Math 407A: Linear Optimization

Lecture 12: The Geometry of Linear Programming

Math Dept, University of Washington

Hyperplanes

Definition: *A hyperplane in* $\mathbb{R}^n$ *is any set of the form*

$$H(a, \beta) = \{x : a^T x = \beta\}$$

where $a \in \mathbb{R}^n \setminus \{0\}$ *and* $\beta \in \mathbb{R}$.

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where $a \in \mathbb{R}^n \setminus \{0\}$ *and* $\beta \in \mathbb{R}$.

Fact: $H \subset \mathbb{R}^n$ is a hyperplane if and only if the set

$$H - x_0 = \{x - x_0 : x \in H\}$$

where $x_0 \in H$ is a subspace of $\mathbb{R}^n$ of dimension $(n-1)$.

Hyperplanes

What are the hyperplanes in $\mathbb{R}$?

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What are the hyperplanes in $\mathbb{R}^n$?

Translates of $(n-1)$ dimensional subspaces.

Hyperplanes

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$$H_+(a, \beta) = \{x \in \mathbb{R}^n : a^T x \geq \beta\}$$

and

$$H_-(a, \beta) = \{x \in \mathbb{R}^n : a^T x \leq \beta\}.$$

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Consider the constraint region to an LP

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$$H_j = \{x : e_j^T x \geq 0\} \qquad \text{for } j = 1, \ldots, n$$

and

$$H_{n+i} = \{x : a_{i\cdot}^T x \leq b_i\} \qquad \text{for } i = 1, \ldots, m,$$

where $a_{i\cdot}$ is the ith row of A.

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Then

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Then

$$\Omega = \bigcap_{k=1}^{n+m} H_k \ .$$

That is, the constraint region of an LP is the intersection of finitely many closed half-spaces.

Convex Polyhedra

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A linear program is simply the problem of either maximizing or minimizing a linear function over a convex polyhedron.

We now develop the geometry of convex polyhedra.

Convex sets

Fact: Given any two points in $\mathbb{R}^n$, say x and y, the line segment connecting them is given by

$$[x, y] = \{(1 - \lambda)x + \lambda y : 0 \leq \lambda \leq 1\}.$$

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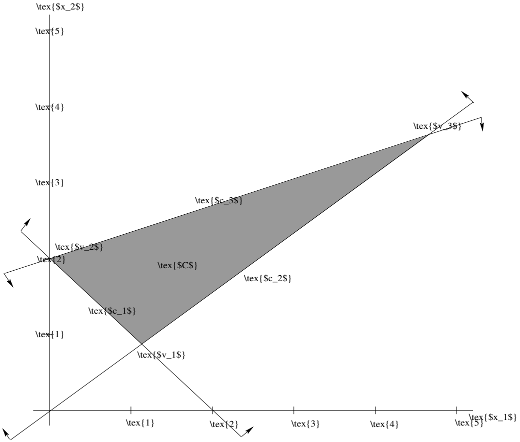
Fact: A convex polyhedron is a convex set.

Example

$$
\begin{array}{rcr}
c_1 & : & -x_1 - x_2 \leq -2 \\
c_2 & : & 3x_1 - 4x_2 \leq 0 \\
c_3 & : & -x_1 + 3x_2 \leq 6
\end{array}
$$



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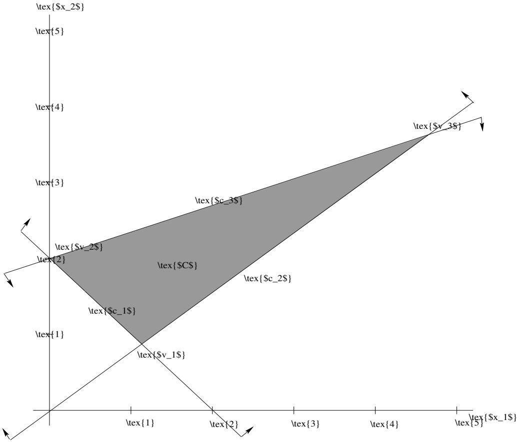

The vertices are $v_1 = \left(\frac{8}{7}, \frac{6}{7}\right)$, $v_2 = (0, 2)$, and $v_3 = \left(\frac{24}{5}, \frac{18}{5}\right)$.

Vertices

Definition: *Let C be a convex polyhedron. We say that $x \in C$ is a vertex of C if whenever $x \in [u, v]$ for some $u, v \in C$, it must be the case that either $x = u$ or $x = v$.*

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The Fundamental Representation Theorem for Vertices
Let $T = (t_{ij})_{m\times n}$, $g \in \mathbb{R}^m$, and consider the convex polyhedron $C := \{x \in \mathbb{R}^n \mid Tx \leq g\}$. A point $x \in C$ is a vertex of C if and only if there exist an index set $\mathcal{I} \subset \{1, \ldots, m\}$ such that x is the unique solution to the system of equations

$$\sum_{j=1}^{n} t_{ij} x_j = g_i \quad i \in \mathcal{I}.$$

Moreover, if x is a vertex, then one can take $|\mathcal{I}| = n$, where $|\mathcal{I}|$ denotes the number of elements in $\mathcal{I}$.

Observations

When does the system of equations

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$|\mathcal{I}| \geq n$; otherwise, one solution implies infinitely many solutions.

If $|\mathcal{I}| > n$, we can select a subset $\mathcal{R} \subset \mathcal{I}$ of the rows $T_{i\cdot}$ of T so that the set of vectors $\{T_{i\cdot} \mid i \in \mathcal{R}\}$ form a basis of the row space of T. Then $|\mathcal{R}| = n$ and x is the unique solution to

$$\sum_{j=1}^{n} t_{ij} x_j = g_i \quad i \in \mathcal{R}.$$

Vertices

Corollary: A point x in the convex polyhedron described by the system of inequalities

$$Ax \le b \qquad \text{and} \qquad 0 \le x,$$

where $A = (a_{ij})_{m \times n}$, is a vertex of this polyhedron if and only if there exist index sets $\mathcal{I} \subset \{1, \ldots, m\}$ and $\mathcal{J} \subset \{1, \ldots, n\}$ with $|\mathcal{I}| + |\mathcal{J}| = n$ such that x is the unique solution to the system of equations

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &= b_i \qquad i \in \mathcal{I}, \qquad \text{and} \\ x_j &= 0 \qquad j \in \mathcal{J}. \end{aligned}$$

Example

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c_1 &: \quad -x_1 - x_2 \leq -2 \\
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(a)The vertex $v_1 = (\frac{8}{7}, \frac{6}{7})$ is given as the solution to the system

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-x_1 - x_2 &= -2 \\
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(b)The vertex $v_2 = (0, 2)$ is given as the solution to the system

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Example

$$\begin{aligned} c_1 &: & -x_1 - x_2 &\le -2 \\ c_2 &: & 3x_1 - 4x_2 &\le 0 \\ c_3 &: & -x_1 + 3x_2 &\le 6 \end{aligned}$$

(c)The vertex $v_3 = \left(\frac{24}{5}, \frac{18}{5}\right)$ is given as the solution to the system

$$\begin{aligned} 3x_1 - 4x_2 &= 0 \\ -x_1 + 3x_2 &= 6. \end{aligned}$$

Application to LPs in Standard Form

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij}x_j & \leq & b_i & i = 1, \ldots, m \\ 0 & \leq & x_j & j = 1, \ldots, n. \end{array}$$

The associated slack variables:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, \ldots, m. \qquad \clubsuit$$

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Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n+m})$ be any solution to the system $\clubsuit$.

$$\mathcal{J} = \{j \in\subset \{1, \ldots, n\} \mid \bar{x}_j = 0\} \qquad \mathcal{I} = \{j \in \{1, \ldots, m\} \mid \bar{x}_{n+i} = 0\}\}$$

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Let $\widehat{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ be the values for the decision variables at $\bar{x}$.

Application to LPs in Standard Form

For each $j \in \mathcal{J} \subset \{1, \ldots, n\}$, $\bar{x}_j = 0$, consequently the hyperplane

$$H_j = \{x \in \mathbb{R}^n : e_j^T x = 0\}$$

is *active* at $\widehat{x}$, i.e., $\widehat{x} \in H_j$.

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is *active* at $\widehat{x}$, i.e., $\widehat{x} \in H_j$.

Similarly, for each $i \in \mathcal{I} \subset \{1, 2, \ldots, m\}$, $\bar{x}_{n+i} = 0$, and so the hyperplane

$$H_{n+i} = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij} x_j = b_i\}$$

is *active* at $\widehat{x}$, i.e., $\widehat{x} \in H_{n+i}$.

Application to LPs in Standard Form

What are the vertices of the system

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij}x_j & \leq & b_i & \qquad i = 1, \ldots, m \\ 0 & \leq & x_j & \qquad j = 1, \ldots, n \end{array}$$

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$\widehat{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is a vertex of this polyhedron if and only if there exist index sets $\mathcal{I} \subset \{1, \ldots, m\}$ and $\mathcal{J} \in \{1, \ldots, n\}$ with $|\mathcal{I}| + |\mathcal{J}| = n$ such that $\widehat{x}$ is the unique solution to the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad i \in \mathcal{I}, \quad \text{and} \quad x_j = 0 \quad j \in \mathcal{J}.$$

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$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad i \in \mathcal{I}, \quad \text{and} \quad x_j = 0 \quad j \in \mathcal{J}.$$

In this case $\bar{x}_{m+i} = 0$ for $i \in \mathcal{I}$ (slack variables).

Vertices

That is, $\widehat{x}$ is a vertex of the polyhedral constraints to an LP in standard form if and only if a total of n of the variables $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+m}\}$ take the value zero, while the value of the remaining m variables are uniquely determined by setting these n variables to the value zero.

Vertices and BFSs

That is, $\widehat{x}$ is a vertex of the polyhedral constraints to an LP in standard form if and only if a total of n of the variables $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+m}\}$ take the value zero, while the value of the remaining m variables are uniquely determined by setting these n variables to the value zero.

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But then, $\widehat{x}$ is a vertex if and only if it is a BFS!

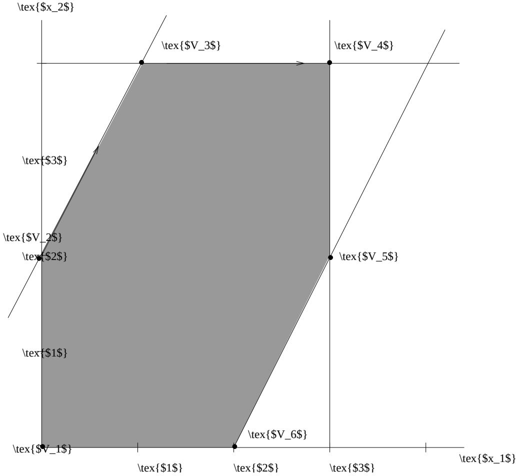
Therefore, one can geometrically interpret the simplex algorithm as a procedure moving from one vertex of the constraint polyhedron to another with higher objective value until the optimal solution exists.

Vertices and BFSs

The simplex algorithm terminates finitely since every vertex is connected to every other vertex by a path of adjacent vertices on the surface of the polyhedron.

Example

$$\begin{array}{lr} \text{maximize} & 3x_1 + 4x_2 \\ \text{subject to} & -2x_1 + x_2 \leq 2 \\ & 2x_1 - x_2 \leq 4 \\ & 0 \leq x_1 \leq 3, \\ & 0 \leq x_2 \leq 4. \end{array}$$



Example

-2	$\boxed{1}$	1	0	0	0	2	vertex
2	-1	0	1	0	0	4	v_1
1	0	0	0	1	0	3	$(0,0)$
0	1	0	0	0	1	4	
3	4	0	0	0	0	0	

-2	1	1	0	0	0	2	vertex
0	0	1	1	0	0	6	v_2
1	0	0	0	1	0	3	$(0,2)$
$\boxed{2}$	0	-1	0	0	1	2	
11	0	-4	0	0	0	-8	

0	1	0	0	0	1	4	vertex
0	0	1	1	0	0	6	v_3
0	0	$\boxed{\frac{1}{2}}$	0	1	$-\frac{1}{2}$	2	$(1,4)$
1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	
0	0	$\frac{3}{2}$	0	0	$\frac{-11}{2}$	-19	

0	1	0	0	0	1	4	vertex
0	0	0	1	-2	1	2	v_4
0	0	1	0	2	-1	4	$(3,4)$
1	0	0	0	1	0	3	
0	0	0	0	-3	-4	-25	

Vertex Pivoting

The BSFs in the simplex algorithm are vertices, and every vertex of the polyhedral constraint region is a BFS.

Phase I of the simplex algorithm is a procedure for finding a vertex of the constraint region, while Phase II is a procedure for moving between adjacent vertices successively increasing the value of the objective function.

The Geometry of Degeneracy

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$$H_{n+i} = \{x : \sum_{j=1}^{n} a_{ij} x_j \le b_i\} \qquad \text{for } i = 1, \ldots, m$$

A basic feasible solution (vertex) is said to be degenerate if one or more of the basic variables is assigned the value zero. This implies that more than n of the hyperplanes H_k, $k = 1, 2, \ldots, n+m$ are active at this vertex.

Example

$$\begin{array}{ll}\text{maximize} & 3x_1 + 4x_2 \\ \text{subject to} & -2x_1 + x_2 \le 2 \\ & 2x_1 - x_2 \le 4 \\ & -x_1 + x_2 \le 3 \\ & x_1 + x_2 \le 7 \\ & 0 \le x_1 \le 3, \\ & 0 \le x_2 \le 4. \end{array}$$

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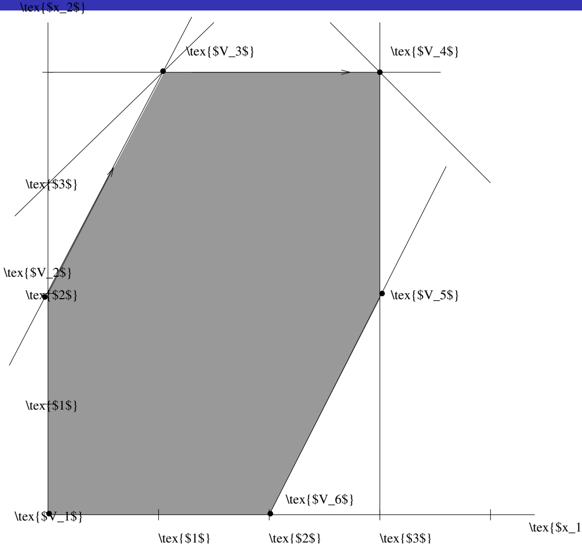

Example

-2	1	1	0	0	0	0	0	2
2	-1	0	1	0	0	0	0	4
-1	1	0	0	1	0	0	0	3
1	1	0	0	0	1	0	0	7
1	0	0	0	0	0	1	0	3
0	1	0	0	0	0	0	1	4
3	4	0	0	0	0	0	0	0

vertex
$V_1 = (0, 0)$

Example

-2	1	1	0	0	0	0	0	2	vertex
2	-1	0	1	0	0	0	0	4	$V_1 = (0, 0)$
-1	1	0	0	1	0	0	0	3	
1	1	0	0	0	1	0	0	7	
1	0	0	0	0	0	1	0	3	
0	1	0	0	0	0	0	1	4	
3	4	0	0	0	0	0	0	0	

-2	1	1	0	0	0	0	0	2	vertex
0	0	1	1	0	0	0	0	6	$V_2 = (0, 2)$
1	0	-1	0	1	0	0	0	1	
3	0	-1	0	0	1	0	0	5	
1	0	0	0	0	0	1	0	3	
2	0	-1	0	0	0	0	1	2	
11	0	-4	0	0	0	0	0	-8	

Example

-2	1	1	0	0	0	0	0	2
0	0	1	1	0	0	0	0	6
1	0	-1	0	1	0	0	0	1
3	0	-1	0	0	1	0	0	5
1	0	0	0	0	0	1	0	3
2	0	-1	0	0	0	0	1	2
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vertex
$V_2 = (0, 2)$

Example

-2	1	1	0	0	0	0	0	2	vertex
0	0	1	1	0	0	0	0	6	$V_2 = (0, 2)$
1	0	-1	0	1	0	0	0	1	
3	0	-1	0	0	1	0	0	5	
1	0	0	0	0	0	1	0	3	
2	0	-1	0	0	0	0	1	2	
11	0	-4	0	0	0	0	0	-8	
0	1	-1	0	2	0	0	0	4	vertex
0	0	1	1	0	0	0	0	6	$V_3 = (1, 4)$
1	0	-1	0	1	0	0	0	1	
0	0	2	0	-3	1	0	0	2	
0	0	1	0	-1	0	1	0	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	7	0	-11	0	0	0	-19	

Example

0	1	-1	0	2	0	0	0	4	vertex
0	0	1	1	0	0	0	0	6	$V_3 = (1,4)$
1	0	-1	0	1	0	0	0	1	
0	0	2	0	-3	1	0	0	2	
0	0	1	0	-1	0	1	0	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	7	0	-11	0	0	0	-19	

Example

0	1	-1	0	2	0	0	0	4	vertex
0	0	1	1	0	0	0	0	6	$V_3 = (1,4)$
1	0	-1	0	1	0	0	0	1	
0	0	2	0	-3	1	0	0	2	
0	0	1	0	-1	0	1	0	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	7	0	-11	0	0	0	-19	

0	1	0	0	0	0	0	1	4	vertex
0	0	0	1	2	0	0	1	6	$V_3 = (1,4)$
1	0	0	0	-1	0	0	1	1	
0	0	0	0	1	1	0	-2	2	
0	0	0	0	1	0	1	-1	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	0	0	3	0	0	-7	-19	

Example

0	1	0	0	0	0	0	1	4	vertex
0	0	0	1	2	0	0	1	6	$V_3 = (1, 4)$
1	0	0	0	-1	0	0	1	1	
0	0	0	0	1	1	0	-2	2	
0	0	0	0	1	0	1	-1	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	0	0	3	0	0	-7	-19	

Example

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0	0	0	1	2	0	0	1	6	$V_3 = (1, 4)$
1	0	0	0	-1	0	0	1	1	
0	0	0	0	1	1	0	-2	2	
0	0	0	0	1	0	1	-1	2	
0	0	1	0	-2	0	0	1	0	degenerate
0	0	0	0	3	0	0	-7	-19	

0	1	0	0	0	0	0	1	4	vertex
0	0	0	1	0	-2	0	5	2	$V_4 = (3, 4)$
1	0	0	0	0	1	0	-1	3	
0	0	0	0	1	1	0	-2	2	optimal
0	0	0	0	0	-1	1	1	0	degenerate
0	0	1	0	0	2	0	-3	4	
0	0	0	0	0	-3	0	-1	-25	

Degeneracy = Multiple Representations of a Vertex

A degenerate tableau occurs when the associated BFS (or vertex) can be represented as the intersection point of more than one subsets of n active hyperplanes.

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A degenerate tableau occurs when the associated BFS (or vertex) can be represented as the intersection point of more than one subsets of n active hyperplanes.

A degenerate pivot occurs when we move between two different representations of a vertex as the intersection of n hyperplanes.

Cycling implies that we are cycling between different representations of the same vertex.

Degeneracy = Multiple Representations of a Vertex

In the previous example, the third tableau represents the vertex $V_3 = (1, 4)$ as the intersection of the hyperplanes

$$\begin{aligned} -2x_1 + x_2 &= 2 && \text{(since } x_3 = 0\text{)} \\ -x_1 + x_2 &= 3. && \text{(since } x_5 = 0\text{)} \end{aligned}$$

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The third pivot brings us to the 4th tableau where the vertex $V_3 = (1, 4)$ is represented as the intersection of the hyperplanes

$$\begin{aligned} -x_1 + x_2 &= 3 && \text{(since } x_5 = 0) \\ x_2 &= 4 && \text{(since } x_8 = 0). \end{aligned} \qquad \text{and}$$

Multiple Dual Optimal Solutions and Degeneracy

0	1	0	0	0	0	0	1	4	primal solution
0	0	0	1	0	-2	0	5	2	$v_4 = (3, 4)$
1	0	0	0	0	1	0	-1	3	
0	0	0	0	1	1	0	-2	2	dual
0	0	0	0	0	-1	1	1	0	solution
0	0	1	0	0	2	0	-3	4	(0,0,0,3,0,1)
0	0	0	0	0	-3	0	-1	-25	

Multiple Dual Optimal Solutions and Degeneracy

0	1	0	0	0	0	0	1	4	primal solution
0	0	0	1	0	-2	0	5	2	$v_4 = (3, 4)$
1	0	0	0	0	1	0	-1	3	
0	0	0	0	1	1	0	-2	2	dual
0	0	0	0	0	-1	1	1	0	solution
0	0	1	0	0	2	0	-3	4	(0,0,0,3,0,1)
0	0	0	0	0	-3	0	-1	-25	

0	1	0	0	0	0	0	0	4	primal solution
0	0	0	1	0	0	-2	3	2	$v4 = (3, 4)$
1	0	0	0	0	0	1	0	3	
0	0	0	0	1	0	1	-1	2	dual
0	0	0	0	0	1	-1	-1	0	solution
0	0	1	0	0	0	2	-1	4	(0,0,0,0,3,4)
0	0	0	0	0	0	-3	-4	-25	

Multiple Dual Optima and Primal Degeneracy

Primal degeneracy in an optimal tableau indicates multiple optimal solutions to the dual which can be obtained with dual simplex pivots.

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Dual degeneracy in an optimal tableau indicates multiple optimal primal solutions that can be obtained with primal simplex pivots.

A tableau is said to be dual degenerate if there is a non-basic variable whose objective row coefficient is zero.

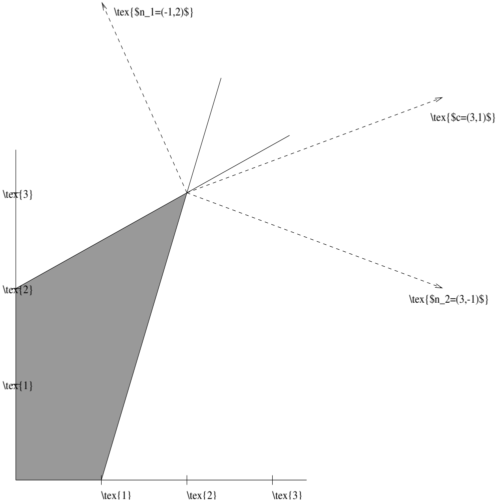
Multiple Primal Optima and Dual Degeneracy

50	0	0	$\underline{100}$	0	1	-10	5	500	
2.5	1	0	2	0	0	$-.1$	.15	15	primal
$-.5$	0	0	0	1	0	0	$-.05$	15	solution
-1	0	1	-1	0	0	.1	$-.1$	10	$(0, 15, 10, 0)$
-100	0	0	0	0	0	-10	-10	-11000	

.5	0	0	1	0	.01	$-.1$	.05	5	
1.5	1	0	0	0	$-.02$	.1	.05	5	primal
$-.5$	0	0	0	1	0	0	$-.05$	15	solution
$-.5$	0	1	0	0	.01	0	$-.05$	15	$(0, 5, 15, 5)$
-100	0	0	0	0	0	-10	-10	-11000	

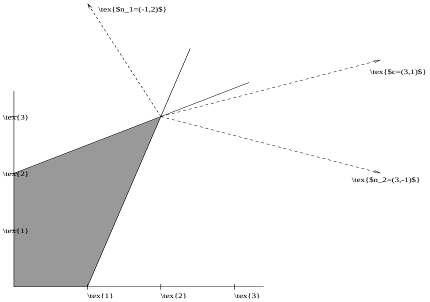
The Geometry of Duality

$$
\begin{array}{ll}
\max & 3x_1 + x_2 \\
\text{s.t.} & -x_1 + 2x_2 \le 4 \\
& 3x_1 - x_2 \le 3 \\
& 0 \le x_1, x_2.
\end{array}
$$



The Geometry of Duality

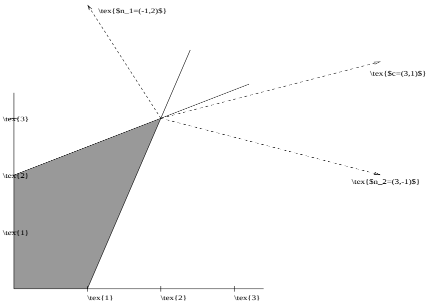
The normal to the hyperplane
$-x_1 + 2x_2 = 4$
is $n_1 = (-1, 2)$.



The Geometry of Duality

The normal to the hyperplane
$-x_1 + 2x_2 = 4$
is $n_1 = (-1, 2)$.

The normal to the hyperplane
$3x_1 - x_2 = 3$
is $n_2 = (3, -1)$.



The Geometry of Duality

The objective normal

$$c = (3, 1)$$

can be written as a non-negative linear combination of the active constraint normals

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$$c = y_1 n_1 + y_2 n_2,$$

Equivalently

$$\begin{aligned} \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= y_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + y_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

The Geometry of Duality

$$
\begin{array}{rr|r}
-1 & 3 & 3 \\
2 & -1 & 1 \\
\hline
1 & -3 & -3 \\
0 & 5 & 7 \\
\hline
1 & -3 & -3 \\
0 & 1 & \frac{7}{5} \\
\hline
1 & 0 & \frac{6}{5} \\
0 & 1 & \frac{7}{5}
\end{array}
$$

The Geometry of Duality

$$
\begin{array}{rr|r}
-1 & 3 & 3 \\
2 & -1 & 1 \\
\hline
1 & -3 & -3 \\
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1 & -3 & -3 \\
0 & 1 & \frac{7}{5} \\
\hline
 & & \\
1 & 0 & \frac{6}{5} \\
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\end{array}
$$

$$y_1 = \frac{6}{5}$$

$$y_2 = \frac{7}{5}$$

$$\begin{array}{rr|r}
-1 & 3 & 3 \\
2 & -1 & 1 \\
\hline
1 & -3 & -3 \\
0 & 5 & 7 \\
\hline
1 & -3 & -3 \\
0 & 1 & \frac{7}{5} \\
\hline
 & & \\
1 & 0 & \frac{6}{5} \\
0 & 1 & \frac{7}{5}
\end{array}$$

$$y_1 = \frac{6}{5}$$

$$y_2 = \frac{7}{5}$$

We claim that $y = (\frac{6}{5}, \frac{7}{5})$ is the optimal solution to the dual!

The Geometry of Duality

$\mathcal{P}$

$$\begin{array}{lll} \max & 3x_1 + x_2 & \\ \text{s.t.} & -x_1 + 2x_2 & \leq 4 \\ & 3x_1 - x_2 & \leq 3 \\ & 0 \leq x_1, & x_2. \end{array}$$

The Geometry of Duality

$$
\begin{array}{lll}
\mathcal{P} & & \\
\max & 3x_1 + x_2 & \\
\text{s.t.} & -x_1 + 2x_2 & \leq 4 \\
 & 3x_1 - x_2 & \leq 3 \\
 & 0 \leq x_1, & x_2.
\end{array}
\qquad
\begin{array}{lll}
\mathcal{D} & & \\
\min & 4y_1 + 3y_2 & \\
\text{s.t.} & -y_1 + 3y_2 & \geq 3 \\
 & 2y_1 - y_2 & \geq 1 \\
 & 0 \leq y_1, & y_2.
\end{array}
$$

The Geometry of Duality

$\mathcal{P}$

$$\begin{array}{lll} \max & 3x_1 + x_2 & \\ \text{s.t.} & -x_1 + 2x_2 & \leq 4 \\ & 3x_1 - x_2 & \leq 3 \\ & 0 \leq x_1, & x_2. \end{array}$$

$\mathcal{D}$

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Primal Solution

$(2, 3)$

The Geometry of Duality

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$$\begin{array}{lll} \max & 3x_1 + x_2 & \\ \text{s.t.} & -x_1 + 2x_2 & \leq 4 \\ & 3x_1 - x_2 & \leq 3 \\ & 0 \leq x_1, & x_2. \end{array}$$

$\mathcal{D}$

$$\begin{array}{lll} \min & 4y_1 + 3y_2 & \\ \text{s.t.} & -y_1 + 3y_2 & \geq 3 \\ & 2y_1 - y_2 & \geq 1 \\ & 0 \leq y_1, & y_2. \end{array}$$

Primal Solution $(2, 3)$ — — Dual Solution $(6/5,\ 7/5)$

$$\begin{array}{lll}
\mathcal{P} & & \\
\max & 3x_1 + x_2 & \\
\text{s.t.} & -x_1 + 2x_2 & \leq 4 \\
 & 3x_1 - x_2 & \leq 3 \\
 & 0 \leq x_1, & x_2.
\end{array}
\qquad\qquad
\begin{array}{lll}
\mathcal{D} & & \\
\min & 4y_1 + 3y_2 & \\
\text{s.t.} & -y_1 + 3y_2 & \geq 3 \\
 & 2y_1 - y_2 & \geq 1 \\
 & 0 \leq y_1, & y_2.
\end{array}$$

$$\begin{array}{ccc}
\text{Primal Solution} & -- & \text{Dual Solution} \\
(2, 3) & & (6/5,\ 7/5)
\end{array}$$

$$\text{Optimal Value} = \ 9$$

Geometric Duality Theorem

Consider the LP $(\mathcal{P})$ $\max\{c^T x \mid Ax \leq b, 0 \leq x\}$, where $A \in \mathbb{R}^{m \times n}$. Given a vector $\bar{x}$ that is feasible for $\mathcal{P}$, define

$$\mathcal{Z}(\bar{x}) = \{j \in \{1, 2, \ldots, n\} : \bar{x}_j = 0\}, \ \mathcal{E}(\bar{x}) = \{i \in \{1, \ldots, m\} : \sum_{j=1}^{n} a_{ij}\bar{x}_j = b_i\}.$$

The indices $\mathcal{Z}(\bar{x})$ and $\mathcal{E}(\bar{x})$ are the *active* indices at $\bar{x}$ and correspond to the active hyperplanes at $\bar{x}$. Then $\bar{x}$ solves $\mathcal{P}$ if and only if there exist non-negative numbers r_j, $j \in \mathcal{Z}(\bar{x})$ and $\bar{y}_i$, $i \in \mathcal{E}(\bar{x})$ such that

$$c = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j + \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet}$$

where for each $i = 1, \ldots, m$, $a_{i\bullet} = (a_{i1}, a_{i2}, \ldots, a_{in})^T$ is the ith column of the matrix A^T, and, for each $j = 1, \ldots, n$, e_j is the jth unit coordinate vector. In addition, if $\bar{x}$ is the solution to $\mathcal{P}$, then the vector $\bar{y} \in \mathbb{R}^m$ given by
$\bar{y}_i = \begin{cases} \bar{y}_i & \text{for } i \in \mathcal{E}(\bar{x}) \\ 0 & \text{otherwise} \end{cases}$, solves the dual problem.

Geometric Duality Theorem: Proof

First suppose that $\bar{x}$ solves $\mathcal{P}$, and let $\bar{y}$ solve $\mathcal{D}$.

Geometric Duality Theorem: Proof

First suppose that $\bar{x}$ solves $\mathcal{P}$, and let $\bar{y}$ solve $\mathcal{D}$.
The Complementary Slackness Theorem implies that

$$(I) \quad \bar{y}_i = 0 \text{ for } i \in \{1, 2, \ldots, m\} \setminus \mathcal{E}(\bar{x}) \quad \left(\textstyle\sum_{j=1}^{n} a_{ij}\bar{x}_j < b_i\right)$$

and

$$(II) \quad \sum_{i=1}^{m} \bar{y}_i a_{ij} = c_j \text{ for } j \in \{1, \ldots, n\} \setminus \mathcal{Z}(\bar{x}) \quad (0 < \bar{x}_j).$$

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Define $r = A^T\bar{y} - c \geq 0$.

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Define $r = A^T\bar{y} - c \geq 0$. By (II), $r_j = 0$ for $j \in \{1, \ldots, n\} \setminus \mathcal{Z}(\bar{x})$, while

$$(III) \; c_j = -r_j + \sum_{i=1}^{m} \bar{y}_i a_{ij} \text{ for } j \in \mathcal{Z}(\bar{x}).$$

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and

$$(II) \quad \sum_{i=1}^{m} \bar{y}_i a_{ij} = c_j \text{ for } j \in \{1, \ldots, n\} \setminus \mathcal{Z}(\bar{x}) \quad (0 < \bar{x}_j).$$

Define $r = A^T\bar{y} - c \geq 0$. By (II), $r_j = 0$ for $j \in \{1, \ldots, n\} \setminus \mathcal{Z}(\bar{x})$, while

$$(III) \; c_j = -r_j + \sum_{i=1}^{m} \bar{y}_i a_{ij} \text{ for } j \in \mathcal{Z}(\bar{x}).$$

(I), (II), and (III) gives

$$c = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \; + \; A^T\bar{y} = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \; + \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet}.$$

Geometric Duality Theorem: Proof

Conversely, suppose $\bar{x}$ is feasible for $\mathcal{P}$ and $0 \le r_j,\ j \in \mathcal{Z}(\bar{x})$ and $0 \le \bar{y}_i,\ i \in \mathcal{E}(\bar{x})$ satisfy

$$c = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \ + \ A^T \bar{y} = -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \ + \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet}$$

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Set $\bar{y}_i = 0 \ \notin \mathcal{E}(\bar{x})$ to obtain $\bar{y} \in \mathbb{R}^m$.

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Set $\bar{y}_i = 0 \ \notin \mathcal{E}(\bar{x})$ to obtain $\bar{y} \in \mathbb{R}^m$. Then

$$A^T \bar{y} = \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet} \geq -\sum_{j \in \mathcal{Z}(\bar{x})} r_j e_j \ + \sum_{i \in \mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet} = c,$$

so that $\bar{y}$ is feasible for $\mathcal{D}$.

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$$c = -\sum_{j\in\mathcal{Z}(\bar{x})} r_j e_j \ + \ A^T\bar{y} = -\sum_{j\in\mathcal{Z}(\bar{x})} r_j e_j \ + \sum_{i\in\mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet}$$

Set $\bar{y}_i = 0 \ \notin \mathcal{E}(\bar{x})$ to obtain $\bar{y} \in \mathbb{R}^m$. Then

$$A^T\bar{y} = \sum_{i\in\mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet} \ge -\sum_{j\in\mathcal{Z}(\bar{x})} r_j e_j \ + \sum_{i\in\mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet} = c,$$

so that $\bar{y}$ is feasible for $\mathcal{D}$. Moreover,

$$c^T\bar{x} = -\sum_{j\in\mathcal{Z}(\bar{x})} r_j e_j^T\bar{x} \ + \sum_{i\in\mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet}^T\bar{x} = \sum_{i\in\mathcal{E}(\bar{x})} \bar{y}_i a_{i\bullet}^T\bar{x} = \bar{y}^T A\bar{x} = \bar{y}^T b,$$

so $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$ by the Weak Duality Theorem.

Example

Does the vector $\bar{x} = (1, 0, 2, 0)^T$ solve the LP

$$\begin{array}{lrrrrl}
\text{maximize} & x_1 & +x_2 & -x_3 & +2x_4 & \\
\text{subject to} & x_1 & +3x_2 & -2x_3 & +4x_4 & \leq -3 \\
 & & 4x_2 & -2x_3 & +3x_4 & \leq 1 \\
 & & -x_2 & +x_3 & -x_4 & \leq 2 \\
 & -x_1 & -x_2 & +2x_3 & -x_5 & \leq 4 \\
 & 0 \leq & x_1, & x_2, & x_3, & x_4 \qquad .
\end{array}$$

Example

Which constraints are active at $\bar{x} = (1, 0, 2, 0)^T$?

$$\begin{array}{rrrrcr} x_1 & +3x_2 & -2x_3 & +4x_4 & \leq & -3 \\ & 4x_2 & -2x_3 & +3x_4 & \leq & 1 \\ & -x_2 & +x_3 & -x_4 & \leq & 2 \\ -x_1 & -x_2 & +2x_3 & -x_5 & \leq & 4 \end{array}$$

Example

Which constraints are active at $\bar{x} = (1, 0, 2, 0)^T$?

$$\begin{array}{rrrrcrr}
x_1 & +3x_2 & -2x_3 & +4x_4 & \leq & -3 & = \\
 & 4x_2 & -2x_3 & +3x_4 & \leq & 1 & \\
 & -x_2 & +x_3 & -x_4 & \leq & 2 & \\
-x_1 & -x_2 & +2x_3 & -x_5 & \leq & 4 &
\end{array}$$

Example

Which constraints are active at $\bar{x} = (1, 0, 2, 0)^T$?

$$\begin{array}{rrrrcrl} x_1 & +3x_2 & -2x_3 & +4x_4 & \leq & -3 & \color{red}{=} \\ & 4x_2 & -2x_3 & +3x_4 & \leq & 1 & \color{red}{< \ \text{so } y_2 = 0} \\ & -x_2 & +x_3 & -x_4 & \leq & 2 & \\ -x_1 & -x_2 & +2x_3 & -x_5 & \leq & 4 & \end{array}$$

Example

Which constraints are active at $\bar{x} = (1, 0, 2, 0)^T$?

$$\begin{array}{rrrrcrl}
x_1 & +3x_2 & -2x_3 & +4x_4 & \leq & -3 & \quad = \\
 & 4x_2 & -2x_3 & +3x_4 & \leq & 1 & \quad < \text{ so } y_2 = 0 \\
 & -x_2 & +x_3 & -x_4 & \leq & 2 & \quad = \\
-x_1 & -x_2 & +2x_3 & -x_5 & \leq & 4 &
\end{array}$$

Example

Which constraints are active at $\bar{x} = (1,0,2,0)^T$?

$$\begin{array}{rrrrcrcl}
x_1 & +3x_2 & -2x_3 & +4x_4 & \leq & -3 & & = \\
 & 4x_2 & -2x_3 & +3x_4 & \leq & 1 & & < \text{ so } y_2 = 0 \\
 & -x_2 & +x_3 & -x_4 & \leq & 2 & & = \\
-x_1 & -x_2 & +2x_3 & -x_5 & \leq & 4 & & < \text{ so } y_4 = 0
\end{array}$$

Example

Which constraints are active at $\bar{x} = (1,0,2,0)^T$?

$$\begin{array}{cccccccl}
x_1 & +3x_2 & -2x_3 & +4x_4 & \leq & -3 & & = \\
 & 4x_2 & -2x_3 & +3x_4 & \leq & 1 & & < \text{ so } y_2 = 0 \\
 & -x_2 & +x_3 & -x_4 & \leq & 2 & & = \\
-x_1 & -x_2 & +2x_3 & -x_5 & \leq & 4 & & < \text{ so } y_4 = 0
\end{array}$$

The 1st and 3rd constraints are active.

Example

Knowing $y_2 = y_4 = 0$ solve for y_1 and y_3 by writing the objective normal as a non-negative linear combination of the constraint outer normals.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & -1 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \\ r_2 \\ r_4 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \end{pmatrix}.$$

Example

Row reducing, we get

$$
\begin{array}{rrrr|r}
y_1 & y_3 & r_2 & r_4 & \\
1 & 0 & 0 & 0 & 1 \\
3 & -1 & -1 & 0 & 1 \\
-2 & 1 & 0 & 0 & -1 \\
4 & -1 & 0 & -1 & 2 \\
\hline
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 2 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 2
\end{array}.
$$

Therefore, $y_1 = 1\, y_3 = 1,\ r_2 = 1$, and $r_4 = 1$. Hence, $\bar{x} = (1, 0, 2, 0)^T$ sopves the pimal and $\bar{y} = (1, 0, 1, 0)^T$ solves the dual.

We now double check to see if the vector $\bar{y} = (1, 0, 1, 0)$ does indeed solve the dual.

Example

Check that $\bar{y} = (1, 0, 1, 0)$ solves the dual problem.

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & -3y_1 & + & y_2 & + & 2y_3 & + & 4y_4 & & \\
\text{subject to} & y_1 & & & & & - & y4 & \geq & 1 \\
& 3y_1 & + & 4y_2 & - & y_3 & - & y_4 & \geq & 1 \\
& -2y_1 & - & 2y_2 & + & y_3 & + & 2y_4 & \geq & -1 \\
& 4y_1 & + & 3y_2 & - & y_3 & - & y_4 & \geq & 2 \\
\end{array}
$$

$$0 \leq y_1, y_2, y_3, y_4 .$$

Example

Check that $\bar{y} = (1, 0, 1, 0)$ solves the dual problem.

$$\begin{array}{lrcrcrcrcrl}
\text{minimize} & -3y_1 & + & y_2 & + & 2y_3 & + & 4y_4 & & & \text{dual slacks} \\
\text{subject to} & y_1 & & & & & - & y4 & \geq & 1 & r_1 = 0 \\
 & 3y_1 & + & 4y_2 & - & y_3 & - & y_4 & \geq & 1 & r_2 = 1 \\
 & -2y_1 & - & 2y_2 & + & y_3 & + & 2y_4 & \geq & -1 & r_3 = 0 \\
 & 4y_1 & + & 3y_2 & - & y_3 & - & y_4 & \geq & 2 & r_4 = 1 \\
 & & & & & 0 \leq y_1, y_2, y_3, y_4. & & & & &
\end{array}$$

Example 2

Does $x = (3, 1, 0)^T$ solve $\mathcal{P}$, where

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 1 & -4 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}.$$

Example 3

Does $x = (1, 2, 1, 0)^T$ solve $\mathcal{P}$, where

$$A = \begin{bmatrix} 3 & 1 & 4 & 2 \\ -3 & 2 & 2 & 1 \\ 1 & -2 & 3 & 0 \\ -3 & 2 & -1 & 4 \end{bmatrix}, \qquad c = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 9 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$