Linear Programming

Lecture 6: The Simplex Algorithm
Language, Notation, and Linear Algebra

1. Dictionaries for LPs in Standard Form
2. The Simplex Algorithm via Matrix Multiplication
3. The Block Structure of the Simplex Algorithm
4. Block Structure and Matrix Multiplication
5. The Block Structure of an Optimal Tableau
6. Block Structure and Duality

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General Dictionaries

A dictionary for $\mathcal{P}$ is any system of the form

$$\begin{aligned} x_i &= \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j \qquad\qquad i \in B \\ z &= \widehat{z} + \sum_{j \in N} \widehat{c}_j x_j \end{aligned} \qquad (D_B)$$

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and such that the systems (D_I) and (D_B) have identical solution sets.

Properties of Dictionaries

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- A feasible dictionary is *optimal* if $\widehat{c}_j \le 0 \ j \in N$.

The Simplex Algorithm via Matrix Multiplication

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First recall that

$$\begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \begin{pmatrix} a \\ \alpha \\ b \end{pmatrix} = \begin{bmatrix} a-a \\ \alpha^{-1}\alpha \\ -b+b \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

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The elimination matrix and its inverse.

$$G = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \qquad G^{-1} = \begin{bmatrix} I & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I \end{bmatrix}$$

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Each simplex pivot can be realized as left multiplication of the simplex tableau by the appropriate Gaussian-Jordan pivot matrix.

The Simplex Algorithm via Matrix Multiplication

$$\left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{-5}{2} & 0 & 1 \end{bmatrix}\left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

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The Block Structure of the Simplex Algorithm

Let T_0 be the initial tableau:

$$T_0 = \left[\begin{array}{rcc|c} 0 & A & I & b \\ -1 & c^{\mathsf{T}} & 0 & 0 \end{array}\right].$$

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Let T_k denote the tableau after k pivots:

$$T_k = \left[\begin{array}{rrr|r} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\mathsf{T}} & -y^{\mathsf{T}} & \widehat{z} \end{array}\right]$$

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T_k is obtained from T_0 by multiplying it on the left by a product of Gaussian pivot matrices $G := G_k G_{k-1} \cdots G_1$:

$$GT_0 = T_k,$$

where G is invertible ($G^{-1} = G_1^{-1} G_2^{-1} \cdots G_k^{-1}$).

The Block Structure of the Simplex Algorithm

Let's investigate the structure of T_k by examining the consequence of this product in terms of the block structure of T_0 and T_k.

$$T_0 = \left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c^{\mathsf{T}} & 0 & 0 \end{array}\right] \qquad T_k = \left[\begin{array}{ccc|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\mathsf{T}} & -y^{\mathsf{T}} & \widehat{z} \end{array}\right]$$

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$$G = \begin{bmatrix} M & u \\ v^\intercal & \beta \end{bmatrix},$$

where $M \in \mathbb{R}^{m\times m}$, $u, v \in \mathbb{R}^m$, and $\beta \in \mathbb{R}$.

Block Structure and Matrix Multiplication

$$\left[\begin{array}{ccc|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\mathsf{T}} & -y^{\mathsf{T}} & \widehat{z} \end{array}\right] = T_k$$

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$$u = 0$$

Block Structure and Matrix Multiplication

$$\left[\begin{array}{ccc|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\mathsf{T}} & -y^{\mathsf{T}} & \widehat{z} \end{array}\right] = \left[\begin{array}{ccc|c} -u & MA + uc^{\mathsf{T}} & M & Mb \\ -\beta & v^{\mathsf{T}}A + \beta c^{\mathsf{T}} & v^{\mathsf{T}} & v^{\mathsf{T}}b \end{array}\right]$$

Equating terms on the left and right gives

$$u = 0 \qquad\qquad \beta = 1$$

Block Structure and Matrix Multiplication

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Equating terms on the left and right gives

$$\begin{aligned} u &= 0 \qquad\qquad & \beta &= 1 \\ M &= R & & \end{aligned}$$

Block Structure and Matrix Multiplication

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$$\begin{array}{ccc} u = 0 & & \beta = 1 \\ M = R & \text{and} & v = -y. \end{array}$$

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Therefore,

$$T_k = \begin{bmatrix} R & 0 \\ -y^{\mathsf{T}} & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c^{\mathsf{T}} & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^{\mathsf{T}} - y^{\mathsf{T}}A & -y^{\mathsf{T}} & -y^{\mathsf{T}}b \end{array}\right],$$

where the matrix R is necessarily invertible.

Block Structure and Matrix Multiplication

$$\left[\begin{array}{ccc|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\top} & -y^{\top} & \widehat{z} \end{array}\right] = \left[\begin{array}{ccc|c} -u & MA + uc^{\top} & M & Mb \\ -\beta & v^{\top}A + \beta c^{\top} & v^{\top} & v^{\top}b \end{array}\right]$$

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where the matrix R is necessarily invertible. (R $\sim$record matrix)

The Block Structure of an Optimal Tableau

$$T_k = \left[\begin{array}{cc} R & 0 \\ -y^\top & 1 \end{array}\right] \left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c^\top & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^\top - y^\top A & -y^\top & -y^\top b \end{array}\right]$$

We say that T_k is an *optimal tableau* if the simplex algorithm terminates at this tableau.

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- it is feasible: $0 \leq Rb$, and
- the z-row has non-positive entries:

$$c - A^{\top}y \leq 0 \quad \text{or equivalently} \quad A^{\top}y \geq c$$
$$-y \leq 0 \quad \text{or equivalently} \quad 0 \leq y.$$

The Block Structure of an Optimal Tableau

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In this case the optimal value $= z = b^\top y$.

The Block Structure of an Optimal Tableau

$$\left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^{\top} - y^{\top}A & -y^{\top} & -y^{\top}b \end{array}\right]$$

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The Block Structure of an Optimal Tableau

$$\left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^\top - y^\top A & -y^\top & -y^\top b \end{array}\right]$$

with

$$0 \leq Rb, \quad A^\top y \geq c, \quad 0 \leq y, \quad c^\top x^* = z = b^\top y,$$

The Block Structure of an Optimal Tableau

$$\left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^{\top} - y^{\top}A & -y^{\top} & -y^{\top}b \end{array}\right]$$

with

$$0 \le Rb, \quad A^{\top}y \ge c, \quad 0 \le y, \quad c^{\top}x^* = z = b^{\top}y,$$

where x^* is the optimal solution to

$$\begin{array}{lll} \mathcal{P} & \max & c^{\top}x \\ & \text{s.t.} & Ax \le b \\ & & 0 \le x \end{array}$$

The Block Structure of an Optimal Tableau

$$\left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^\top - y^\top A & -y^\top & -y^\top b \end{array}\right]$$

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The Block Structure of an Optimal Tableau

$$\left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^{\top} - y^{\top}A & -y^{\top} & -y^{\top}b \end{array}\right]$$

with

$$0 \leq Rb, \quad A^{\top}y \geq c, \quad 0 \leq y, \quad c^{\top}x^{*} = z = b^{\top}y,$$

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WEAK DUALITY THM. $\Rightarrow$ Y SOLVES $\mathcal{D}$!!!

Optimal Tableaus Yield Optimal Solutions

$$
\begin{array}{llll}
\mathcal{P} & \max & c^{\mathsf{T}}x \\
 & \text{s.t.} & Ax \le b \\
 & & 0 \le x
\end{array}
\qquad\qquad
\begin{array}{llll}
\mathcal{D} & \min & b^{\mathsf{T}}y \\
 & \text{s.t.} & A^{\mathsf{T}}y \ge c \\
 & & 0 \le y
\end{array}
$$

Theorem:[Optimal Tableau Theorem]
If the simplex tableau

$$
\left[\begin{array}{ccc|c}
0 & RA & R & Rb \\
-1 & c^{\mathsf{T}} - y^{\mathsf{T}}A & -y^{\mathsf{T}} & -y^{\mathsf{T}}b
\end{array}\right]
$$

is optimal for $\mathcal{P}$, i.e. if x^* is the associated BFS and

$$0 \le Rb, \quad A^{\mathsf{T}}y \ge c, \quad 0 \le y, \quad c^{\mathsf{T}}x^* = z = b^{\mathsf{T}}y,$$

then x^* is an optimal solution to $\mathcal{P}$ and y is an optimal solution to $\mathcal{D}$.

Plastic Cup Factory Reprised

$$
\begin{array}{lll}
\mathcal{P} & \text{maximize} & 25B + 20C \\
 & \text{subject to} & 20B + 12C \leq 1800 \\
 & & \frac{1}{15}B + \frac{1}{15}C \leq 8 \\
 & & 0 \leq B, C
\end{array}
$$

Plastic Cup Factory Reprised

$$\mathcal{P}\quad \begin{array}{ll} \text{maximize} & 25B + 20C \\ \text{subject to} & 20B + 12C \leq 1800 \\ & \frac{1}{15}B + \frac{1}{15}C \leq 8 \\ & 0 \leq B, C \end{array}$$

$$\mathcal{D}\quad \begin{array}{ll} \text{minimize} & 1800R + 8L \\ \text{subject to} & 20R + \frac{1}{15}L \geq 25 \\ & 12R + \frac{1}{15}L \geq 20 \\ & 0 \leq R, L \end{array}$$

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$$\left[\begin{array}{cccc|c} 20 & 12 & 1 & 0 & 1800 \\ \frac{1}{15} & \frac{1}{15} & 0 & 1 & 8 \\ \hline 25 & 20 & 0 & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} 20 & 12 & 1 & 0 & 1800 \\ \frac{1}{15} & \frac{1}{15} & 0 & 1 & 8 \\ \hline 25 & 20 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1/8 & -75/2 & 45 \\ 0 & 1 & -1/8 & 75/2 & 75 \\ \hline 0 & 0 & -5/8 & -375/2 & -2625 \end{array}\right]$$

Plastic Cup Factory Reprised

$$\mathcal{P} \quad \begin{array}{ll} \text{maximize} & 25B + 20C \\ \text{subject to} & 20B + 12C \le 1800 \\ & \frac{1}{15}B + \frac{1}{15}C \le 8 \\ & 0 \le B, C \end{array} \qquad \mathcal{D} \quad \begin{array}{ll} \text{minimize} & 1800R + 8L \\ \text{subject to} & 20R + \frac{1}{15}L \ge 25 \\ & 12R + \frac{1}{15}L \ge 20 \\ & 0 \le R, L \end{array}$$

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$$(B, C)^* = (45, 75),$$

Plastic Cup Factory Reprised

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & 25B + 20C \\ & \text{subject to} & 20B + 12C \leq 1800 \\ & & \frac{1}{15}B + \frac{1}{15}C \leq 8 \\ & & 0 \leq B, C \end{array} \qquad \begin{array}{lll} \mathcal{D} & \text{minimize} & 1800R + 8L \\ & \text{subject to} & 20R + \frac{1}{15}L \geq 25 \\ & & 12R + \frac{1}{15}L \geq 20 \\ & & 0 \leq R, L \end{array}$$

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$$(B, C)^* = (45, 75), \quad (R, L)^* = \left(\frac{5}{8}, \frac{375}{2}\right),$$

Plastic Cup Factory Reprised

$$\begin{array}{llll} \mathcal{P} & \text{maximize} & 25B + 20C \\ & \text{subject to} & 20B + 12C \leq 1800 \\ & & \frac{1}{15}B + \frac{1}{15}C \leq 8 \\ & & 0 \leq B, C \end{array} \qquad \begin{array}{llll} \mathcal{D} & \text{minimize} & 1800R + 8L \\ & \text{subject to} & 20R + \frac{1}{15}L \geq 25 \\ & & 12R + \frac{1}{15}L \geq 20 \\ & & 0 \leq R, L \end{array}$$

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$$(B, C)^* = (45, 75), \quad (R, L)^* = \left(\frac{5}{8}, \frac{375}{2}\right), \quad z^* = 2625$$

Another Duality Example

$$
\begin{array}{lll}
\mathcal{P} & \max & 4x_1 + 5x_2 + 3x_3 \\
 & \text{s.t.} & x_1 + 4x_2 + 2x_3 \quad \leq 11 \\
 & & 3x_1 + 2x_2 + x_3 \quad \leq 5 \\
 & & 4x_1 + 2x_2 + 2x_3 \quad \leq 8 \\
 & & 0 \leq x_1, x_2, x_3
\end{array}
$$

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 & 3x_1 + 2x_2 + x_3 & \leq 5 \\
 & 4x_1 + 2x_2 + 2x_3 & \leq 8 \\
 & 0 \leq x_1, x_2, x_3 &
\end{array}
\qquad
\mathcal{D} \quad \begin{array}{lll}
\min & 11y_1 + 5y_2 + 8y_3 & \\
\text{s.t.} & y_1 + 3y_2 + 4y_3 & \geq 4 \\
 & 4y_1 + 2y_2 + 2y_3 & \geq 5 \\
 & 2y_1 + y_2 + 2y_3 & \geq 3 \\
 & 0 \leq y_1, y_2, y_3 &
\end{array}
$$

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 & & 3x_1 + 2x_2 + x_3 & \le 5 \\
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 & & 0 \le x_1, x_2, x_3 &
\end{array}
\qquad
\begin{array}{llll}
\mathcal{D} & \min & 11y_1 + 5y_2 + 8y_3 & \\
 & \text{s.t.} & y_1 + 3y_2 + 4y_3 & \ge 4 \\
 & & 4y_1 + 2y_2 + 2y_3 & \ge 5 \\
 & & 2y_1 + y_2 + 2y_3 & \ge 3 \\
 & & 0 \le y_1, y_2, y_3 &
\end{array}$$

$$T_0 = \left[\begin{array}{cccccc|c}
1 & 4 & 2 & 1 & 0 & 0 & 11 \\
3 & 2 & 1 & 0 & 1 & 0 & 5 \\
4 & 2 & 2 & 0 & 0 & 1 & 8 \\
\hline
4 & 5 & 3 & 0 & 0 & 0 & 0
\end{array}\right]$$

Another Duality Example

$$T_0 = \left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$T_{\text{opt}} = \left[\begin{array}{cccccc|c} -5 & 0 & 0 & 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & \frac{-1}{2} & 1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 3 \\ \hline -4 & 0 & 0 & 0 & -2 & \frac{-1}{2} & -14 \end{array}\right]$$

Another Duality Example

$$T_0 = \left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

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$$x^* = (0, 1, 3), \quad y^* = (0, 2, 1/2), \quad z^* = 14$$

Another Duality Example

Check

$$
\begin{array}{llll}
\mathcal{P} & \max & 4x_1 + 5x_2 + 3x_3 & \\
 & \text{s.t.} & x_1 + 4x_2 + 2x_3 & \leq 11 \\
 & & 3x_1 + 2x_2 + x_3 & \leq 5 \\
 & & 4x_1 + 2x_2 + 2x_3 & \leq 8 \\
 & & 0 \leq x_1, x_2, x_3 &
\end{array}
\qquad
\begin{array}{llll}
\mathcal{D} & \min & 11y_1 + 5y_2 + 8y_3 & \\
 & \text{s.t.} & y_1 + 3y_2 + 4y_3 & \geq 4 \\
 & & 4y_1 + 2y_2 + 2y_3 & \geq 5 \\
 & & 2y_1 + y_2 + 2y_3 & \geq 3 \\
 & & 0 \leq y_1, y_2, y_3 &
\end{array}
$$

$$x^* = (0, 1, 3), \quad y^* = (0, 2, 1/2), \quad z^* = 14$$

Strong Duality

If we can now show that the simplex algorithm works, then we have an algorithm that simultaneously solves both the primal and dual problems.

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If we can now show that the simplex algorithm works, then we have an algorithm that simultaneously solves both the primal and dual problems.

Moreover, the optimal value in the primal and dual coincides giving equality in the weak duality inequality.

We now focus on the details of the simplex algorithm to determine if and when it works.

More Tableau Terminology

The block structure formula for simplex tableaus.

$$T_k = \begin{bmatrix} R & 0 \\ -y^{\top} & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c^{\top} & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^{\top} - y^{\top}A & -y^{\top} & -y^{\top}b \end{array}\right]$$

T_k is *primal feasible* if $Rb \geq 0$.

T_k is *dual feasible* if $0 \leq y$ and $A^T y \geq c$.

T_k is optimal if it is both primal and dual feasible in which case (x^*, y) is a Primal-Dual optimal pair where x^* is the BFS associated with T_k. Moreover, the optimal value of the Primal equals that of the Dual.