Linear Programming

Lecture 1: Linear Algebra Review

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

columns rows

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1} & a_{\bullet 2} & \dots & a_{\bullet n} \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1}^T \\ a_{\bullet 2}^T \\ \vdots \\ a_{\bullet n}^T \end{bmatrix} = \begin{bmatrix} a_{1\bullet}^T & a_{2\bullet}^T & \dots & a_{m\bullet}^T \end{bmatrix}$$

Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1 \, \mathbf{a}_{\bullet 1} \; + \; x_2 \, \mathbf{a}_{\bullet 2} \; + \; \cdots \; + \; x_n \, \mathbf{a}_{\bullet n}$$

A linear combination of the columns.

The Range of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).

Range of A

$$\mathrm{Ran}\,(A) \;=\; \{y \in \mathbb{R}^m \mid \exists\, x \in \mathbb{R}^n \text{ such that } y = Ax\}$$

$$\mathrm{Ran}\,(A) = \text{ the linear span of the columns of } A$$

Two Special Subspaces

Let $v_1, \ldots, v_k \in \mathbb{R}^n$.

- The linear span of $v_1, \ldots, v_k$:

$$\mathrm{Span}\,[v_1, \ldots, v_k] = \{y \mid y = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k,\ \xi_1, \ldots, \xi_k \in \mathbb{R}\}$$

- The subspace orthogonal to $v_1, \ldots, v_k$:

$$\{v_1, \ldots, v_k\}^{\perp} = \{z \in \mathbb{R}^n \mid z \bullet v_i = 0,\ i = 1, \ldots, k\}$$

Facts:

$$\{v_1, \ldots, v_k\}^{\perp} = \mathrm{Span}\,[v_1, \ldots, v_k]^{\perp}$$

$$\mathrm{Span}\,[v_1, \ldots, v_k] = \left[\mathrm{Span}\,[v_1, \ldots, v_k]^{\perp}\right]^{\perp}$$

Matrix Vector Multiplication

A row space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \bullet x \\ a_{2\bullet} \bullet x \\ \vdots \\ a_{m\bullet} \bullet x \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} x_i \\ \sum_{i=1}^{n} a_{2i} x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} x_i \end{bmatrix}$$

The dot product of x with the rows of A.

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).
Null Space of A

$$\mathrm{Nul}\,(A) \;=\; \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$$\begin{aligned}
\mathrm{Nul}\,(A) &= \text{subspace orthogonal to the rows of } A \\
&= \mathrm{Span}\,[a_{1\bullet}, a_{2\bullet}, \ldots, a_{m\bullet}]^{\perp} \\
&= \mathrm{Ran}\,\left(A^T\right)^{\perp}
\end{aligned}$$

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\end{aligned}$$

Fundamental Theorem of the Alternative:

$$\mathrm{Nul}\,(A) = \mathrm{Ran}\,\left(A^T\right)^{\perp} \qquad \mathrm{Ran}\,(A) = \mathrm{Nul}\,\left(A^T\right)^{\perp}$$

Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

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where

$$B = \begin{bmatrix} 3 & -4 & 1 \\ 2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \quad \text{so take} \quad M = \begin{bmatrix} X \\ Y \end{bmatrix},$$

$$\text{where} \quad X = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}.$$

Multiplication of Block Structured Matrices

$$AM = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix}$$

Multiplication of Block Structured Matrices

$$
\begin{aligned}
AM &= \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 2 & -11 \\ 2 & 12 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 4 & -12 \\ 6 & 15 \\ 1 & -2 \\ 4 & 1 \\ -4 & -1 \end{bmatrix}.
\end{aligned}
$$

Solving Systems of Linear equations

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If a solution $x_0 \in \mathbb{R}^n$ exists, then the set of solutions is given by

$$x_0 \ + \ \mathrm{Nul}\,(A) \ .$$

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \,|\, b]$$

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1. Interchange any two rows.
2. Multiply any row by a non-zero constant.
3. Replace any row by itself plus a multiple of any *other* row.

These elementary row operations can be interpreted as multiplying the augmented matrix on the left by a special nonsingular matrix.

Exchange and Permutation Matrices

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Multiplying any $4 \times n$ matrix on the left by the exchange matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

will exchange the second and fourth rows of the matrix.

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A permutation matrix is obtained by permuting the columns of the identity matrix.

Notes on Matrix Multiplication

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For example, left multiplication by a permutation matrix permutes the rows of the matrix.

However, mechanically, left multiplication corresponds to matrix vector multiplication on the columns.

$$MA \; = \; M\,[a_{\bullet 1} \; a_{\bullet 2} \; \cdots \; a_{\bullet n}] \; = \; [Ma_{\bullet 1} \; Ma_{\bullet 2} \; \cdots \; Ma_{\bullet n}]$$

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$$AN = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix} N = \begin{bmatrix} a_{1\bullet} N \\ a_{2\bullet} N \\ \vdots \\ a_{m\bullet} N \end{bmatrix}$$

Gaussian Elimination Matrices

The key step in Gaussian elimination is to transform a vector of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form

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This can be accomplished by left matrix multiplication as follows.

Gaussian Elimination Matrices

$a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

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The matrix

$$G = \begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}$$

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This matrix is invertible with inverse

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Note that a Gaussian elimination matrix and its inverse are both lower triangular matrices.

Matrix Sub-Algebras

Lower (upper) triangular matrices in $\mathbb{R}^{n\times n}$ are said to form a *sub-algebra* of $\mathbb{R}^{n\times n}$.

A subset S of $\mathbb{R}^{n\times n}$ is said to be a sub-algebra of $\mathbb{R}^{n\times n}$ if

- S is a subspace of $\mathbb{R}^{n\times n}$,
- S is closed wrt matrix multiplication, and
- if $M \in S$ is invertible, then $M^{-1} \in S$.

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix. Here $k = 0$ (so there is no vector a), $\alpha = 1$, and $b = (2, -1)^T$. Hence, $-\alpha^{-1} b = (-2, 1)^T$.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

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Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

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Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

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Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

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