Math 407A: Linear Optimization

Lecture 8: Initialization and the Two Phase Simplex Algorithm

Math Dept, University of Washington

1. Initialization

2. The Auxilliary Problem

3. The Two Phase Simplex Algorithm

Initialization

We have shown that if we are given a feasible dictionary (tableau) for an LP, then the simplex algorithm will terminate finitely if it is employed with a anti-cycling rule.

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- The LP is determined to be unbounded.
- An optimal BFS is found.

We now address the question of how to determine an initial feasible dictionary (tableau).

The Auxiliary Problem

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject tp} & Ax \leq b, \quad 0 \leq x. \end{array}$$

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Consider an auxiliary LP of the form

$$\begin{array}{lll} \mathcal{Q} & \text{minimize} & x_0 \\ & \text{subject to} & Ax - x_0\mathbf{1} \leq b, \quad 0 \leq x_0, x \ . \end{array}$$

where $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.

The Auxiliary Problem

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^Tx \\ & \text{subject tp} & Ax \leq b, \quad 0 \leq x. \end{array}$$

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where $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.
The ith row of the system of inequalities $Ax - x_0\mathbf{1} \leq b$ is

$$a_{i1}x_1 \ + \ a_{i2}x_2 \ + \ \ldots \ + \ a_{in}x_n \ \leq \ b_i + x_0 \ .$$

The Auxiliary Problem

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$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i + x_0 \ .$$

In block matrix form we write

$$\left[\begin{array}{cc} -\mathbf{1} & A \end{array}\right] \begin{pmatrix} x_0 \\ x \end{pmatrix} \leq b \ .$$

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If the optimal value in the auxiliary problem is zero, then at the optimal solution $(\tilde{x}_0, \tilde{x})$ we have $\tilde{x}_0 = 0$.

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Plugging into $Ax - x_0\mathbf{1} \le b$, we get $A\tilde{x} \le b$, i.e. $\tilde{x}$ is feasible for $\mathcal{P}$.

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Plugging into $Ax - x_0\mathbf{1} \leq b$, we get $A\tilde{x} \leq b$, i.e. $\tilde{x}$ is feasible for $\mathcal{P}$.

On the other hand, if $\hat{x}$ is feasible for $\mathcal{P}$, then $(\hat{x}_0, \hat{x})$ with $\hat{x}_0 = 0$ is feasible for $\mathcal{Q}$, so $(\hat{x}_0, \hat{x})$ is optimal for $\mathcal{Q}$.

The Auxiliary Problem

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject tp} & Ax \leq b, \\ & & 0 \leq x \end{array} \qquad \begin{array}{lll} \mathcal{Q} & \text{minimize} & x_0 \\ & \text{subject to} & Ax - x_0\mathbf{1} \leq b, \\ & & 0 \leq x_0, x \end{array}$$

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- $\mathcal{P}$ is feasible $\Leftrightarrow$ the optimal value in $\mathcal{Q}$ is zero.

The Auxiliary Problem

$$\begin{array}{lllllll} \mathcal{P} & \text{maximize} & c^T x & \qquad & \mathcal{Q} & \text{minimize} & x_0 \\ & \text{subject tp} & Ax \leq b, & & & \text{subject to} & Ax - x_0\mathbf{1} \leq b, \\ & & 0 \leq x & & & & 0 \leq x_0, x \end{array}$$

- $\mathcal{P}$ is feasible $\Leftrightarrow$ the optimal value in $\mathcal{Q}$ is zero.
- $\mathcal{P}$ is infeasible $\Leftrightarrow$ the optimal value in $\mathcal{Q}$ is positive.

Two Phase Simplex Algorithm

The auxiliary problem $\mathcal{Q}$ is also called the *Phase I* problem since solving it is the first phase of a two phase process of solving general LPs.

Two Phase Simplex Algorithm

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In Phase I we solve the auxiliary problem to obtain an initial feasible tableau for $\mathcal{P}$, and in Phase II we solve the original LP starting with the feasible tableau provided in Phase I.

Initializing the Auxiliary Problem

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Problem: The initial dictionary for $\mathcal{Q}$ is infeasible!

Initializing the Auxiliary Problem

$$\mathcal{Q} \quad \begin{array}{ll} \text{minimize} & x_0 \\ \text{subject to} & Ax - x_0\mathbf{1} \le b, \quad 0 \le x_0, x \ . \end{array}$$

Problem: The initial dictionary for $\mathcal{Q}$ is infeasible!

Solution: Set $x_0 = -\min\{b_i : i = 0, \dots, n\}$ with $b_0 = 0$,
then $b + x_0\mathbf{1} \ge 0$ since

$$\begin{aligned} \min\{b_i + x_0 : i = 1, \dots, m\} &= \min\{b_i : i = 1, \dots, m\} + x_0 \\ = \min\{b_i : i = 1, \dots, m\} &- \min\{b_i : i = 0, \dots, m\} \ge 0. \end{aligned}$$

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Hence, $x_0 = -\min\{b_i : i = 0, \ldots, m\}$ and $x = 0$ is feasible for $\mathcal{Q}$.

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Hence, $x_0 = -\min\{b_i : i = 0, \ldots, m\}$ and $x = 0$ is feasible for $\mathcal{Q}$.
It is also a BFS for $\mathcal{Q}$.

Initializing the Auxiliary Problem

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The initial dictionary for $\mathcal{Q}$ is

$$\begin{array}{rcl} x_{n+i} & = & b_i + x_0 - \sum_{j=1}^{m} a_{ij}x_j \\ z & = & -x_0. \end{array}$$

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$$b_{i_0} = \min\{b_i : i = 0, 1, \ldots, m).$$

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Let i_0 be such that

$$b_{i_0} = \min\{b_i : i = 0, 1, \ldots, m).$$

If $i_0 = 0$, the LP has feasible origin and so the initial dictionary is optimal.

Initializing the Auxiliary Problem

If $i_0 > 0$, then pivot on this row bringing x_0 into the basis yielding

$$\begin{aligned} x_0 &= -b_{i_0} + x_{n+i_0} + \sum_{j=1}^{m} a_{i_0 j} x_j \\ x_{n+i} &= b_i - b_{i_0} + x_{n+i_0} - \sum_{j=1}^{m} (a_{ij} - a_{i_0 j}) x_j, \quad i \neq i_0 \\ z &= b_{i_0} - x_{n+i_0} - \sum_{j=1}^{m} a_{i_0 j} x_j. \end{aligned}$$

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x_{n+i} &= b_i - b_{i_0} + x_{n+i_0} - \sum_{j=1}^{m} (a_{ij} - a_{i_0 j}) x_j, \quad i \neq i_0 \\
z &= b_{i_0} - x_{n+i_0} - \sum_{j=1}^{m} a_{i_0 j} x_j .
\end{aligned}$$

This dictionary is feasible for $\mathcal{Q}$.

Initializing the Auxiliary Problem: Example

$$\begin{array}{lrcrcrcr}
\text{max} & x_1 & - & x_2 & + & x_3 & & \\
\text{s.t.} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
 & & & \multicolumn{3}{c}{0 \leq x_1, x_2, x_3} & & .
\end{array}$$

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$$\begin{array}{lrcrcrcr}
\max & x_1 & - & x_2 & + & x_3 & & \\
\text{s.t.} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
\multicolumn{8}{c}{0 \leq x_1, x_2, x_3 \ .}
\end{array}$$

$$\begin{array}{lrcrcrcrcr}
\max & -x_0 & & & & & & & & \\
\text{s.t.} & -x_0 & + & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & -x_0 & + & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_0 & - & x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
\multicolumn{10}{c}{0 \leq x_0, x_1, x_2, x_3 \ .}
\end{array}$$

Example

$$\begin{array}{lrcrcrcrcr} \max & -x_0 \\ \text{s.t.} & -x_0 & + & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\ & -x_0 & + & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\ & -x_0 & - & x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\ \end{array}$$

$$0 \leq x_0, x_1, x_2, x_3\ .$$

Example

$$
\begin{array}{lrcrcrcrcr}
\max & -x_0 \\
\text{s.t.} & -x_0 & + & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & -x_0 & + & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_0 & - & x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
\end{array}
$$

$$0 \leq x_0, x_1, x_2, x_3 \ .$$

$$
\begin{array}{rrrrrrr|r}
-1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\
-1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

First Pivot

	x_0							
	−1	2	−1	2	1	0	0	4
	−1	2	−3	1	0	1	0	-5
	−1	−1	1	−2	0	0	1	−1
z	0	1	−1	1	0	0	0	0
w	−1	0	0	0	0	0	0	0

First Pivot

$$\begin{array}{c|ccccccc|c|l}
 & x_0 & & & & & & & & \\
 & -1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 & \\
 & \enclose{circle}{-1} & 2 & -3 & 1 & 0 & 1 & 0 & -5 & \text{most negative} \\
 & -1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 & \\
\hline
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & \\
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\hline
\end{array}$$

First Pivot

	x_0							
	−1	2	−1	2	1	0	0	4
	$\textcircled{-1}$	2	−3	1	0	1	0	-5
	−1	−1	1	−2	0	0	1	−1
z	0	1	−1	1	0	0	0	0
w	−1	0	0	0	0	0	0	0

	0	0	2	1	1	−1	0	9
	1	−2	3	−1	0	−1	0	5
	0	−3	4	−3	0	−1	1	4
z	0	1	−1	1	0	0	0	0
w	0	−2	3	−1	0	−1	0	5

First Pivot

$$
\begin{array}{c|ccccccc|c}
 & x_0 & & & & & & & \\
 & -1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\
 & -1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 \\
 & -1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 \\
\hline
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & 0 & -3 & \textcircled{4} & -3 & 0 & -1 & 1 & 4 \\
\hline
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\hline
\end{array}
$$

Second Pivot

$$\begin{array}{cccccccc|c}
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & 0 & -3 & \textcircled{4} & -3 & 0 & -1 & 1 & 4 \\
\cline{2-9}
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\cline{2-9}
\end{array}$$

Second Pivot

$$
\begin{array}{c|ccccccc|c}
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & 0 & -3 & \textcircled{4} & -3 & 0 & -1 & 1 & 4 \\
\hline
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccccc|c}
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline
\end{array}
$$

Second Pivot

$$\begin{array}{cccccccc|c}
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & 0 & -3 & \textcircled{4} & -3 & 0 & -1 & 1 & 4 \\
\hline
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\hline
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \textcircled{\frac{5}{4}} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline
\end{array}$$

Third Pivot

$$
\begin{array}{c|ccccccc|c}
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \textcircled{$\frac{5}{4}$} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline
\hline
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\
\hline
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Third Pivot

$$
\begin{array}{cccccccc|c}
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \enclose{circle}{\frac{5}{4}} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline\hline
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\
\hline
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Auxiliary
problem
solved.

Auxiliary Problem Solution

	-2	1	0	0	1	0	1	3
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$
w	-1	0	0	0	0	0	0	0

Original LP
is feasible.

Extract Initial Feasible Tableau

$$\begin{array}{c|ccccccc|c}
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\
\hline
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

Extract
initial.
feasible
tableau.

Third Pivot

$$\begin{array}{cccccccc|c}
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\ \cline{2-9}
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\ \cline{2-9}
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cline{2-9}
\end{array}$$

$$\begin{array}{cc|ccccccc|c}
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\ \cline{3-9}
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\ \cline{3-9}
w & \multicolumn{1}{c}{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Extract
initial.
feasible
tableau.

Phase II

$$\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & \textcircled{1} & 3 \\
\frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
-\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\ \hline
\frac{1}{5} & 0 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \frac{3}{5} \\ \hline\hline
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\ \hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}$$

Phase II: Solution

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}
$$

Phase II: Solution

$$\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}$$

Optimal primal and dual solutions are

Phase II: Solution

$$\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix}$$

Phase II: Solution

$$\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0 \end{pmatrix}$$

Phase II: Solution

$$\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0 \end{pmatrix}$$

with optimal value $z = .6$.

Steps for Phase I of the Two Phase Simplex Algorithm

We assume $b_{i_0} = \min\{b_i \ : \ i = 1, \dots, m\} < 0$.

1. Form the standard initial tableau: $\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c & 0 & 0 \end{array}\right]$.
2. Border the initial tableau: $\left[\begin{array}{cccc|c} \mathbf{-1} & 0 & A & I & b \\ \hline 0 & -1 & c & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array}\right]$.
3. In the first pivot, the pivot row is the i_0 row and the pivot column is the first column (the x_0 column).
4. Apply simplex algorithm on the w row until optimality.
5. If optimal value is positive, stop the original LP is not feasible.
6. If the optimal value is zero, extract feasible tableau for the original problem and pivot to optimality.

Example: Two Phase Simplex Algorithm

Use the two phase simplex method to solve the following LP:

$$\begin{array}{lrcrcr} \text{maximize} & 3x_1 & + & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & -1 \\ & -x_1 & - & x_2 & \leq & -3 \\ & 2x_1 & + & x_2 & \leq & 4 \\ & & 0 & \leq & x_1, & x_2 \end{array}$$

Hint: A complete solution is possible in 3 pivots.