Triple Integrals for Volumes of Some Classic Shapes

In the following pages, I give some worked out examples where triple integrals are used to find some classic shapes volumes (boxes, cylinders, spheres and cones) For all of these shapes, triple integrals aren't needed, but I just want to show you how you could use triple integrals to find them. The methods of cylindrical and spherical coordinates are also illustrated. I hope this helps you better understand how to set up a triple integral. Remember that the volume of a solid region E is given by $\iiint_E 1\ dV$.

A Rectangular Box

A rectangular box can be described by the set of inequalities $a \leq x \leq b$, $c \leq y \leq d$, $p \leq z \leq q$. So that the volume comes out to be length times width times height as expected:

$$\iiint_E 1\ dV = \int_a^b \int_c^d \int_p^q 1\, dz dy dx = (b-a)(d-c)(q-p).$$

A Circular Cylinder

The equation for the outer edge of a circular cylinder of radius a is given by $x^2 + y^2 = a^2$. If we want to consider the volume inside such a cylinder with height h, then we are considering the region where $x^2 + y^2 \leq a^2$ and $0 \leq z \leq h$ (in other words between the planes $z = 0$ and $z = h$). We already have bounds on z, so let's use that as the innermost integral. Now we need bounds for the circular $x^2 + y^2 \leq a^2$ in the xy-plane. We can do that in a few different ways:

1. *In Cartesian Coordinates*:
   The solid can be described by the inequalities $-a \leq x \leq a$, $-\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}$, $0 \leq z \leq h$. So we find the volume is:

   $$\iiint_E 1\ dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h 1\, dz dy dx = \int_{-a}^{a} 2h\sqrt{a^2 - x^2}\, dx = 2h\frac{1}{2}\pi a^2 = \pi a^2 h.$$

   Note: I skipped some steps in the integration. You would need to see the last integration geometrically (that the last integral represents the area of exactly half a circle), or you would have to use trig substitution.

2. *In Cylindrical Coordinates*: A circular cylinder is perfect for cylindrical coordinates! The region $x^2 + y^2 \leq a^2$ is very easily described, so that all together the solid can be described by the inequalities $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$, $0 \leq z \leq h$. So we find the volume is:

   $$\iiint_E 1\ dV = \int_0^{2\pi} \int_0^a \int_0^h r\, dz dr d\theta = \int_0^{2\pi} d\theta \int_0^a r\, dr \int_0^h dz = 2\pi \frac{1}{2} a^2 h = \pi a^2 h.$$

   Either way, we see that we get the expected volume formula.

A Sphere

The equation for the outer edge of a sphere of radius a is given by $x^2 + y^2 + z^2 = a^2$. If we want to consider the volume inside, then we are considering the regions $x^2 + y^2 + z^2 \leq a^2$. We will set up the inequalities in three ways.

1. *In Cartesian Coordinates*: Solving for z gives $-\sqrt{a^2 - x^2 - y^2} \leq z \leq \sqrt{a^2 - x^2 - y^2}$. Then the projection of the sphere onto the xy-plane (i.e. the equation you get when you have $z = 0$ in the sphere equation) is just the circle $x^2 + y^2 = a^2$. Now we must describe this with inequalities. All together, the solid can be described by the inequalities $-a \leq x \leq a$, $-\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}$, $-\sqrt{a^2 - x^2 - y^2} \leq z \leq \sqrt{a^2 - x^2 - y^2}$. So we can find the volume:

$$\iiint_E 1\ dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1\, dzdydx = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= \int_{-a}^{a} 2\frac{1}{2}\pi(a^2 - x^2)\, dx = \pi(2a^3 - \frac{2}{3}a^3) = \frac{4}{3}\pi a^3.$$

   Note: Same note as I made for the circular cylinder concerning skipped steps in the integration.

2. *In Cylindrical Coordinates*: The bound on z would still be the same, but we would use polar for x and y. All together, the solid can be described by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$, $-\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}$. And we get a volume of:

$$\iiint_E 1\ dV = \int_0^{2\pi} \int_0^{a} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\, dzdrd\theta = 2\pi \int_0^{a} 2r\sqrt{a^2 - r^2}\, dr$$

$$= 2\pi \int_0^{a^2} \sqrt{u}\, du = 2\pi\frac{2}{3}a^3 = \frac{4}{3}\pi a^3$$

3. *In Spherical Coordinates*: In spherical coordinates, the sphere is all points where $0 \leq \phi \leq \pi$ (the angle measured down from the positive z axis ranges), $0 \leq \theta \leq 2\pi$ (just like in polar coordinates), and $0 \leq \rho \leq a$. And we get a volume of:

$$\iiint_E 1\ dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{a} \rho^2 \sin(\phi) d\rho d\theta d\phi = \int_0^{\pi} \sin(\phi)\, d\phi \int_0^{2\pi} d\theta \int_0^{a} \rho^2 d\rho = (2)(2\pi)\left(\frac{1}{3}a^3\right) = \frac{4}{3}\pi a^3$$

   In all three cases, we see that we get the expected volume formula.

A Cone

The equation $a^2z^2 = h^2x^2 + h^2y^2$ gives a cone with a point at the origin that opens upward (and downward), such that if the height is $z = h$ then radius of the circle at that height is a (you can see this by pluggin in $z = h$ and simplifying). So let's find the volume inside this cone which has height h and radius of a at that height.

1. *In Cartesian Coordinates*: First we have $\frac{h}{a}\sqrt{x^2+y^2} \leq z \leq h$ (I got the first bound by solving for z in the equation for the cone and simplifying). The projection down on the xy-plane would be the intersection of $z = h$ and the cone, which is the disc $x^2 + y^2 \leq a^2$. So the solid can be described by the inequalities $-a \leq x \leq a$, $-\sqrt{a^2-x^2} \leq y \leq \sqrt{a^2-x^2}$, $\frac{h}{a}\sqrt{x^2+y^2} \leq z \leq h$. We find the volume is:

$$\iiint_E 1\ dV = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{\frac{h}{a}\sqrt{x^2+y^2}}^{h} 1\, dzdydx = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} h - \frac{h}{a}\sqrt{x^2+y^2}\, dydx$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} h\, dydx - \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{h}{a}\sqrt{x^2+y^2}\, dydx = h\pi a^2 - \frac{2}{3}\pi h a^2 = \frac{1}{3}\pi h a^2.$$

Note: Again I skipped steps in the integration (this would be a messy/hard integration problem, Cartesian coordinates give messy integrals when working with spheres and cones).

2. *In Cylindrical Coordinates*: The solid can be described by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$, $\frac{h}{a}r \leq z \leq h$. And we get a volume of:

$$\iiint_E 1\ dV = \int_0^{2\pi}\int_0^{a}\int_{\frac{h}{a}r}^{h} r\, dzdrd\theta = 2\pi\int_0^a hr - \frac{h}{a}r^2\, dr = 2\pi(\frac{1}{2}ha^2 - \frac{h}{3a}a^3) = \frac{1}{3}\pi h a^2.$$

3. *In Spherical Coordinates*: In spherical coordinates, we need to find the angle, ϕ, that the cone makes with the positive z-axis and we need to find the range on ρ. Viewing the cone from the side, the angle ϕ is part of a right triangle with side lengths a and h. So $\tan(\phi) = \frac{a}{h}$ on the edge of the cone. Thus, the range is $0 \leq \phi \leq \tan^{-1}\left(\frac{a}{h}\right)$. The range on ρ depends on ϕ. We do know the $0 \leq z \leq h$. And since $z = \rho\cos(\phi)$, we can say that $0 \leq \rho \leq \frac{h}{\cos(\phi)} = h\sec(\phi)$.

   So all together we have $0 \leq \phi \leq \tan^{-1}\left(\frac{a}{h}\right)$, $0 \leq \theta \leq 2\pi$, and $0 \leq \rho \leq h\sec(\phi)$. And we get a volume of:

$$\iiint_E 1\ dV = \int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{2\pi}\int_0^{h\sec(\phi)} \rho^2\sin(\phi) d\rho d\theta d\phi = 2\pi\int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{h\sec(\phi)} \rho^2\sin(\phi) d\rho d\phi$$

$$= 2\pi\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \frac{1}{3}h^3\sec^3(\phi)\sin(\phi)\, d\phi = \frac{2}{3}\pi h^3\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \sec^2(\phi)\tan(\phi)\, d\phi = \frac{2}{3}\pi h^3\frac{1}{2}\left(\frac{a}{h}\right)^2 = \frac{1}{3}\pi h a^2$$

In all three cases, we see that we get the expected volume formula.