Linear Programming

Lecture 13: Sensitivity Analysis

Sensitivity Analysis

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We begin our study of sensitivity analysis with a concrete toy example.

SILICON CHIP CORPORATION

A Silicon Valley firm specializes in making four types of silicon chips for personal computers. Each chip must go through four stages of processing before completion. First the basic silicon wafers are manufactured, second the wafers are laser etched with a micro circuit, next the circuit is laminated onto the chip, and finally the chip is tested and packaged for shipping. The production manager desires to maximize profits during the next month. During the next 30 days she has enough raw material to produce 4000 silicon wafers. Moreover, she has 600 hours of etching time, 900 hours of lamination time, and 700 hours of testing time. Taking into account depreciated capital investment, maintenance costs, and the cost of labor, each raw silicon wafer is worth \$1, each hour of etching time costs \$40, each hour of lamination time costs \$60, and each hour of inspection time costs \$10.

The production manager has formulated her problem as a profit maximization

Initial Tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

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testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

x_1, x_2, x_3, x_4 represent the number of 100 chip batches of the four types of chips.

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lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

x_1, x_2, x_3, x_4 represent the number of 100 chip batches of the four types of chips. The objective row coefficients correspond to dollars profit per 100 chip batch.

Optimal tableau:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

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-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	-5	0	-100	-50	$-145,000$

The optimal production schedule is

$$(x_1, x_2, x_3, x_4) = (0,\ 25,\ 10,\ 5),$$

and the optimal value is \$145,000.

Break-even Prices and Reduced Costs

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
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In this solution type 1 chip is not efficient to produce.

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At what sale price is it efficient to produce type 1 chip?

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At what sale price is it efficient to produce type 1 chip?

That is, what is the sale price p below which type 1 chip does not appear in the optimal production mix, and above which it does appear in the optimal mix?

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0	0	1	0	$-.02$	0	.1	0	10
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In this solution type 1 chip is not efficient to produce.

At what sale price is it efficient to produce type 1 chip?

That is, what is the sale price p below which type 1 chip does not appear in the optimal production mix, and above which it does appear in the optimal mix?

This is called the **breakeven sale price** of type 1 chip.

Break-even Prices and Reduced Costs

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$$\begin{aligned}
\text{chip cost} &= \text{no. chips} \times \text{cost per chip} = 100 \times 1 = 100 \\
\text{etching cost} &= \text{no. hours} \times \text{cost per hour} = 10 \times 40 = 400 \\
\text{lamination cost} &= \text{no. hours} \times \text{cost per hour} = 20 \times 60 = 1200 \\
\text{inspection cost} &= \text{no. hours} \times \text{cost per hour} = 20 \times 10 = 200 \ .
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The cost per batch of 100 type 1 chips is \$1900.
The current sale price of each batch of 100 type 1 chips is \$2000 + \$1900 = \$3900, or equivalently, \$39 per chip.

Break-even Prices and Reduced Costs

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Let θ denote the increase in sale price of type 1 chip needed for it to enter the optimal production mix.

With this change to the sale price of type 1 chip the initial tableau for the LP becomes

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	$2000 + \theta$	3000	5000	4000	0	0	0	0	0

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Break-even Prices and Reduced Costs

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	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
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lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	$2000+\theta$	3000	5000	4000	0	0	0	0	0

.

Suppose we repeat on this tableau all of the pivots that led to the previously optimal tableau.

What will the new tableau look like? That is, how does θ appear in this new tableau?

We answer this question by recalling the basic principle of simplex pivoting.

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The initial tableau is the augmented matrix

$$\begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} .$$

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Pivoting to an optimal tableau corresponds to left multiplication by a matrix of the form

$$G = \begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix}.$$

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The nonsingular matrix R is called the *record matrix*.

Break-even Prices and Reduced Costs

The optimal tableau has the form

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ (c - A^T y)^T & -y^T & -b^T y \end{bmatrix},$$

where $0 \leq y,\ A^T y \geq c$, and the optimal value is $b^T y$.

Break-even Prices and Reduced Costs

Changing the value of one (or more) of the objective coefficients c corresponds to replacing c by a vector of the form $c + \Delta c$.

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The corresponding new initial tableau is

$$\begin{bmatrix} A & I & b \\ (c + \Delta c)^T & 0 & 0 \end{bmatrix}.$$

Performing the same simplex pivots on this tableau as before simply corresponds to left multiplication by the matrix G.

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That is, we just add Δc to the objective row in the old optimal tableau.

$$T = \begin{bmatrix} RA & R & Rb \\ \Delta c^T + (c - A^T y)^T & -y^T & -b^T y \end{bmatrix}$$

Note that T may no longer be a simplex tableau since by adding Δc to $(c - A^T y)$ we may have introduced a non-zero entry into the objective row associated with a basic column. These non-zero entries must be pivoted to zero to recover a tableau.
On the other hand, if T is a tableau, then T remains optimal if and only if

$$\Delta c + (c - A^T y) \leq 0$$

or equivalently,

$$\Delta c \leq -(c - A^T y) \ .$$

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These inequalities place restrictions on how large the entries of Δc can be before one must pivot to obtain the new optimal tableau.

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	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
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	$2000+\theta$	3000	5000	4000	0	0	0	0	0

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	$2000+\theta$	3000	5000	4000	0	0	0	0	0

$$c = \begin{pmatrix} 2000 \\ 3000 \\ 5000 \\ 4000 \end{pmatrix}, \quad \Delta c = \begin{pmatrix} \theta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \theta e_1, \quad c + \Delta c = \begin{pmatrix} 2000+\theta \\ 3000 \\ 5000 \\ 4000 \end{pmatrix} = c + \theta e_1$$

$$\left[\begin{array}{ccc} RA & R & Rb \\ (c - A^T y)^T & -y^T & -b^T y \end{array}\right] =$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

$$\Delta c + (c - A^T y) = \theta e_1 + (c - A^T y) = \begin{pmatrix} \theta \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1500 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta - 1500 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
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Thus, to preserve optimality, we need $\theta \leq 1500$.

That is, as soon as θ increases beyond 1500, type 1 chip enters the optimal production mix, and for $\theta = 1500$ we obtain multiple optimal solutions where type 1 chip may be in the optimal production mix if we so choose.

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In general, the negative of the objective row coefficient for decision variables in the optimal tableau are the reduced costs of these variables.

Break-even Prices and Reduced Costs

That is, as soon as θ increases beyond 1500, type 1 chip enters the optimal production mix, and for $\theta = 1500$ we obtain multiple optimal solutions where type 1 chip may be in the optimal production mix if we so choose.

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For non-basic variables the break-even sale price can be read off from the reduced costs in the optimal tableau.

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For non-basic variables the break-even sale price can be read off from the reduced costs in the optimal tableau.

break-even price = current price + reduced cost = \$39 + \$15 = \$54 .

Break-even Prices and Reduced Costs

Now consider a more intuitive and simpler explanation of break-even sale prices.

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One way to determine these prices, is to determine by how much our profit is reduced if we produce one batch of these chips.

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Recall that the objective row coefficients in the optimal tableau correspond to the following expression for the objective variable z:

$$z = 145000 - 1500x_1 - 5x_5 - 100x_7 - 50x_8.$$

Break-even Prices and Reduced Costs

Now consider a more intuitive and simpler explanation of break-even sale prices.

One way to determine these prices, is to determine by how much our profit is reduced if we produce one batch of these chips.

Recall that the objective row coefficients in the optimal tableau correspond to the following expression for the objective variable z:

$$z = 145000 - 1500x_1 - 5x_5 - 100x_7 - 50x_8.$$

Hence, if we make one batch of type 1 chip, we reduce our optimal value by \$1500. Thus, to recoup this loss we must charge \$1500 more for these chips yielding a break-even sale price of $\$39 + \$15 = \$54$ per chip.

Range Analysis for Objective Coefficients

Range analysis is a tool for understanding the effects of both objective coefficient variations as well as resource availability variations.

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We now examine objective coefficient variations.

Range Analysis for Objective Coefficients

Recall that to compute a breakeven price one needs to determine the change in the associated objective coefficient that make it efficient to introduce this activity into the optimal production mix, or equivalently, to determine the smallest change in the objective coefficient of this currently non-basic decision variable that requires one to bring it into the basis in order to maintain optimality.

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A related question is *what is the range of variation of a given objective coefficient that preserves the current basis as optimal*?

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A related question is *what is the range of variation of a given objective coefficient that preserves the current basis as optimal*?

The answer to this question is an interval, possibly unbounded, on the real line within which a given objective coefficient can vary but these variations do not effect the currently optimal basis.

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raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

Opt. Tableau	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
	0.5	1	0	0	.015	0	0	$-.05$	25
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Silicon Chip Corp optimal tableau.

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0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
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In the Silicon Chip Corp problem the decision variable x_3 associated with type 3 chips is in the optimal basis.

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In the Silicon Chip Corp problem the decision variable x_3 associated with type 3 chips is in the optimal basis.

For what range of variations in $c_3 = 5000$ does the current optimal basis $\{x_2, x_3, x_4, x_6\}$ remain optimal?

Range Analysis for Objective Coefficients

To answer this question we perturb the objective coef. of type 3 chip and write $c_3 = 5000 + \theta$.

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0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	θ	0	-5	0	-100	-50	$-145,000$

This is no-longer a simplex tableau. To recover a tableau we must pivot on the x_3 column.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	θ	0	-5	0	-100	-50	$-145,000$

To recover a proper simplex tableau we must eliminate θ from the objective row entry under x_3.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
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-1500	0	θ	0	-5	0	-100	-50	$-145,000$

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Multiply the 3^{rd} row by $-\theta$ and add it to the objective row to eliminate θ.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	$-5 + 0.02\theta$	0	$-100 - 0.1\ \theta$	-50	$-145,000 - 10\theta$

.

To remain optimal the objective row must remain non-positive.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	$-5 + 0.02\theta$	0	$-100 - 0.1\,\theta$	-50	$-145,000 - 10\theta$

To remain optimal the objective row must remain non-positive.

$$\begin{array}{lll} -5 + 0.02\theta \leq 0, & \text{or equivalently,} & \theta \leq 250 \\ -100 - 0.1\theta \leq 0, & \text{or equivalently,} & -1000 \leq \theta \end{array}.$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
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which implies

$$4000 \leq c_3 \leq 5250.$$

since originally $c_3 = 5000$.

What is the range of the objective coefficient for type 4 chips that preserves the current basis as optimal?

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-5	0	0	0	$-.05$	1	0	$-.5$	50
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0.5	0	0	1	.015	0	$-.1$	.05	5
$-1500-0.5\theta$	0	0	0	$-5-0.015\theta$	0	$-100+0.1\theta$	$-50-0.05\theta$	$-145,000-5\theta$

To preserve dual feasibility we must have

$$\begin{array}{rll} -1500-0.5\theta \leq 0, & \text{or equivalently,} & -3000 \leq \theta \\ -5-0.015\theta \leq 0, & \text{or equivalently,} & -333.\bar{3} \leq \theta \\ -100+0.1\theta \leq 0, & \text{or equivalently,} & \theta \leq 1000 \\ -50-0.05\theta \leq 0, & \text{or equivalently,} & -1000 \leq \theta \end{array}.$$

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Thus,

$$-333.\bar{3} \leq \theta \leq 1000,$$

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$$\begin{array}{rll} -1500 - 0.5\theta \leq 0, & \text{or equivalently,} & -3000 \leq \theta \\ -5 - 0.015\theta \leq 0, & \text{or equivalently,} & -333.\bar{3} \leq \theta \\ -100 + 0.1\theta \leq 0, & \text{or equivalently,} & \theta \leq 1000 \\ -50 - 0.05\theta \leq 0, & \text{or equivalently,} & -1000 \leq \theta \end{array} .$$

Thus,

$$-333.\bar{3} \leq \theta \leq 1000,$$

and the range for c_4 is

$$3666.\bar{6} \leq c_4 \leq 5000$$

since originally $c_4 = 4000$.

What is the range for the objective coefficient for x_2?

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
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-1500	0	0	0	-5	0	-100	-50	$-145,000$

$$-333.\bar{3} \leq \theta \leq 1000$$

and

$$1666.\bar{6} \leq c_2 \leq 3000$$

Resource Variations, Marginal Values, and Range Analysis

We now consider questions concerning the effect of resource variations on the optimal solution.

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We begin with standard questions for the Silicon Chip Corp.

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- What is the most that we should pay for them?

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- How many should we purchase?
- What is the most that we should pay for them?
- After the purchase, what is the new optimal production schedule?

The technique we develop for answering these questions is similar to the technique used to determine objective coefficient ranges.

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We begin by introducing a variable θ for the number of silicon wafers that will be purchased, and then determine how this variable appears in the tableau after using the same simplex pivots encoded in the matrix G given above.

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	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	$4000+\theta$
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

.

To effect the same simplex pivots we multiply the perturbed initial tableau by the elimination matrx G.

General RHS Perturbations

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$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b + \Delta b \\ c^T & 0 & 0 \end{bmatrix}$$

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The new tableau is dual feasible.
This tableau is optimal if it is primal feasible.
That is, the new tableau is optimal as long as

$$0 \leq Rb + R\Delta b \;\Leftrightarrow\; -Rb \leq R\Delta b \; .$$

$$\left[\begin{array}{cc|c} RA & R & Rb \\ \hline (c - A^T y)^T & -y^T & -y^T b \end{array}\right] =$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

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0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

$$R = \begin{bmatrix} .015 & 0 & 0 & -.05 \\ -.05 & 1 & 0 & -.5 \\ -.02 & 0 & .1 & 0 \\ .015 & 0 & -.1 & .05 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} RA & R & Rb \\ \hline (c - A^T y)^T & -y^T & -y^T b \end{array}\right] =$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
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−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145, 000

$$R = \begin{bmatrix} .015 & 0 & 0 & -.05 \\ -.05 & 1 & 0 & -.5 \\ -.02 & 0 & .1 & 0 \\ .015 & 0 & -.1 & .05 \end{bmatrix} \qquad y = \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} RA & R & Rb \\ \hline (c - A^T y)^T & -y^T & -y^T b \end{array}\right] =$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

$$R = \begin{bmatrix} .015 & 0 & 0 & -.05 \\ -.05 & 1 & 0 & -.5 \\ -.02 & 0 & .1 & 0 \\ .015 & 0 & -.1 & .05 \end{bmatrix} \quad y = \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix} \quad Rb = \begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix}$$

How do variations the raw wafer resource effect the optimal tableau?

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	$4000 + \theta$
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

.

$$b + \Delta b = b + \theta e_1 = \begin{pmatrix} 4000 \\ 600 \\ 900 \\ 700 \end{pmatrix} + \theta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

General RHS Perturbations

$$\left[\begin{array}{cc|c} RA & R & Rb + R\Delta b \\ \hline (c - A^T y)^T & -y^T & -y^T b - y^T \Delta b \end{array}\right] =$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	
0.5	1	0	0	.015	0	0	$-.05$	25	
-5	0	0	0	$-.05$	1	0	$-.5$	50	
0	0	1	0	$-.02$	0	.1	0	10	$+R\Delta b$
0.5	0	0	1	.015	0	$-.1$	.05	5	
-1500	0	0	0	-5	0	-100	-50	$-145,000$	$-y^T\Delta b$

$$\left[\begin{array}{cc|c} RA & R & Rb + R{\color{red}\Delta b} \\ \hline (c - A^T y)^T & -y^T & -y^T b - y^T {\color{red}\Delta b} \end{array}\right] =$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	
0.5	1	0	0	.015	0	0	−.05	25	
−5	0	0	0	−.05	1	0	−.5	50	
0	0	1	0	−.02	0	.1	0	10	$+R\Delta b$
0.5	0	0	1	.015	0	−.1	.05	5	
−1500	0	0	0	−5	0	−100	−50	−145,000	$-y^T\Delta b$

$$Rb + R\Delta b = Rb + \theta R e_1 = \begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix} + \theta \begin{pmatrix} 0.015 \\ -0.05 \\ -0.02 \\ 0.015 \end{pmatrix} = \begin{pmatrix} 25 + \theta\, 0.015 \\ 50 - \theta\, 0.05 \\ 10 - \theta\, 0.02 \\ 5 + \theta\, 0.015 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} RA & R & Rb + R\color{red}{\Delta b} \\ \hline (c - A^T y)^T & -y^T & -y^T b - y^T \color{red}{\Delta b} \end{array}\right] =$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	
0.5	1	0	0	.015	0	0	$-.05$	25	
-5	0	0	0	$-.05$	1	0	$-.5$	50	
0	0	1	0	$-.02$	0	.1	0	10	$+R\Delta b$
0.5	0	0	1	.015	0	$-.1$	.05	5	
-1500	0	0	0	-5	0	-100	-50	$-145,000$	$-y^T\Delta b$

$$0 \leq Rb + R\Delta b = Rb + \theta R e_1 = \begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix} + \theta \begin{pmatrix} 0.015 \\ -0.05 \\ -0.02 \\ 0.015 \end{pmatrix} = \begin{pmatrix} 25 + \theta\, 0.015 \\ 50 - \theta\, 0.05 \\ 10 - \theta\, 0.02 \\ 5 + \theta\, 0.015 \end{pmatrix}$$

General RHS Perturbations

To preserve primal feasibility we need $-Rb \leq \theta R\Delta b$, i.e.

$$-\begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix} \leq \theta \begin{pmatrix} 0.015 \\ -0.05 \\ -0.02 \\ 0.015 \end{pmatrix},$$

or equivalently,

$$\begin{array}{rcrlcrr} -25 & \leq & .015\theta & \text{implies} & \theta & \geq & -5000/3 \\ -50 & \leq & -.05\theta & \text{implies} & \theta & \leq & 1000 \\ -10 & \leq & -.02\theta & \text{implies} & \theta & \leq & 500 \\ -5 & \leq & .015\theta & \text{implies} & \theta & \geq & -1000/3 \end{array}$$

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This reduces to the simple inequality

$$-\frac{1000}{3} \leq \theta \leq 500.$$

The interval $3666.\bar{6} \leq b_1 \leq 4500$ is called the *range of the raw chip resource in the optimal solution*.

General RHS Perturbations

If $-\frac{1000}{3} \leq \theta \leq 500$, then the optimal solution is given by

$$\begin{pmatrix} x_2 \\ x_6 \\ x_3 \\ x_4 \end{pmatrix} = Rb + R\Delta b = \begin{pmatrix} 25 + .015\theta \\ 50 - .05\theta \\ 10 - .02\theta \\ 5 + .015\theta \end{pmatrix}$$

with optimal value

$$y^T b + y^T \Delta b = 145000 + 5\theta.$$

Now examine the profit expression

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The marginal value is the per unit increased value of this resource due to the production process.

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The marginal value is the per unit increased value of this resource due to the production process.
Since we currently pay \$1 per wafer. If another vendor sells them at \$2.50 per wafer, then we should buy them since our unit increase in profit with this purchase price is $\$5 - \$1.5 = \$3.5$ since \$2.5 is \$1.5 greater than the \$1 we now pay.

Thus we should purchase 500 raw wafers at a purchase price of no more than $\$5 + \$1 = \$6$ dollars per wafer.

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The new optimal production schedule is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 25 + .015\theta \\ 10 - .02\theta \\ 5 + .015\theta \end{pmatrix}_{\theta=500} = \begin{pmatrix} 0 \\ 32.5 \\ 0 \\ 12.5 \end{pmatrix} .$$

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Should we purchase more than 500 chips?

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	$25 + .015\theta$
-5	0	0	0	$-.05$	1	0	$-.5$	$50 - .05\theta$
0	0	1	0	$-.02$	0	.1	0	$10 - .02\theta$
0.5	0	0	1	.015	0	$-.1$	.05	$5 + .015\theta$
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	
0.5	1	0	0	.015	0	0	$-.05$	$25 + .015\theta$	
-5	0	0	0	$-.05$	1	0	$-.5$	$50 - .05\theta$	
0	0	1	0	$-.02$	0	.1	0	$10 - .02\theta$	$\leftarrow$
0.5	0	0	1	.015	0	$-.1$	.05	$5 + .015\theta$	
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$	

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	
0.5	1	0	0	.015	0	0	$-.05$	$25 + .015\theta$	
-5	0	0	0	$-.05$	1	0	$-.5$	$50 - .05\theta$	
0	0	1	0	$-.02$	0	.1	0	$10 - .02\theta$	$\leftarrow$
0.5	0	0	1	.015	0	$-.1$	.05	$5 + .015\theta$	
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$	

0.5	1	.75	0	0	0	.075	$-.05$	32.5
-5	0	-2.5	0	0	1	$-.25$	$-.5$	25
0	0	-50	0	1	0	-5	0	$-500 + \theta$
0.5	0	.75	1	0	0	$-.025$	.05	12.5
-1500	0	-250	0	0	0	-125	-50	-147500

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	
0.5	1	0	0	.015	0	0	$-.05$	$25 + .015\theta$	
-5	0	0	0	$-.05$	1	0	$-.5$	$50 - .05\theta$	
0	0	1	0	$-.02$	0	.1	0	$10 - .02\theta$	$\leftarrow$
0.5	0	0	1	.015	0	$-.1$	.05	$5 + .015\theta$	
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$	

0.5	1	.75	0	0	0	.075	$-.05$	32.5
-5	0	-2.5	0	0	1	$-.25$	$-.5$	25
0	0	-50	0	1	0	-5	0	$-500 + \theta$
0.5	0	.75	1	0	0	$-.025$	.05	12.5
-1500	0	-250	0	0	0	-125	-50	-147500

Do not purchase more than 500 since the wafer resource becomes slack.

RHS Range Analysis: Etching Time

Let us now do a range analysis on the etching time resource b_2.

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	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

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testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

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lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

.

$$b + \Delta b = b + \theta e_2 = \begin{pmatrix} 4000 \\ 600 \\ 900 \\ 700 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

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	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	$600+\theta$
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

$$b + \Delta b = b + \theta e_2 = \begin{pmatrix} 4000 \\ 600 \\ 900 \\ 700 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The new rhs in the opt. tableau is $Rb + \theta Re_2$ since $\Delta b = \theta e_2$.

RHS Range Analysis: Etching Time

$$Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 + \theta \\ 10 \\ 5 \end{pmatrix}$$

RHS Range Analysis: Etching Time

$$0 \leq \; Rb + R\Delta b = \begin{pmatrix} 25 \\ 50+\theta \\ 10 \\ 5 \end{pmatrix}$$

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To preserve primal feasibility we only require

$$0 \leq 50 + \theta,$$

or equivalently,

$$-50 \leq \theta.$$

RHS Range Analysis: Etching Time

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To preserve primal feasibility we only require

$$0 \le 50 + \theta,$$

or equivalently,

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Therefore, the range for b_2 is

$$[550, \; +\infty) \; .$$

RHS Range Analysis: Etching Time

What is the shadow price for etching, and what does it mean?

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	-5	0	-100	-50	$-145,000$

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x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

The shadow price, or marginal value, is 0 since we have surplus etching time in the optimal solution.

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x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	-5	0	-100	-50	$-145,000$

The shadow price, or marginal value, is 0 since we have surplus etching time in the optimal solution.
Additional hours of etching time do not change current profit levels.

What is the range for lamination time, and what is its marginal value?

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	-5	0	-100	-50	$-145,000$

RHS Range Analysis: Lamination Time

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

$$0 \le Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 \\ 10 + 0.1\theta \\ 5 - 0.1\theta \end{pmatrix}$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
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$$0 \leq Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 \\ 10 + 0.1\theta \\ 5 - 0.1\theta \end{pmatrix}$$

$$\begin{array}{lll} 0 \leq 10 + 0.1\theta, & \text{or equivalently,} & -100 \leq \theta \\ 0 \leq 5 - 0.1\theta, & \text{or equivalently,} & \theta \leq 50 \end{array} .$$

RHS Range Analysis: Lamination Time

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
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$$\begin{array}{lll} 0 \leq 10 + 0.1\theta, & \text{or equivalently,} & -100 \leq \theta \\ 0 \leq 5 - 0.1\theta, & \text{or equivalently,} & \theta \leq 50 \end{array} .$$

Therefore,

$$800 \leq b_3 \leq 950 .$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
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The shadow price, or marginal value, for lamination time is \$100.

RHS Range Analysis: Lamination Time

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
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Each additional hour of lamination time improves profitability by $100.

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0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
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RHS Range Analysis: Lamination Time

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-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
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\$5,000

Pricing Out New Products

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(a) Is it efficient to produce?

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Suppose it can be sold for $ 33.10 per chip.

(a) Is it efficient to produce?

(b) If it is efficient to produce, what is the new optimal production schedule?

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Second, determine how the change to the initial tableau propagates through to the optimal tableau.

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This propagation is determined by multiplying the new initial tableau through by the pivot matrix G.

Pricing Out New Products

The initial tableau is altered by the addition of a new column:

$$\begin{bmatrix} a_{\mathrm{new}} & A & I & b \\ c_{\mathrm{new}} & c^T & 0 & 0 \end{bmatrix}.$$

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Multiplying on the left by the matrix G gives

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} a_{\text{new}} & A & I & b \\ c_{\text{new}} & c^T & 0 & 0 \end{bmatrix} =$$
$$\begin{bmatrix} Ra_{\text{new}} & RA & R & Rb \\ c_{\text{new}} - a_{\text{new}}^T y & (c - A^T y)^T & -y^T & -y^T b \end{bmatrix}.$$

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The expression $(c_{\text{new}} - a_{\text{new}}^T y)$ determines whether this new tableau is optimal or not.

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$$\begin{bmatrix} Ra_{\text{new}} & RA & R & Rb \\ c_{\text{new}} - a_{\text{new}}^T y & (c - A^T y)^T & -y^T & -y^T b \end{bmatrix}.$$

The expression $(c_{\text{new}} - a_{\text{new}}^T y)$ determines whether this new tableau is optimal or not. If $0 < (c_{\text{new}} - a_{\text{new}}^T y)$, then the new tableau is not optimal. In this case the new product is efficient to produce.

Pricing Out New Products

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If $(c_{\text{new}} - a_{\text{new}}^T y) < 0$, then the new product **does not price out**, and we do not produce it since in this case the new tableau is optimal with the new product non-basic.

Pricing Out New Products

The act of forming the expression $(c_{\text{new}} - a_{\text{new}}^T y)$ is called *pricing out* the new product.

If $(c_{\text{new}} - a_{\text{new}}^T y) < 0$, then the new product **does not price out**, and we do not produce it since in this case the new tableau is optimal with the new product non-basic.

If$(c_{\text{new}} - a_{\text{new}}^T y) > 0$, then we say that the new product **does price out** and it should be introduced into the optimal production mix.

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The value $a_{\mathrm{new}}^T y$ represents the increase in value of the resources consumed by one unit of this activity due to the current production schedule. If this value exceeds the profitability of this activity, then it is not efficient to introduce this activity into the production mix.

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The value $a_{\text{new}}^T y$ represents the increase in value of the resources consumed by one unit of this activity due to the current production schedule. If this value exceeds the profitability of this activity, then it is not efficient to introduce this activity into the production mix.

The new optimal production mix is found by applying the standard primal simplex algorithm to the tableau since this tableau is primal feasible but not dual feasible.

Returning to the Silicon Chip Corp. problem, the new chip under consideration has

$$a_{\text{new}} = \begin{pmatrix} 100 \\ 10 \\ 10 \\ 10 \end{pmatrix} .$$

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The stated sale price or revenue for each 100 chip batch of the new chip is \$3310, so

$$c_{\mathrm{new}} = 3310 - \begin{pmatrix} 1 \\ 40 \\ 60 \\ 10 \end{pmatrix}^T \begin{pmatrix} 100 \\ 10 \\ 10 \\ 10 \end{pmatrix} = 3310 - 1200 = 2110.$$

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We need to subtract from this number the cost of producing each 100 chip batch.

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$$\begin{aligned} 100 \times 1 & \quad \text{(cost of the raw wafers)} \\ +\,10 \times 40 & \quad \text{(cost of etching time)} \end{aligned}$$

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+\,10 \times 40 & \text{(cost of etching time)} \\
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+\,10 \times 10 & \text{(cost of testing time)} \\
\hline
1200 & \text{(total cost)}\ .
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Hence the profit on each 100 chip batch of these new chips is $\$3310 - \$1200 = \$2110$, or \$21.10 per chip, and so

$$c_{\mathrm{new}} = 2110.$$

Pricing Out New Products

Pricing out the new chip gives

$$c_{\text{new}} - a_{\text{new}}^T y = 2110 - \begin{pmatrix} 100 \\ 10 \\ 10 \\ 10 \end{pmatrix}^T \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix} = 2110 - 2000 = 110 \ .$$

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The new chip prices out positive, and so it will be efficient to produce.

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The new column in the tableau associated with this chip is

$$\begin{pmatrix} Ra_{\text{new}} \\ c_{\text{new}} - a_{\text{new}}^T y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 110 \end{pmatrix} \ .$$

Pricing Out New Products

x_{new}	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
1	0.5	1	0	0	.015	0	0	$-.05$	25
0	-5	0	0	0	$-.05$	1	0	$-.5$	50
-1	0	0	1	0	$-.02$	0	.1	0	10
1	0.5	0	0	1	.015	0	$-.1$	.05	5
110	-1500	0	0	0	-5	0	-100	-50	$-145,000$

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1	0.5	0	0	1	.015	0	$-.1$	.05	5
110	-1500	0	0	0	-5	0	-100	-50	$-145,000$
0	0	1	0	-1	0	0	.1	$-.1$	20
0	-5	0	0	0	$-.05$	1	0	$-.5$	50
0	.5	0	1	1	$-.005$	0	0	.05	15
1	0.5	0	0	1	.015	0	$-.1$	.05	5
0	-1555	0	0	-110	-6.65	0	-88.9	-55.5	-145550

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1	0.5	0	0	1	.015	0	−.1	.05	5
110	−1500	0	0	0	−5	0	−100	−50	−145,000
0	0	1	0	−1	0	0	.1	−.1	20
0	−5	0	0	0	−.05	1	0	−.5	50
0	.5	0	1	1	−.005	0	0	.05	15
1	0.5	0	0	1	.015	0	−.1	.05	5
0	−1555	0	0	−110	−6.65	0	−88.9	−55.5	−145550

The new optimal solution is $(x_{\text{new}}, x_1, x_2, x_3, x_4) = (5,\ 0,\ 20,\ 15,\ 0)$.

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110	-1500	0	0	0	-5	0	-100	-50	$-145,000$
0	0	1	0	-1	0	0	.1	$-.1$	20
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The new optimal solution is $(x_{\text{new}}, x_1, x_2, x_3, x_4) = (5,\ 0,\ 20,\ 15,\ 0)$.

The new shadow prices are $(y_1, y_2, y_3, y_4) = (6.65,\ 0,\ 89.9,\ 55.5)$.

Pricing Out New Products

Consider a different new chip.

This chip requires 15 hours each of etching and testing, and 30 hours of lamination time per 100 chip batch.

What is the breakeven sale price of this new chip?

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	$-.05$	25
-5	0	0	0	$-.05$	1	0	$-.5$	50
0	0	1	0	$-.02$	0	.1	0	10
0.5	0	0	1	.015	0	$-.1$	.05	5
-1500	0	0	0	-5	0	-100	-50	$-145,000$

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Costs of production are

$$\text{costs} = \begin{pmatrix} 1 \\ 40 \\ 60 \\ 10 \end{pmatrix}^T \begin{pmatrix} 100 \\ 15 \\ 30 \\ 15 \end{pmatrix} = \$2650$$

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Breakeven sale price = \$2650 + \$4250 = \$6900. Or equivalently, \$69 per chip.

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$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} a_{\text{new1}} & a_{\text{new2}} & A & I & b \\ c_{\text{new1}} & c_{\text{new2}} & c^T & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} Ra_{\text{new1}} & Ra_{\text{new2}} & RA & R & Rb \\ c_{\text{new1}} - a_{\text{new1}}^T y & c_{\text{new2}} - a_{\text{new2}}^T y & (c - A^T y)^T & -y^T & -y^T b \end{bmatrix}$$

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Then pivot to optimality.

The Fundamental Theorem on Sensitivity Analysis

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject to} & Ax \leq b, \ 0 \leq x \ . \end{array}$$

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We associate to $\mathcal{P}$ the *optimal value function* $V : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\begin{array}{lll} V(u) = & \text{maximize} & c^T x \\ & \text{subject to} & Ax \le b + u,\ 0 \le x \end{array}$$

for all $u \in \mathbb{R}^m$.

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$$\mathcal{F}(u) = \{x \in \mathbb{R}^n \mid Ax \leq b + u,\ 0 \leq x\}$$

denote the feasible region for the LP associated with value $V(u)$.

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denote the feasible region for the LP associated with value $V(u)$.

If $\mathcal{F}(u) = \emptyset$ for some $u \in \mathbb{R}^m$, we define $V(u) = -\infty$.

The Fundamental Theorem on Sensitivity Analysis

Theorem: *If $\mathcal{P}$ is primal nondegenerate, i.e. the optimal value is finite and no basic variable in any optimal tableau takes the value zero, then the dual solution y^* is unique and there is an $\epsilon > 0$ such that*

$$V(u) = b^T y^* + u^T y^* \quad \text{whenever } |u_i| \leq \epsilon, \; i = 1, \dots, m \; .$$

Thus, in particular, the optimal value function V is differentiable at $u = 0$ with $\nabla V(0) = y^*$.

The Fundamental Theorem on Sensitivity Analysis: Proof

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* \end{bmatrix}$$

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$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b + u \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb + Ru \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* - u^T y^* \end{bmatrix}$$

$$\begin{bmatrix} RA & R & Rb + Ru \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* - u^T y^* \end{bmatrix}$$

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Non-degeneracy implies that $Rb > 0$ so there is an $\epsilon > 0$ such that

$$Rb > \epsilon \mathbf{1} \ .$$

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Non-degeneracy implies that $Rb > 0$ so there is an $\epsilon > 0$ such that

$$Rb > \epsilon \mathbf{1} \ .$$

By continuity, there is a $\delta > 0$ such that

$$|(Ru)_i| \le \epsilon \quad \text{whenever} \quad |u_i| \le \delta \ \forall \, i = 1, 2, \ldots, n.$$

$$\begin{bmatrix} RA & R & Rb + R{\color{red}u} \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* - u^T y^* \end{bmatrix}$$

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Hence $Rb + Ru > 0$ whenever $|u_i| \le \delta \ \forall\, i = 1, 2, \ldots, n$.