y = x$:

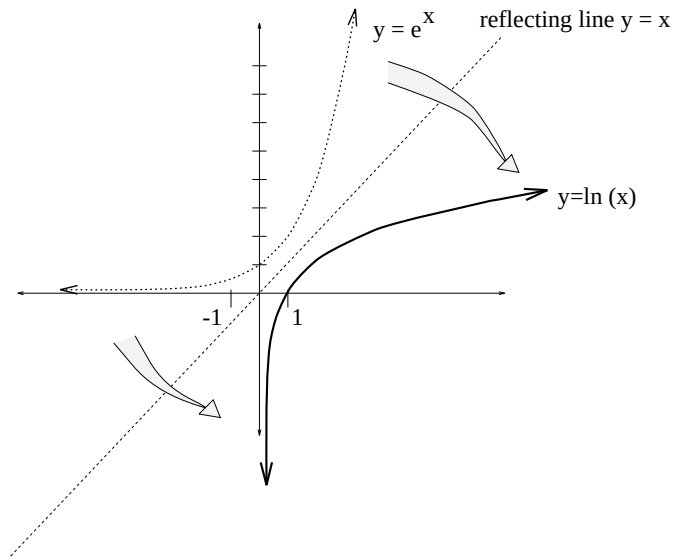


Figure 5.3.1

Graphical Features of Natural Log 5.3.2: *The function y=ln(x) has these features:*

- *The largest domain is the set of positive numbers; e.g. ln(-1) makes no sense.*
- *The graph has x-intercept 1 and is increasing.*
- *The graph becomes closer and closer to the vertical axis as we approach $x = 0$; i.e. the y-axis is a vertical asymptote for the graph.*
- *The graph is unbounded as we move to the right.*

Anytime we are working with an inverse function, symbolic properties are useful. Here are the important ones related to the natural logarithm.

Natural Log Properties 5.3.3: *We have the following properties:*
(a) For any real number x, $\ln(e^x)$=x.
(b) For any positive number x, $e^{\ln(x)} = x$.
(c) $\ln(b^t) = t\ln(b)$, for $b > 0$ and t any real number;
(d) $\ln(ba) = \ln(a) + \ln(b)$, for all $a, b > 0$;
(e) $\ln(\frac{b}{a}) = \ln(b) - \ln(a)$, for all $a, b > 0$.

The properties (c)-(e) are related to three of the rules of exponents in (5.1.4). Here are the kinds of basic symbolic maneuvers you can pull off using these properties:

Examples 5.3.4:

(i) $\ln(8^3)$=3ln(8)= 6.2383; $\ln(6\pi)$=ln(6) + $\ln(\pi)$=2.9365; $\ln(\frac{3}{5})$=ln(3)-ln(5)= -0.5108.

(ii) $\ln(\sqrt{x})$=$\ln(x^{1/2})$=$\frac{1}{2}\ln(x)$; $\ln(x^2-1) = \ln((x-1)(x+1))$=$\ln(x-1)+\ln(x+1)$; $\ln(\frac{x^5}{x^2+1})$ = $\ln(x^5)$ - $\ln(x^2+1)$ = 5ln(x) - $\ln(x^2+1)$.

Examples 5.3.5:

(i) Given the equation 3^{x+1}=12, we can solve for x:

$$\begin{aligned} 3^{x+1} &= 12 \\ \ln(3^{x+1}) &= \ln(12) \\ (x+1)\ln(3) &= \ln(12) \\ x &= \frac{\ln(12)}{\ln(3)} - 1 = 1.2619. \end{aligned}$$

(ii) Given the equation $e^{\tan(x)} = 2$, we can solve for x:

$$\begin{aligned} e^{\tan(x)} &= 2 \\ \ln(e^{\tan(x)}) &= \ln(2) \\ \tan(x) &= \ln(2) = 0.6931 \\ x &= 0.6061 + k\pi, \end{aligned}$$

where k is any integer $0, \pm 1, \pm 2, \ldots$.

Example 5.3.6: *If \$2,000 is invested in a continuously compounding savings account and we want the value after 12 years to be \$130,000, what is the required annual interest rate? If, instead, the same \$2,000 is invested in a continuously compounding savings account with $r = 6.4\%$ annual interest, when will the exact account value be be \$130,000?*

Solution. In the first scenario,

$$\begin{aligned} 130000 &= 2000e^{12r} \\ 65 &= e^{12r} \\ \ln(65) &= \ln(e^{12r}) \\ \ln(65) &= 12r \\ r &= \frac{\ln(65)}{12} = .3479. \end{aligned}$$

This gives an annual interest rate of 34.79%. In the second scenario, we study the equation

$$\begin{aligned} 130000 &= 2000e^{(0.064)t} \\ 65 &= e^{(0.064)t} \\ t &= \frac{ln(65)}{0.064} = 65.22. \end{aligned}$$

So, it takes over 65 years to accumulate \$130,000 under the second scheme. □

Alternate form for functions of exponential type.

The standard model for an exponential function is $A(t) = A_o b^t$, for some $b > 0, b \neq 1$ and $A_o \neq 0$. Using the properties of the natural logarithm function,

$$b^t = (e^{\ln(b)})^t = e^{t\ln(b)}.$$

This means that every function as in (5.1.6) can be re-written using the exponential function e^t. Another way of saying this is that you really only need the function keys "e^t" and "$\ln(t)$" on your calculator.

Observation 5.3.7: *A function of exponential type can be written in the form*

$$A(t) = A_o e^{at},$$

for some constants $A_o \neq 0$ and $a \neq 0$.

By studying the sign of the constant a, we can determine whether the function exhibits exponential growth or decay. For example, given the function $A(t) = e^{at}$, if $a > 0$ (resp. $a < 0$), then the function exhibits exponential growth (resp. decay).

Examples 5.3.8: (a) The function $A(t) = 200(2^t)$ exhibits exponential growth and can be re-written as:

$$A(t) = 200(e^{t\ln(2)}) = 200e^{0.69315t}$$

(b) The function $A(t) = 4e^{-0.2t}$ exhibits exponential decay and can be re-written as:

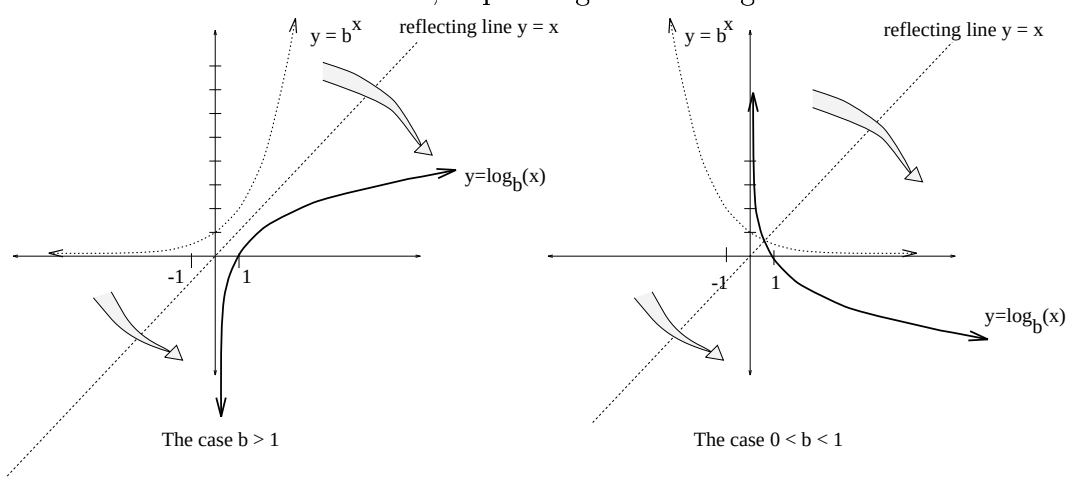
$$A(t) = 4e^{-0.2t} = 4(e^{-0.2})^t = 4(0.81873^t).$$

The Inverse Function of $y = b^x$.

For some topics in Chemistry and Physics (e.g. acid base equilibria and acoustics) it is useful to have on hand an inverse function for $y = b^x$, where $b > 0$ and $b \neq 1$. Just as above, we would show that $f(x) = b^x$ is one-to-one, the range is all positive numbers and obtain the graph using ideas in Figure 5.3.1. We will refer to the inverse rule as the *logarithm function base b*, denoted $\log_b(x)$, defined by the rule:

$$\log_b(c) = \begin{cases} \text{the unique solution of } c = b^x, & \text{if } c > 0 \\ \text{undefined,} & \text{if } c \leq 0. \end{cases}$$

We will need to consider two cases, depending on the magnitude of b:



The important qualitative features of the logarithm function y=$\log_b(x)$ mirror (5.3.2):

Graphical Features of General Logs 5.3.9: *The function $y = \log_b(x)$ has these features:*

- *The largest domain is the set of positive numbers; e.g. $\log_b(-1)$ is not defined.*
- *The graph has x-intercept 1 and is increasing if $b > 1$ (resp. decreasing if $0 < b < 1$).*
- *The graph becomes closer and closer to the vertical axis as we approach $x = 0$; this says the y-axis is a vertical asymptote for the graph.*
- *The graph is unbounded as we move to the right.*

Log Properties 5.3.10: *Fix a positive base b, $b \neq 1$.*
(a) For any real number x, $\log_b(b^x) = x$.
(b) For any positive number x, $b^{\log_b(x)} = x$.
(c) $\log_b(r^t) = t\ \log_b(r)$, for $r > 0$ and t any real number;
(d) $\log_b(rs) = \log_b(r) + \log_b(s)$, for all $r, s > 0$;
(e) $\log_b(\frac{r}{s}) = \log_b(r)$ - $\log_b(s)$, for all $r, s > 0$.

It is common to simplify terminology and refer to the function $\log_b(x)$ as the *log base b* function, dropping the longer phrase "logarithm". Some scientific calculators will have a key devoted to this function. Other calculators may have a key labeled "$\log(x)$", which is usually understood to mean the log base 10. However, many calculators only have the key "$\ln(x)$". This is not cause for alarm, since it is always possible to express $\log_b(x)$ in terms of the natural log function. Let's see how to do this, since it is a great application of the Log Properties just listed:

Suppose we start with $y = \log_b(x)$. We will rewrite this in terms of the natural log by carrying out a sequence of algebraic steps below; make sure you see why each step is justified.

$$\begin{aligned} y &= \log_b(x) \\ b^y &= x \\ \ln(b^y) &= \ln(x) \\ y\ln(b) &= \ln(x) \\ y &= \ln(x)/\ln(b). \end{aligned}$$

We have just verified a useful conversion formula:

Log Conversion Formula 5.3.11: *For x a positive number and $b > 0, b \neq 1$ a base,*

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}.$$

For example,

$$\log_{10}(5) = \frac{\ln(5)}{\ln(10)} = 0.699$$

$$\log_{0.02}(11) = \frac{\ln(11)}{\ln(0.02)} = -0.613$$

$$\log_{20}(\frac{1}{2}) = \frac{\ln(\frac{1}{2})}{\ln(20)} = -0.2314$$

The conversion formula allows one to proceed slightly differently when solving equations involving functions of exponential type. This is illustrated in the next example.

Example 5.3.12: *Ten years ago, you purchased a house valued at \$80,000. Your plan is to sell the house at some point in the future, when the value is at least \$1,000,000. Assume that the future value of the house can be computed using quarterly compounding and an annual interest rate of 4.8%. How soon can you sell the house?*

Solution. We can use the future value formula to obtain the equation

$$\begin{aligned} 1,000,000 &= 80,000(1+\frac{0.048}{4})^{4t} \\ 12.5 &= (1.012)^{4t} \end{aligned}$$

Using the log base $b = 1.012$,

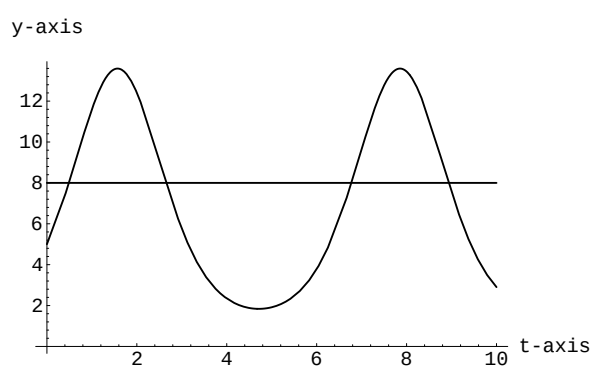
$$\begin{aligned} \log_{1.012}(12.5) &= \log_{1.012}((1.012)^{4t}) \\ \log_{1.012}(12.5) &= 4t \\ t &= \frac{\ln(12.5)}{4\ln(1.012)} = 52.934. \end{aligned}$$

Since you have already owned the house for 10 years, you would need to wait nearly 43 years to sell at the desired price. □

Let's try our hand at a problem that uses both the inverse trigonometric functions and the inverse of the exponential function. We will use preliminary graphical reasoning as an aide.

Example 5.3.13: *The voltage output of a circuit at time t is given by the function $y = f(t) = 5e^{\sin(t)}$. During the first 10 seconds, when is the voltage equal to 8 volts?*

Solution. If we look at a software plot of $y = f(t)$ and $y = 8$ in the same coordinate system, we see the two graphs cross four times on the domain $0 \le t \le 10$. This means there will be four solutions to the equation $8 = f(t)$, corresponding to the t-coordinates of the four intersection points:



The computation is easy to begin:

$$\begin{aligned} 5e^{\sin(t)} &= 8 \\ \ln(5e^{\sin(t)}) &= \ln(8) \\ \ln(5) + \ln(e^{\sin(t)}) &= \ln(8) \\ \sin(t) &= 0.47. \end{aligned}$$

We now use the technique explained in §3.6 to solve this equation. This requires we find the principal solution using the inverse sine function:

$$t = \sin^{-1}(0.47) = 0.4893.$$

Next, we find the symmetry solution:

$$t = -\sin^{-1}(0.47) + \pi = 2.6523.$$

Finally, we conclude EVERY solution has the form:

$$t = 0.4893 + 2k\pi \quad \text{or} \quad 2.6523 + 2k\pi \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

We need to find which of these solutions are between 0 and 10. This is just a calculation:

$$t = 0.4893, 2.6523, 6.7725, 8.9355.$$

□

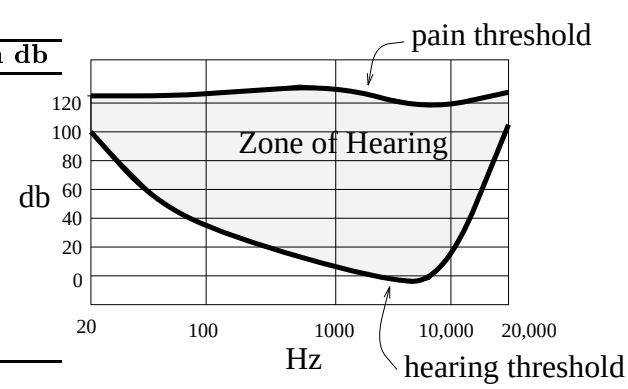
Measuring the Loudness of Sound.

As we noted earlier, the reception of a sound wave by the ear gives rise to a vibration of the eardrum with a definite frequency and a definite amplitude. This vibration may also be described in terms of the variation of air pressure at the same point, which causes the eardrum to move. The perception that rustling leaves and a jet aircraft sound different involves two concepts: (1) the fact that the frequencies involved may differ; (2) the intuitive notion of "loudness". This loudness is directly related to the force being exerted on the eardrum, which we refer to as the intensity of the sound. We can try to measure the intensity using some sort of scale. This becomes challenging, since the human ear is an amazing instrument, capable of hearing a large range of sound intensities. For that reason, a logarithmic scale becomes most useful. The *sound pressure level* β of a sound is defined by the equation

$$\beta = 10\log_{10}(\frac{I}{I_o}), \tag{5.3.14}$$

where I_o is an arbitrary reference intensity which is taken to correspond with the average faintest sound which can be heard and I is the intensity of the sound being measured. The units used for β are called *decibels*, abbreviated *db*. (Historically, the units of loudness were called *bels*, in honor of Alexander Graham Bell, referring to the quantity $\log_{10}(\frac{I}{I_o})$.) Notice, in the case of sound of intensity $I = I_o$, we have a sound pressure level of $\beta = 10\log_{10}(\frac{I_o}{I_o})$ $=10\log_{10}(1) = 10(0) = 0$. We refer to any sound of intensity I_o as having a sound pressure level at the *threshold of hearing*. At the other end of the scale, a sound of intensity the maximum the eardrum can tolerate has an average sound pressure level of about 120 db. The table below gives a hint of the sound pressure levels associated to some common sounds.

Source of noise	Sound pressure level in db
Threshold of pain	120
Riveter	95
Busy Street Traffic	70
Ordinary Conversation	65
Quiet Auto	50
Background Radio	40
Whisper	20
Rustle of Leaves	10
Threshold of Hearing	0



It turns out that the above comments on the threshold of hearing and pain are really only averages and depend upon the frequency of the given sound. In fact, while the threshold of pain is on average close to 120 db across all frequencies between 20 Hz and 20,000 Hz, the threshold of hearing is much more sensitive to frequency. For example, for a tone of 20 Hz (something like the ground-shaking rumble of a passing freight train), the sound pressure level needs to be relatively high to be heard; 100 db on average. As the frequency increases, the required sound pressure level for hearing tends to drop down to 0 db around 2000 Hz. An examination by a hearing specialist can determine the precise sensitivities of your ear across the frequency range, leading to a plot of your "envelope of hearing"; a sample plot is given above. Such a plot would differ from person to person and is helpful in isolating hearing problems.

Example 5.3.15: *A loudspeaker manufacturer advertises that their model no. 801 speaker produces a sound pressure level of 87 db when a reference test tone is applied. A competing speaker company advertises that their model X-1 speaker produces a sound pressure level of 93 db when fed the same test signal. What is the ratio of the two sound intensities produced by these speakers? If you wanted to find a speaker which produces a sound of intensity twice that of the no. 801 when fed the test signal, what is its sound pressure level?*

Solution. If we let I_1 and I_2 refer to the sound intensities of the two speakers reproducing the test signal, then we have two equations:

$$\begin{aligned} 87 &= 10\log_{10}(\frac{I_1}{I_o}) \\ 93 &= 10\log_{10}(\frac{I_2}{I_o}) \end{aligned}$$

Using log properties, we can solve the first equation for I_1:

$$\begin{aligned} 87 &= 10\log_{10}(\frac{I_1}{I_o}) = 10\log_{10}(I_1) - 10\log_{10}(I_o) \\ \log_{10}(I_1) &= 8.7 + \log_{10}(I_o) \\ 10^{\log_{10}(I_1)} &= 10^{8.7+\log_{10}(I_o)} \\ I_1 &= 10^{8.7}10^{\log_{10}(I_o)} = 10^{8.7}I_o. \end{aligned}$$

Similarly, we find that $I_2 = 10^{9.3}I_o$. This means that the ratio of the intensities will be

$$\frac{I_2}{I_1} = \frac{10^{9.3}I_o}{10^{8.7}I_o} = 10^{0.6} = 3.98.$$

This means that the test signal on the $X-1$ speaker produces a sound pressure level nearly 4 times that of the same test signal on the no. 801 speaker.

To finish the problem, imagine a third speaker which produces a sound pressure level β, which is twice that of the first speaker. If I_3 is the corresponding intensity of the sound, then

as above, $I_3 = 10^{(\beta/10)} I_o$. We are assuming that $I_3 = 2I_1$, so this gives us the equation

$$\begin{aligned}
I_1 &= \frac{1}{2} I_3 \\
10^{8.7} I_o &= \frac{1}{2} 10^{(\beta/10)} I_o \\
\log_{10}(10^{8.7}) &= \log_{10}(\frac{1}{2} 10^{(\beta/10)}) \\
8.7 &= \log_{10}(\frac{1}{2}) + \log_{10}(10^{(\beta/10)}) \\
8.7 &= -0.30103 + (\beta/10) \\
90 &= \beta
\end{aligned}$$

So, the test signal on the third speaker must produce a sound pressure level of 90 db. □

Problems

1. These problems will help you develop your skills with logarithms.
   (a) Compute: $\log_5 3, \log_e 11, \log_{\sqrt{2}} \pi, \log_2 10, \log_{10} 2$.
   (b) Solve for x: $35 = e^x, \log_3 x = e, \log_3 5 = xe^3$.
   (c) Solve each of these equations for x in terms of y: $y = 10^x, 3y = 10^x, y = 10^{3x}$.
2. Solve the following equations for x:
   (a) $\log_3(5) = \log_2(x)$
   (b) $10^{\log_2(x)} = 3$
   (c) $3^{5^x} = 7$
   (d) $\log_2(\ln(x)) = 3$
   (e) $e^x = 10^5$
   (f) $2^{3x+5} = 3^2$
   (g) $e^{\sin(x)} = \frac{1}{2}$
3. Return to the exponential models for Men's and Women's Earning Power in Exercise 5.2.3: $M(x) = 9521(1.0662066)^x$ and $W(x) = 5616(1.0727495)^x$. Determine when the earning power of women will exceed the earning power of men.
4. In 1987, the population of Mexico was estimated at 82 million people, with an annual growth rate of 2.5%. The 1987 population of the United States was estimated at 244 million with an annual growth rate of 0.7 %. Assume the future population of each country is predicted by the "continuous compounding" formula in (5.2.6).
   (a) When will Mexico double its 1987 population?
   (b) When will the United States and Mexico have the same population?
5. The average tenure of a Professor at the University of Washington is 31.6 years. The administration believes that a Professor's salary after 31.6 years of service should be 2.5

times his/her "hiring in" salary. Assume the Professor's salary grows with continuous compounding according to this constraint.

(a) What is the annual rate r of salary growth?

(b) Assume inflation grows at an annual rate of $r = 3.4\%$, compounded continuously. If a Professor is hired at \$30,000, what is the inflation adjusted buying power of his/her salary at retirement after 40 years of service?

6. *MacForever* Magazine has published data on the average downtime due to system crashes for the new Windows 95 PC operating system. One month after installation, the user average downtime was 2%. Six months after installation, the user average downtime was 3.2%. The magazine goes on to claim "...downtime is increasing exponentially...". If this is the case, find an exponential model $d(t)$ which predicts the average amount of downtime t months after installation of the Windows 95 operating system. What is the downtime two years after installation? When will the computer be down 100% of the time?

7. Rewrite each function in the form $y = A_o e^{at}$, for appropriate constants A_o and a.

(a) $y = 13(3^t)$

(b) $y = -7(1.567)^{t-3}$

c. $y = -17(2.005)^{-t}$

d. $y = 3(14.24)^{4t}$

8. Complete the table:

$P(t) = P_o e^{rt}$	$P(t) = P_o b^t$	$P(0)$	point on $y = P(t)$	point on $y = P(t)$
$P(t) = 5000e^{0.03t}$			$(1,\quad\quad)$	$(\quad\quad, 40034)$
	$P(t) = 84(1/4)^t$		$(-3,\quad\quad)$	$(\quad\quad, 2)$
		34	$(7, 120)$	$(\quad\quad, 1)$
			$(4, 20)$	$(19, 3)$

9. A *cancerous cell* lacks normal biological growth regulation and can divide continuously. Suppose a single mouse skin cell is cancerous and its *mitotic cell cycle* (the time for the cell to divide once) is 20 hours. The number of cells at time t grows according to an exponential model.

(a) Find a formula $C(t)$ for the number of cancerous skin cells after t hours.

(b) Assume a typical mouse skin cell is spherical of radius 50×10^{-4} cm. Find the combined volume of all cancerous skin cells after t hours. When will the volume of cancerous cells be 1 cm^3?

Appendix A

Selected Solutions

Important: When a new text is under development, changes, revisions and human error can lead to mistakes in the solutions. If you are in doubt about a particular answer, ask your TA, your Professor or visit the Math Study Center. In addition, your professor may have a website link that tabulates current typos and corrections to the text.

Solutions from Chapter 1

1.1: #1. (a) 5.5 min/mi = 5:30 pace. (b) $14\frac{2}{3}$ ft/sec. (c) Adrienne.

#2. Go for the 15 inch pie.

#4. 1080 pies. 8:22pm

#8. Formula simplifies to $\frac{r(1+x)}{1+2x}$.

1.2: #1. (a) $d(t) = (65.3)t$;(b) 227 minutes, 168.4 miles; (c) $t = 80.86$ seconds.

#2. (a) port origin: (200,210), boat origin: (-100,710) (b) port origin: (550,-425), boat origin:(250,75) (c) port origin:(300,-500), boat origin: (0,0) (d) port origin:(0,0), boat origin: (-300,500)

#3. (a)Adrienne at (-16,0); Allyson at (0,20). (b) No, the distance between the girls is only 25.612 ft. (c) $t = 2.3426$ seconds. (d) $t = 5.5$ seconds.

#5. (a)$\sqrt{8}, \sqrt{17/2}, \sqrt{10}$. (d)$\sqrt{2t^2+8}$. (e)$t = \sqrt{14}$, $P(\sqrt{14}) = (1+\sqrt{14}, 1)$, $Q(\sqrt{14}) = (-1, -1+\sqrt{14})$.

1.3: #1. (a)$(x+3)^2 + (y-4)^2 = 9$; (b) $(x-3)^2 + (y+(11/3))^2 = (1/16)$; (c) Lots of possible answers. One way is to start with the point (1,1), then add or subtract 2 from one of the coordinates; this gives the points: (1,3),(3,1),(-1,1),(1,-1). We have moved in the North, South, East and West directions to get the centers of four circles of radius 2 passing thru (1,1). Here are the equations: $(x-1)^2+(y-3)^2 = 4$, $(x-3)^2+(y-1)^2 = 4$, $(x+1)^2+(y-1)^2 = 4$,

$(x-1)^2+(y+1)^2=4$; (d) Plug in and check (1,1), (1,-3), $(1+\sqrt{3},0),(0,-1-\sqrt{3})$ all satisfy the equation;

#2. (a) Let's impose coordinates so the x-axis is ground level and the y-axis is along the tower. The wheel is modeled by the equation: $x^2+(y-62)^2=(60)^2$. (The other natural choice would have the origin at the center of the wheel; in this case, the equations will change, but the final answers would agree.) (b) Solve the system consisting the the above circle equation and the equation $y=100$. End up with solutions $x=46.43,-46.43$. From the picture, conclude cone lands 46.43 ft to the right of tower base. (c) Solve the system consisting of the circle equation and the equation $x=-24$. End up with solutions $y=117,7$. This means that at the rider locations (-24,7) and (-24,117) a dropped cone nails the operator.

#3. (a) Impose coordinates with the sprinkler the origin; sidewalks modeled by $y=100, y=110$. (b) 32 minutes; (c) 52 minutes; (d) 20 minutes; (f) 7257-400=6857 sq. ft.

1.4: #1. (a) $y=(-5/3)(x-1)-1$; (b) $y=40(x+1)-2$; (c) $y=-2x-2$; (d) $y=11$; (e) $m=3/5, y=(3/5)(x-1)+1$; (f) $y=40x-14$; (g) $m=-3/4$; (h) $y=x-1$.

#2. (a) $y=6850(x-1970)+38000$. (b) $y=8000(x-1970)+8400$. (d) 1983: Sea=\$127,050, PT=\$112,400. 1998: Sea=\$229,800, PT=\$232,400. (e) 1995.74,\$214,313. (f) 1982.7; Sea=\$124,965. (g) NO.

#4. (a) Allyson at (0,70); Adrienne at (-44,0). Bungee is 83.1401 ft. long. (b) Occurs at time $t=7.7546$ seconds and Allyson is at (0,77.546).Allyson's final location is 77.546 ft. from her starting point.

#6. (a)$\sqrt{5t^2-38t+74}$. (b)$t=2.11477, 5.48523$. At time $t=2.11477$, ant= $(1.77046, 2)$, spider= $(-1,-0.88523)$.

#7. Line of travel is $y=(0.667)(x-35)$. (a) (-13.46,-32.31); (b) 3.19 sec.; (c) 5.82 sec.; (d) (10.77,-16.22) at about $t=6.1$ sec.

Solutions from Chapter 2

2.1: #1. (a) $y=f(x)=x+1$ is a function on any domain of x values. (b) $y=f(x)=\frac{2}{3}x-\frac{5}{3}$ is a function on any domain of x values. (c) Not a function in the independent variable x.

#4. (a) $y=-1\pm\sqrt{\frac{1}{4}(1-\frac{x^2}{9})}$, $-3\le x\le 3$. (b) $y=2\pm b\sqrt{1-(\frac{x+1}{a})^2}, -a-1\le x\le a-1$. (c) $y=-4\pm b\sqrt{(\frac{x-1}{a})^2-1}, x\ge a+1$ or $x\le -a+1$.

#5. (a) Not a function, by the vertical line test. Slice ellipse symmetrically into upper and lower halves to get two seperate curves, each of which will be the graph of a function. (o) This is a function, by the vertical line test.

2.2: #2. (a) The depth of the water is 156 inches (13 feet), so the width is 60.92 feet. (b) 0.3 minutes. 8.04 minutes. 116.2 minutes. (c) For the water to rise the full twenty feet (240 inches) you must wait 120 minutes (2 hours).

#4.(a) $j(t)=62.2(t+1)$. (b) $s(t)=280-70t$.

2.3: #1.(a) $2(x-4)^2+9$, Vertex: (4, 9), Axis: $x=4$. (b)$(x-3/14)^2+2539/196$, Vertex: (3/14, 2539/196), Axis: $x=3/14$. (c)$2(x-0)^2-0$, Vertex: (0, 0), Axis: $x=0$.

#2. (a) Find the vertex; $y=100$. (b) Check out prob 1.6.11 if you have difficulties - be careful - it's a little different. $y=156.25$. (c) $x=625$ ft. (d) Using the quadratic formula, you get $x=54.81$ ft and $x=570.19$ ft.

2.4:

2.5: #2. (b) No.(f) c=11/2.(g) c=5/2.(h) c=1/6.

#3. $-\frac{1}{3}\le x\le 0$

2.6:

2.7: #1. Cutouts $x=10-5\sqrt{2}=2.93$ and $x=5$ are the only sizes feasible.

2.8:

2.9: #2.(a) $y=0.03x^2-1.1x+14$ (b) $w=0.03x-1.1+\frac{14}{x}$ (c) 70 or $6\frac{2}{3}$

#3.(a) $k=400$ (b) $I(t)=\frac{400}{484t^2-1452t+1189}$ (c) $t=1.5$ (d) $t=1.05$ and 1.95

Solutions from Chapter 3

3.1: #1. (a) $13^o24'$ or 0.233874 rads. (b) 1.0788 degs or .01882 rads. (c) 5.7296 degs or $5^o43'46.5"$.

#2. (a) 1413.7 square feet. (c) 4.244 secs .

#4. 2160 miles

3.2: #1. (a) $\omega=22\pi$ rad/min and $v=484\pi$ ft/min. (b) $v=125.66$ in/min, $\tau=2.5$ RPM and $\omega=0.2618$ rad/sec. (c) $v=1036.7$ mph and $\omega=0.2618$ rad/hour. (d) $\omega=95.33$ rad/sec and $\tau=910.4$ RPM.

#2. (a) Radius of tires and wheels is 7 + 5.5 in = 12.5 in, but it's supposed to be 11.5 in. Your speedometer figures out speed by angular speed of the tires with right radius. So the ang. speed, using (2.2.2), is (1144 in/sec) /(11.5 in) = 99.478 rad/sec. But actual speed, using (2.2.2), is (99.478 rad/sec)(12.5 in) = 1243.478 in/sec = 70.65 mph. (b) 32.72 mph.

#3. 20.31 rad/sec = 193.942 RPM.

#4. (a) At 3/4 inch, the revolutional speed is 601.61 RPM and at 5.5 inches it's 82.04 RPM. (b) The radius is 4.51 inches.

#6. (b) 700 feet. (c) 15,000 ft^2.

#7. (a) $v=293\frac{1}{3}$ ft/sec, $r=233.427$ ft. (b) $\theta=0.06283$ rad. (c) $2\pi/5$ rad counterclockwise from P.

3.3: #2. Dam is 383 feet high.

#3. (b) $(23.882, 2.375)$.

#5. (a) $Q=(96.186,-175.352)$. (d) Aaron crosses the track at $(-65.87,-188.84)$ and Michael's location at this instant is $(186.09, 73.29)$.

#7. (a) If you impose coordinates with the center of the wheel at $(0, 237.427)$, then ground level coincides with the x-axis. (a) $T(t) = (x(t), y(t))$, where $x(t) = 233.427\cos(\frac{2\pi}{5}t - 0.06283)$ and $y(t) = 233.427\sin(\frac{2\pi}{5}t - 0.06283) + 237.427$. (b) $T(6) = (85.93, 454.45)$. (c) First find the slope of a radial line from the wheel center out to Tiff's launch point.

3.4: #1. (a) 7/25 or -7/25. (b) -0.6. (c) $\pm\sqrt{(}40)/7$.

3.5: #1. (a) $1, \pi, \frac{\pi}{2}, 1$. (b) $6, 2, 0, -1$. (c) $2, \pi, 0, 3$.

#4. (a) A = 25, B = 5 seconds, C = 1.75, D = 28. (b) t=1.75 and 4.25 seconds

#6. (a) $b(t) = 0.6\sin(\frac{2\pi}{100}(t - 30)) + 1.2$.

#7. $E(t) = 27\sin(\frac{2\pi}{366}(t - 80.5)) + 45$.

3.6: #1. (a)(i) 0, 1.5708, -1.5708, 1.0472, 0.7168, -0.2762, not defined.

Solutions from Chapter 4

4.1: #1. (a) P(0)=(1,0), P(1)=(-0.4161,0.9093), P(2)=(-0.6536,-0.7568), P(3)=(0.9602,-0.2794), P(4)=(-0.1455,0.9894), P(5)=(-0.8391,-0.5440). (b) $t = 0.7854$ seconds. (c) $t = 0.8861$ seconds.

#2. Just to make sure you are on the right track, the types of curves are: (a) vertical line segment , (b) horizontal line segment, (c) line segment, (d) line segment.

#3. (a) $\omega = 0.2764$ rad/sec. $v = 18.85$ mph. (b) $x(t) = 100\cos(3\pi/2 + 0.2764t)$, $y(t) = 103 + 100\sin(3\pi/2 + 0.2764t)$. (c) 200 ft.

#5. (a) 6 feet. (b)$t = 0.553155$ seconds. (c) No. (d) At times $t = 0.676666, 1.47783$ seconds it is 22 feet above the floor. (e) Maximum height is 24.5675 feet. (f) Allyson teaches Lee a lesson. (g) $y = f(x) = 6 + 1.19177x - 0.0191238x^2$.

4.2: #1. (a) $x(t) = 6 - 0.8t, y(t) = 3 + 0.4t$. (b)$v_x = -0.8, v_y = 0.4, v = 0.894$. (c) 5.814 feet.

#2. (a) $x(t) = 30 - (30/7)t, y(t) = 24 - (24/7)t$ (b) 11.78 ft. (c) Enters green when t = 2.08 sec at $(21.0857, 16.869)$. Leaves green when $t = 11.92$ sec at $(-21.0857, -16.869)$. (d) $t = 6.8862$ sec, $x = 0.4878$, $y = 0.3902$. (e) $d(t) = \sqrt{(x(t) - 4)^2 + (y(t) + 4)^2} = \sqrt{(1476/49)t^2 - (2904/7)t + 1460}$ (f) Two answers: $t = 5.849$ sec, so $x = 4.932$, $y = 3.946$; t = 7.923, so x = -3.956, y = -3.165.

#3. Michael $M(t) = (5.547t, 8.32t)$, Tim $T(t) = (400-8.944t, 50+4.472t)$. Study distance SQUARED from $M(t)$ to $T(t)$. Closest when $t = 26.641$ sec and distance is 54.3 ft.

Solutions from Chapter 5

5.1: #1. (a) 31.5443; (b) 355.1134; (c) 36.4622; (d) 0.0616; (e) 51,168; (f) 0.009794; (g)-84.2; (h) $x = 0.1136 + 2k\pi, 3.0280 + 2k\pi, k$ an integer .

#2. (a) $x = 5$; (b) no solution; (c) $x = 0$; (d) $x = 2$; (e) $x = 0, 5/2$; (f) $x = 1/2$.

5.2: #1. (a) $p(x) = 860(1.070674)^x$, $l(x) = 70x + 860$. (b) $p(10) = 1702, l(10) = 1560$. (d)When $1.11336 \leq x \leq 3.9821$ the linear model population exceeds the exponential model by at least 10.

#2. (a) $14.34 lb/in^2$; no explosion. (b) $8.32 lb/in^2$; explosion. (c) 54.6 degrees.

5.3: #1. (a) 0.6826; 2.3979; 3.3030; 3.3219;0.3010. (b) 3.555; 19.8; 0.0729. (c) $x = \log_{10} y$; $x = \log_{10}(3y)$; $x = (1/3)\log_{10}(y)$.

#2. (a) $x = 2.7606$. (b) 1.392. (c) 0.3552. (d) $x = e^8$. (e) $x = 11.513$. (f)-0.61. (g) $\sin(x) = -0.6931$; $x = -0.7658 + 2k\pi$ or $x = 3.9074 + 2k\pi$.

#3. Equal after 86.29 years; in the year 2056.

#6. $d(t) = (0.0182056)(1.0986)^t$; 17.38%; the machine dies in 42.6 months.

Appendix B

Formulas, Constants and Abbreviations

Abbreviations

inch	=in	millimeter	=mm	second	=sec = s	ounce	=oz
feet	=ft	centimeter	=cm	minute	=min	pound	=lb
yard	=yd	meter	=m	hour	=hr	gram	=g
mile	=mi	kilometer	=km	year	=yr	kilogram	=kg
quart	=qt	milliliter	=ml	Joule	=J	atmosphere	=atm
gallon	=gal	liter	=L	calorie	=cal	Coulomb	= C

radian	=rad	miles per hour = mi/hr = mph	meters per second = m/sec = m/s
degree	=deg= $^\circ$	feet per second = ft/sec = ft/s	revolutions per minute= rev/min = RPM

Conversion Factors

Length	**Volume**	**Energy**	**Mass**
1 in = 2.54 cm	1 gallon = 3.7854 L	1 J = 1 kg m^2/s^2	1 oz = 28.3495 g
1 ft = 0.3048 m	1 quart = 0.946353 L	1 cal = 4.184 J	1 lb = 0.453592 kg
1 mi = 1.609344 km			

Formulas from Plane and Solid Geometry

Rectangle:
Perimeter=$2\ell + 2w$; Area= ℓw

Triangle:
Perimeter= $a + b + c$; Area= $\frac{1}{2}bh$

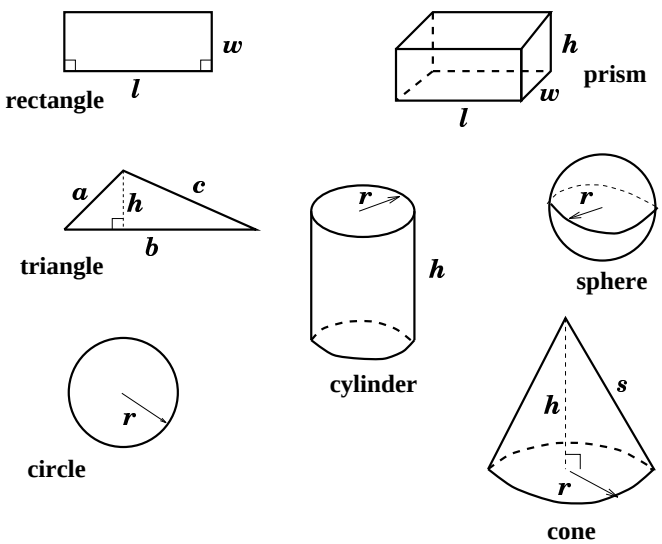
Circle:
Perimeter= $2\pi r$; Area= πr^2

Rectangular prism:
Surface Area=$2(\ell w + \ell h + wh)$; Volume=ℓhw

Right circular cylinder:
Surface Area=$2\pi r^2 + 2\pi rh$; Volume = $\pi r^2 h$

Sphere:
Surface Area=$4\pi r^2$; Volume = $\frac{4}{3}\pi r^3$

Right circular cone:
Surface Area=$\pi r^2 + \pi rs$; Volume = $\frac{1}{3}\pi r^2 h$



Constants

Avogadro's number = N = 6.02×10^{23}
speed of light = $c = 3 \times 10^8$ m/s
density of water = 1 g/cm^3

mass of earth = 5.98×10^{24} kg
earth equator radius = 3960 mi= 6.38×10^6 m

acceleration of gravity (earth's surface) = 32 ft/s^2 = 9.8 m/s^2

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