Math 407A: Linear Optimization

Lecture 3: LP Modeling [1]

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LP Modeling

Model 10: Detergent Production

Model 9: Investing Over Time

Model 4: Blending

LP Modeling

The four basic steps of LP modeling.

1. Identify and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function*.
3. Determine the *explicit constraints* and write a functional expression for each of them.
4. Determine the *implicit constraints*.

Model 10: Detergent Production

The Rosseral Company is a small detergent manufacturing company. It is one of several companies that produce a new, nonpolluting "washday whitener" called NPW. Rosseral can sell NPW to other detergent manufacturers for \$0.80 per gallon. Rosseral itself manufactures detergent that uses NPW. This NPW can be purchased outside for \$1.20 per gallon (shipping and handling charges have been added) or be obtained from Rosseral's own production. Each gallon of detergent produced requires .1 gallon of NPW. Production costs for NPW and detergent are respectively \$0.50 and \$0.60 per gallon. Detergent production costs does not include the cost for the .1 gallon of NPW used in each gallon of detergent. Detergent can be sold for \$0.70 per gallon. Production capacities at Rosseral are: NPW – 10,000 gallons per month; detergent – 120,000 gallons per month. Formulate the problem of maximizing profit as a linear program.

Model 10: Detergent Production

What are the decision variables?

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- DP = gallons of detergent to produce per month (sold)
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Model 10: Detergent Production

What are the decision variables?

What decisions are required?

- DP = gallons of detergent to produce per month (sold)
- NP = gallons of NPW to produce per month
- NB = gallons of NPW to buy each month
- NS = gallons of NPW to sell each month
- ND = gallons of NPW used in the production of detergent each month.

Model 10: Detergent Production

What is the objective?

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Maximize Profit = Revenue − Costs

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Maximize Profit $=$ Revenue $-$ Costs

- Revenue $= 0.8NS + 0.7DP$

Model 10: Detergent Production

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Maximize Profit = Revenue − Costs

- Revenue $= 0.8NS + 0.7DP$
- Costs $= 0.5NP + 0.6DP + 1.2NB$

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Production Bounds:

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- NPW:

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NPW-Detergent Relationship:

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NPW Variable Inter-Relationships:

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$$ND + NS \leq NP + NB$$

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Production Bounds:

- Detergent: $DP \leq 120,000$
- NPW: $NP \leq 10000$

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NPW Variable Inter-Relationships:

$$ND + NS \leq NP + NB$$

Implicit Constraints: $0 \leq DP, NP, NS, NB, ND$

Model 10: Detergent Production

The complete model.

$$\begin{array}{ll} \text{maximize} & .8NS + .7DP - .5NP - .6DP - 1.2NB \\ \\ \text{subject to} & NP \leq 10,000 \\ & DP \leq 120,000 \\ & DP = 10ND \\ & ND + NS \leq NP + NB \\ & 0 \leq DP, NP, NS, NB, ND \end{array}$$

Model 9: Investing Over Time

An investor has money-making activities A and B available at the beginning of each of the next 5 years. Each dollar invested in A at the beginning of 1 year returns \$1.40 (a profit of \$0.40) 2 years later (in time for immediate re-investment). Each dollar invested in B at the beginning of 1 year returns \$1.70 3 years later.

In addition, investments C and D will each be available at one time in the future. Each dollar investment in C at the beginning of year 2 returns \$1.90 at the end of year 5. Each dollar invested in D at the beginning of year 5 returns \$1.30 at the end of year 5.

Money uninvested in a given year earns 3% per annum.

The investor begins with \$50,000 and wishes to know which investment plan maximizes cash balance at the beginning of year 6. Formulate the linear programming model for this problem.

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What are the decision variables?

- Dollars invested in A yrs 1-4: $A_1,\ A_2,\ A_3,\ A_4$
- Dollars invested in B yrs 1-3: $B_1,\ B_2,\ B_3$
- Dollars invested in C yr 2: C_2

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- Dollars invested in A yrs 1-4: $A_1, \ A_2, \ A_3, \ A_4$
- Dollars invested in B yrs 1-3: $B_1, \ B_2, \ B_3$
- Dollars invested in C yr 2: C_2
- Dollars invested in D yr 5: D_5

Model 9: Investing Over Time

What are the decision variables?

- Dollars invested in A yrs 1-4: A_1, A_2, A_3, A_4
- Dollars invested in B yrs 1-3: B_1, B_2, B_3
- Dollars invested in C yr 2: C_2
- Dollars invested in D yr 5: D_5
- Dollars at 3% in yrs 1-5: S_1, S_2, S_3, S_4, S_5

Model 9: Investing Over Time

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How can we represent this cash balance using the decision variables given?

This is not easy to do at this point in our modeling.

So we make it easy by adding in a new decision variable.

$$T \; = \text{Cash balance at the beginning of year 6}$$

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– Year 1: $A_1 + B_1 + S_1 = 50,000$

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$$\text{Outgoing Money} = \text{Incoming Money}$$

- Year 1: $A_1 + B_1 + S_1 = 50,000$
- Year 2: $A_2 + B_2 + C_2 + S_2 = 1.03S_1$

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- Year 1: $A_1 + B_1 + S_1 = 50,000$
- Year 2: $A_2 + B_2 + C_2 + S_2 = 1.03S_1$
- Year 3: $A_3 + B_3 + S_3 = 1.4A_1 + 1.03S_2$

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- Year 1: $A_1 + B_1 + S_1 = 50,000$
- Year 2: $A_2 + B_2 + C_2 + S_2 = 1.03S_1$
- Year 3: $A_3 + B_3 + S_3 = 1.4A_1 + 1.03S_2$
- Year 4: $A_4 + S_4 = 1.4A_2 + 1.7B_1 + 1.03S_3$

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What are the explicit constraints?

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Outgoing Money = Incoming Money

- Year 1: $A_1 + B_1 + S_1 = 50,000$
- Year 2: $A_2 + B_2 + C_2 + S_2 = 1.03S_1$
- Year 3: $A_3 + B_3 + S_3 = 1.4A_1 + 1.03S_2$
- Year 4: $A_4 + S_4 = 1.4A_2 + 1.7B_1 + 1.03S_3$
- Year 5: $D_5 + S_5 = 1.4A_3 + 1.7B_2 + 1.03S_4$

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What are the explicit constraints?

Write the constraints for each year separately.

$$\text{Outgoing Money} = \text{Incoming Money}$$

- Year 1: $A_1 + B_1 + S_1 = 50,000$
- Year 2: $A_2 + B_2 + C_2 + S_2 = 1.03S_1$
- Year 3: $A_3 + B_3 + S_3 = 1.4A_1 + 1.03S_2$
- Year 4: $A_4 + S_4 = 1.4A_2 + 1.7B_1 + 1.03S_3$
- Year 5: $D_5 + S_5 = 1.4A_3 + 1.7B_2 + 1.03S_4$
- Year 6: $T = 1.4A_4 + 1.7B_3 + 1.9C_2 + 1.3D_5 + 1.03S_5$

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All variables are non-negative!

Model 4: Blending

A company makes a blend consisting of two chemicals, 1 and 2, in the ratio 5:2 by weight. These chemical can be manufactured by three different processes using two different raw materials and a fuel. Production data are given in the table below. For how much time should each process be run in order to maximize the total amount of *blend* manufactured?

	Requirements per Unit Time			*Output per Unit Time*	
Process	*Raw Mat. 1 (units)*	*Raw Mat. 2 (units)*	*Fuel (units)*	*Chem. 1 (units)*	*Chem. 2 (units)*
1	9	5	50	9	6
2	6	8	75	7	10
3	4	11	100	10	6
Amount available	200	400	1850		

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What are the decision variables?

$$T_i \;=\; \text{number of time units running process } i = 1, 2, 3.$$

What is the objective function?

$$\text{maximize Blend} \;=\; B$$

Model 4: Blending

Explicit Constraints:

Resources

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Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Model 4: Blending

Explicit Constraints:

Resources

$$\text{Raw Material 1: } 9T_1 + 6T_2 + 4T_3 \leq 200$$

$$\text{Raw Material 2: } 5T_1 + 8T_2 + 11T_3 \leq 400$$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Chemical 1 = $C_1 = 9T_1 + 7T_2 + 10T_3$

Model 4: Blending

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Raw Material 1: $9T_1 + 6T_2 + 4T_3 \ \leq \ 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \ \leq \ 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \ \leq \ 1850$

Chemicals Produced:

Chemical 1 $= \ C_1 \ = \ 9T_1 + 7T_2 + 10T_3$

Chemical 2 $= \ C_2 \ = \ 6T_1 + 10T_2 + 6T_3$

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Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

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Chemicals Produced:

Chemical 1 $= C_1 = 9T_1 + 7T_2 + 10T_3$

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Blend:

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Explicit Constraints:

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Raw Material 1: $9T_1 + 6T_2 + 4T_3 \ \leq \ 200$

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Chemicals Produced:

$$\text{Chemical 1} = \ C_1 \ = \ 9T_1 + 7T_2 + 10T_3$$

$$\text{Chemical 2} = \ C_2 \ = \ 6T_1 + 10T_2 + 6T_3$$

Blend: $\frac{5}{7}B \ \leq C_1$

Model 4: Blending

Explicit Constraints:

Resources

Raw Material 1: $9T_1 + 6T_2 + 4T_3 \leq 200$

Raw Material 2: $5T_1 + 8T_2 + 11T_3 \leq 400$

Fuel: $50T_1 + 75T_2 + 100T_3 \leq 1850$

Chemicals Produced:

Chemical 1 $=$ $C_1 = 9T_1 + 7T_2 + 10T_3$

Chemical 2 $=$ $C_2 = 6T_1 + 10T_2 + 6T_3$

Blend: $\frac{5}{7}B \leq C_1$ and $\frac{2}{7}B \leq C_2$

Model 4: Blending

Explicit Constraints:

Resources

$$\text{Raw Material 1: } 9T_1 + 6T_2 + 4T_3 \leq 200$$

$$\text{Raw Material 2: } 5T_1 + 8T_2 + 11T_3 \leq 400$$

$$\text{Fuel: } 50T_1 + 75T_2 + 100T_3 \leq 1850$$

Chemicals Produced:

$$\text{Chemical 1} = C_1 = 9T_1 + 7T_2 + 10T_3$$

$$\text{Chemical 2} = C_2 = 6T_1 + 10T_2 + 6T_3$$

Blend: $\frac{5}{7}B \leq C_1$ and $\frac{2}{7}B \leq C_2$

Implicit Constraints: $0 \leq B,\ 0 \leq T_i,\ i = 1,2,3,\ 0 \leq C_j,\ j = 1,2$