Math 407: Linear Optimization

The Fundamental Theorem of Linear Programming
The Strong Duality Theorem
Complementary Slackness

The Two Phase Simplex Algorithm

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(iii) *If it is feasible and bounded, then it has an optimal basic feasible solution.*

Duality Theory

$$\mathcal{P} \quad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b, \ 0 \leq x \end{array} \qquad \mathcal{D} \quad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c, \ 0 \leq y \end{array}$$

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What is the dual to the dual?

The Dual of the Dual

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c, \\ & 0 \leq y \end{array}$$

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The dual of the dual is the primal.

The Weak Duality Theorem

Theorem:
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T Ax \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

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We combine the Weak Duality Theorem with the Fundamental Theorem of Linear Programming to obtain the *Strong Duality Theorem*.

The Strong Duality Theorem

Theorem:
If either $\mathcal{P}$ or $\mathcal{D}$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both $\mathcal{P}$ and $\mathcal{D}$ exist.

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Remark: In general a finite optimal value does not imply the existence of a solution.

$$\min f(x) = e^x$$

The optimal value is zero, but no solution exists.

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The optimal tableau is

$$\left[\begin{array}{cc|c} RA & R & Rb \\ \hline c^T - y^T A & -y^T & -y^T b \end{array}\right],$$

where we have already seen that y solves $\mathcal{D}$, and the optimal values coincide.

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This concludes the proof.

Complementary Slackness

Theorem: [WDT]
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T A x \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

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The SDT implies that x solves $\mathcal{P}$ and y solves $\mathcal{D}$ if and only if (x, y) is a $\mathcal{P}$-$\mathcal{D}$ feasible pair and

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We now examine the consequence of this equivalence.

Complementary Slackness

The equation $c^T x = y^T Ax$ implies that

$$0 = x^T(A^T y - c) = \sum_{j=1}^{n} x_j(\sum_{i=1}^{m} a_{ij} y_i - c_j). \qquad (\clubsuit)$$

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$\mathcal{P}$-$\mathcal{D}$ feasibility gives

$$0 \leq x_j \qquad \text{and} \qquad 0 \leq \sum_{i=1}^{m} a_{ij} y_i - c_j \qquad \text{for } j = 1, \ldots, n.$$

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Hence, ($\clubsuit$) can only hold if

$$x_j(\sum_{i=1}^{m} a_{ij} y_i - c_j) = 0 \quad \text{for} \quad j = 1, \ldots, n, \quad \text{or equivalently,}$$

$$x_j = 0 \qquad \text{or} \qquad \sum_{i=1}^{m} a_{ij} y_i = c_j \qquad \text{or both for } j = 1, \ldots, n.$$

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Similarly, the equation $y^T Ax = b^T y$ implies that

$$0 = y^T(b - Ax) = \sum_{i=1}^{m} y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j). \quad \left(\begin{array}{c} 0 \leq y_i \\ 0 \leq b_i - \sum_{j=1}^{n} a_{ij}x_j \end{array} \right)$$

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Hence,

$$y_i = 0 \qquad \text{or} \qquad \sum_{j=1}^{n} a_{ij}x_j = b_i \qquad \text{or both for } i = 1, \ldots, m.$$

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- $y_i = 0 \quad \text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad$ or both for $i = 1, \ldots, m$.

Complementary Slackness Theorem

Theorem:
The vector $x \in \mathbb{R}^n$ *solves* $\mathcal{P}$ *and the vector* $y \in \mathbb{R}^m$ *solves* $\mathcal{D}$ *if and only if* x *is feasible for* $\mathcal{P}$ *and* y *is feasible for* $\mathcal{D}$ *and*

(i) *either* $0 = x_j$ *or* $\sum_{i=1}^{m} a_{ij} y_i = c_j$ *or both for* $j = 1, \ldots, n$, *and*

(ii) *either* $0 = y_i$ *or* $\sum_{j=1}^{n} a_{ij} x_j = b_i$ *or both for* $i = 1, \ldots, m$.

Corollary to the Complementary Slackness Theorem

Corollary:
The vector $x \in \mathbb{R}^n$ solves $\mathcal{P}$ if and only if x is feasible for $\mathcal{P}$ and there exists a vector $y \in \mathbb{R}^m$ feasible for $\mathcal{D}$ and such that

(i) *if* $\sum_{j=1}^{n} a_{ij} x_j < b$, *then* $y_i = 0$, *for* $i = 1, \ldots, m$ *and*

(ii) *if* $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

Testing Optimality via Complementary Slackness

Does

$$x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$$

solve the LP

$$\begin{array}{lrcrcrcrcrcl}
\text{maximize} & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\
\text{subject to} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & 4 \\
& 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 3 \\
& 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 5 \\
& 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 \\
\end{array}$$

$$0 \leq x_1, x_2, x_3, x_4, x_5.$$

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Testing Optimality via Complementary Slackness

The point

$$x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$$

must be feasible for the LP.

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Plugging into the constraints we get

$$\begin{array}{rcrcrcrcrcr}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 \\
4(0) & + & 2\left(\frac{4}{3}\right) & - & 2\left(\frac{2}{3}\right) & + & \left(\frac{5}{3}\right) & + & (0) & = & 3 \\
2(0) & + & 4\left(\frac{4}{3}\right) & + & 4\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 5(0) & < & 5 \\
3(0) & + & \left(\frac{4}{3}\right) & + & 2\left(\frac{2}{3}\right) & - & \left(\frac{5}{3}\right) & - & 2(0) & = & 1.
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\end{array}$$

Can we use this information to construct a solution to the dual problem, (y_1, y_2, y_3, y_4)?

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Recall that

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Recall that
if $\sum_{j=1}^{n} a_{ij}x_j < b$, *then* $y_i = 0$, *for* $i = 1, \ldots, m$.
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$$\begin{array}{ccccccccccccc}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 & : & y_1 \\
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Also recall that

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Hence,

$$3y_1 + 2y_2 + 4y_3 + y_4 = 6 \qquad \text{(since } x_2 = \tfrac{4}{3} > 0\text{)}$$

Testing Optimality via Complementary Slackness

Also recall that

if $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

Hence,

$$\begin{array}{rcrcrcrcl} 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 \qquad (\text{since } x_2 = \tfrac{4}{3} > 0) \\ 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 \qquad (\text{since } x_3 = \tfrac{2}{3} > 0) \end{array}$$

Testing Optimality via Complementary Slackness

Also recall that

if $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

Hence,

$$\begin{array}{rcrcrcrcr}
3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 \\
5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 \\
-2y_1 & + & y_2 & - & 2y_3 & - & y_4 & = & -2
\end{array}
\quad
\begin{array}{l}
(\text{since } x_2 = \frac{4}{3} > 0) \\
(\text{since } x_3 = \frac{2}{3} > 0) \\
(\text{since } x_4 = \frac{5}{3} > 0)
\end{array}$$

Testing Optimality via Complementary Slackness

Combining these observations gives the system

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -2 \\ 0 \end{pmatrix},$$

which any dual solution must satisfy.

Testing Optimality via Complementary Slackness

Combining these observations gives the system

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -2 \\ 0 \end{pmatrix},$$

which any dual solution must satisfy.
This is a square system that we can try to solve for y.

Testing Optimality via Complementary Slackness

3	2	4	1	6	
5	−2	4	2	5	
−2	1	−2	−1	−2	
0	0	1	0	0	
3	2	0	1	6	$r_1 - 4r_4$
5	−2	0	2	5	$r_2 - 4r_4$
−2	1	0	−1	−2	$r_3 + 2r_4$
0	0	1	0	0	
1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
−2	1	0	−1	−2	
0	0	1	0	0	

Testing Optimality via Complementary Slackness

3	2	4	1	6	
5	-2	4	2	5	
-2	1	-2	-1	-2	
0	0	1	0	0	
3	2	0	1	6	$r_1 - 4r_4$
5	-2	0	2	5	$r_2 - 4r_4$
-2	1	0	-1	-2	$r_3 + 2r_4$
0	0	1	0	0	
1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
-2	1	0	-1	-2	
0	0	1	0	0	

1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
-2	1	0	-1	-2	
0	0	1	0	0	
0	3	0	0	3	$r_1 - r_2$
1	0	0	0	1	
0	1	0	-1	0	$r_3 + 2r_2$
0	0	1	0	0	
1	0	0	0	1	r_2
0	1	0	0	1	$\frac{1}{3}r_1$
0	0	1	0	0	r_4
0	0	0	1	1	$-r_3 + \frac{1}{3}r_1$

Testing Optimality via Complementary Slackness

3	2	4	1	6	
5	-2	4	2	5	
-2	1	-2	-1	-2	
0	0	1	0	0	
3	2	0	1	6	$r_1 - 4r_4$
5	-2	0	2	5	$r_2 - 4r_4$
-2	1	0	-1	-2	$r_3 + 2r_4$
0	0	1	0	0	
1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
-2	1	0	-1	-2	
0	0	1	0	0	

1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
-2	1	0	-1	-2	
0	0	1	0	0	
0	3	0	0	3	$r_1 - r_2$
1	0	0	0	1	
0	1	0	-1	0	$r_3 + 2r_2$
0	0	1	0	0	
1	0	0	0	1	r_2
0	1	0	0	1	$\frac{1}{3}r_1$
0	0	1	0	0	r_4
0	0	0	1	1	$-r_3 + \frac{1}{3}r_1$

This gives the solution $(y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$.

Testing Optimality via Complementary Slackness

3	2	4	1	6	
5	−2	4	2	5	
−2	1	−2	−1	−2	
0	0	1	0	0	
3	2	0	1	6	$r_1 - 4r_4$
5	−2	0	2	5	$r_2 - 4r_4$
−2	1	0	−1	−2	$r_3 + 2r_4$
0	0	1	0	0	
1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
−2	1	0	−1	−2	
0	0	1	0	0	

1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
−2	1	0	−1	−2	
0	0	1	0	0	
0	3	0	0	3	$r_1 - r_2$
1	0	0	0	1	
0	1	0	−1	0	$r_3 + 2r_2$
0	0	1	0	0	
1	0	0	0	1	r_2
0	1	0	0	1	$\frac{1}{3}r_1$
0	0	1	0	0	r_4
0	0	0	1	1	$-r_3 + \frac{1}{3}r_1$

This gives the solution $(y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$.
Is this dual feasible?

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & & 0 \leq y_1, y_2, y_3, y_4. & & & & &
\end{array}$$

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & & 0 \leq y_1, y_2, y_3, y_4. & & & & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & & 0 \leq y_1, y_2, y_3, y_4. & & & & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
& 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
& 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
& -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
& 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
& & & & 0 \leq y_1, y_2, y_3, y_4. & & & & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: $1 + 4 + 0 + 3 = 8 > 7$

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
& 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
& 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
& -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
& 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
& & & & 0 \leq y_1, y_2, y_3, y_4. & & & & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: $1 + 4 + 0 + 3 = 8 > 7$

Fifth: $2 + 1 + 0 - 2 = 1 \not\geq 3$, the fifth dual inequality is violated.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & & & 0 \leq y_1, y_2, y_3, y_4. & & & &
\end{array}
$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: $1 + 4 + 0 + 3 = 8 > 7$

Fifth: $2 + 1 + 0 - 2 = 1 \not\geq 3$, the fifth dual inequality is violated.

Hence, $x = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$ cannot be optimal!

Example: Testing Optimality via Complementary Slackness

Does the point $x = (1, 1, 1, 0)$ solve the following LP?

$$
\begin{array}{lrrrrcl}
\text{maximize} & 4x_1 & +2x_2 & +2x_3 & +4x_4 & & \\
\text{subject to} & x_1 & +3x_2 & +2x_3 & +\ x_4 & \leq & 7 \\
 & x_1 & +\ x_2 & +\ x_3 & +2x_4 & \leq & 3 \\
 & 2x_1 & & +\ x_3 & +\ x_4 & \leq & 3 \\
 & x_1 & +\ x_2 & & +2x_4 & \leq & 2 \\
 & 0 \leq & x_1, x_2, & x_3, x_4 & & &
\end{array}
$$