

Taylor Polynomials and Taylor Series

Contents

1 Taylor Polynomials	1
Example 1.1.....	2
Example 1.2.....	4
Example 1.3.....	5
1 Exercises	6
1 Exercise Solutions	8
2 Taylor's Inequality.....	11
Example 2.1.....	12
Example 2.2.....	12
2 Exercises	13
2 Exercise Solutions	15
3 Infinite Series.....	16
Introduction.....	16
Geometric Series.....	19
Example 3.1	19
3 Exercises	21
3 Exercise Solutions	22
Appendix A. Proof of the Sum Formula for a Geometric Series	23
4 Taylor Series	24
Example 4.0.....	24
Taylor Series	24
Example 4.1.....	25
Example 4.2.....	26
Example 4.3.....	27
Example 4.4.....	28
Summary to this Point	29
Taylor Polynomials.....	29
Taylor Series.....	29
Radius of Convergence.....	30
Example 4.5.....	31
4 Exercises	32
4 Exercise Solutions	33
Common Taylor Series & Radii of Convergence.....	34
5 Operations with Taylor Series.....	35
Example 5.1.....	35
Example 5.2.....	37
Example 5.3.....	37
Example 5.4.....	37
Example 5.5.....	38
5 Exercises	39
5 Exercise Solutions	42

TAYLOR POLYNOMIALS AND TAYLOR SERIES

The following notes are based in part on material developed by Dr. Ken Bube of the University of Washington Department of Mathematics in the Spring, 2005.

1 Taylor Polynomials

The tangent line to the graph of $y = f(x)$ at the point $x = a$ is the line going through the point $(a, f(a))$ that has slope $f'(a)$. By the point-slope form of the equation of a line, its equation is

$$\begin{aligned}y - f(a) &= f'(a)(x - a) \\y &= f(a) + f'(a)(x - a)\end{aligned}$$

As you have seen in Math 124, the tangent line is a very good approximation to $y = f(x)$ near $x = a$, as shown in Figure 1.

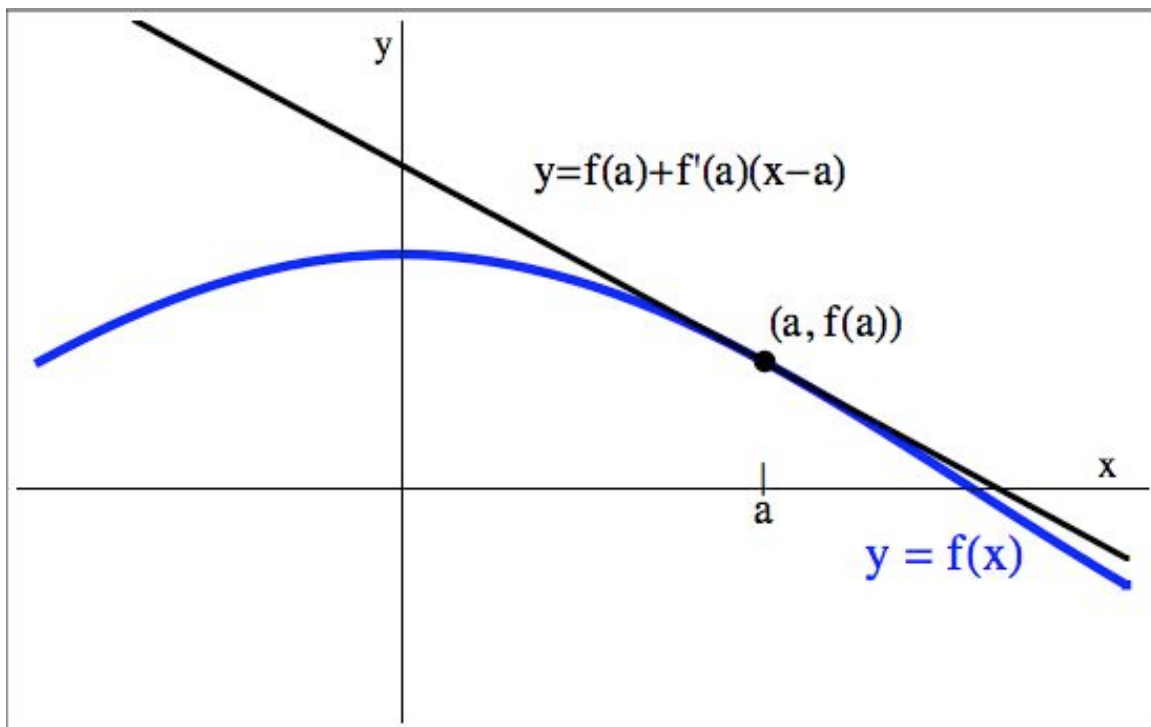


FIGURE 1. The line $y = f(a) + f'(a)(x - a)$ tangent to the graph of $y = f(x)$ at the point $(a, f(a))$.

We will give a name, $T_1(x)$, to the function corresponding to the tangent line:

$$T_1(x) = f(a) + f'(a)(x - a)$$

For x near $x = a$, we have

$$f(x) \approx T_1(x).$$

The tangent line function $T_1(x)$ is called the Taylor polynomial of degree one for $f(x)$, centered at $x = a$. Notice that it satisfies the two conditions

$$T_1(a) = f(a) \quad \text{and} \quad T_1'(a) = f'(a).$$

In other words, $T_1(x)$ is the polynomial of degree one that has the same function value at $x = a$ and the same first derivative value at $x = a$ as the original functions $f(x)$.

We can get a better approximation, $T_2(x)$ near $x = a$, using a parabola (a polynomial of degree two). The formula for $T_2(x)$ is

$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

$T_2(x)$ is called the Taylor polynomial of degree two for $f(x)$, centered at $x = a$.

Since $T_2'(x) = f'(a) + f''(a)(x - a)$ and $T_2''(x) = f''(a)$, $T_2(x)$ satisfies the three conditions

$$T_2(a) = f(a), \quad T_2'(a) = f'(a) \quad \text{and} \quad T_2''(a) = f''(a).$$

In other words, $T_2(x)$ is the polynomial of degree two that has the same function value at $x = a$, the same first derivative value at $x = a$, and the same second derivative value at $x = a$ as the original function $f(x)$.

Example 1.1

Find the Taylor polynomials of degrees one and two for $f(x) = e^x$, centered at $x = 0$.

Solution: Since $f(x) = f'(x) = f''(x) = e^x$, we have $f(0) = f'(0) = f''(0) = e^0 = 1$, so the Taylor polynomial of degree one (the tangent line to $y = e^x$ at the point $(0, 1)$) is

$$T_1(x) = f(0) + f'(0)(x - 0) = 1 + x.$$

The Taylor polynomial of degree two (the parabola that best fits $y = e^x$ near $x = 0$) is

$$T_2(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2}(x - 0)^2 = 1 + x + \frac{x^2}{2}.$$

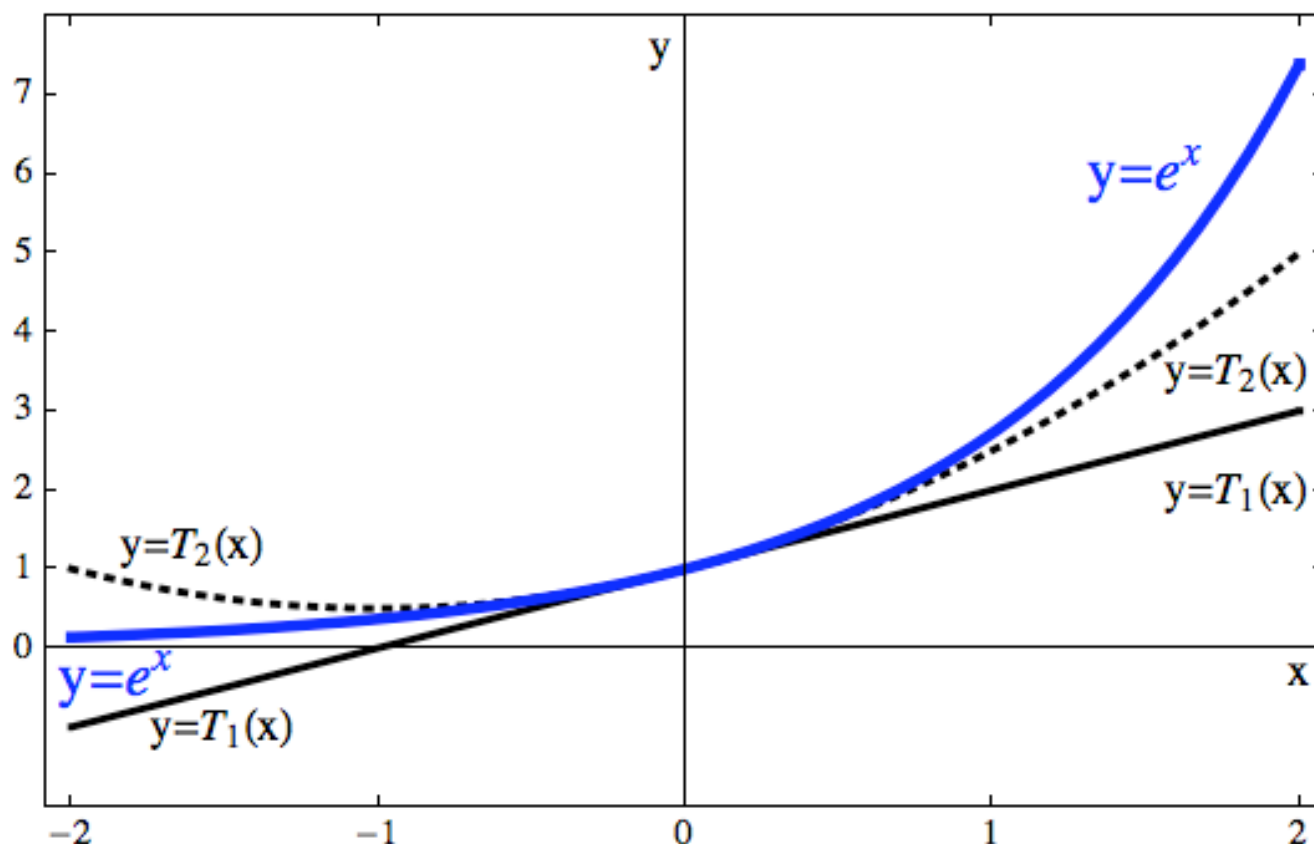


FIGURE 2. The Taylor polynomials $T_1(x)$ and $T_2(x)$ for $f(x) = e^x$, centered at $x = 0$. Notice that $T_2(x)$ does a better job of matching $f(x)$ near $x = 0$.

We can get an even better approximation, $T_3(x)$ near $x = a$, using a cubic (a polynomial of degree three). The formula for $T_3(x)$ is

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

$T_3(x)$ is called the Taylor polynomial of degree three for $f(x)$, centered at $x = a$. A short computation (Exercise 1.1, problem 3) shows that $T_3(x)$ satisfies the four conditions

$$T_3(x) = f(a), \quad T_3'(x) = f'(a), \quad T_3''(x) = f''(a), \quad \text{and} \quad T_3'''(x) = f'''(a).$$

In other words, $T_3(x)$ is the polynomial of degree three that has the same function value at $x = a$, the same first derivative value at $x = a$, the same second derivative value at $x = a$, and the same third derivative value at $x = a$, as the original function $f(x)$.

Example 1.2

Find the Taylor polynomial of degree three for $f(x) = \sin x$, centered at $x = 5\pi/6$.

Solution:

$$\begin{aligned} f(x) &= \sin x, & f\left(\frac{5\pi}{6}\right) &= \frac{1}{2}, \\ f'(x) &= \cos x, & f'\left(\frac{5\pi}{6}\right) &= -\frac{3}{2}, \\ f''(x) &= -\sin x, & f''\left(\frac{5\pi}{6}\right) &= -\frac{1}{2}, \\ f'''(x) &= -\cos x, & f'''\left(\frac{5\pi}{6}\right) &= \frac{\sqrt{3}}{2}, \end{aligned}$$

The Taylor polynomial of degree three (the cubic that best fits $y = \sin x$ near $x = 5\pi/6$) is

$$\begin{aligned} T_3(x) &= f\left(\frac{5\pi}{6}\right) + f'\left(\frac{5\pi}{6}\right)\left(x - \frac{5\pi}{6}\right) + \frac{f''\left(\frac{5\pi}{6}\right)}{2}\left(x - \frac{5\pi}{6}\right)^2 + \frac{f'''\left(\frac{5\pi}{6}\right)}{6}\left(x - \frac{5\pi}{6}\right)^3 = \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{5\pi}{6}\right) - \frac{1}{4}\left(x - \frac{5\pi}{6}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{5\pi}{6}\right)^3. \end{aligned}$$

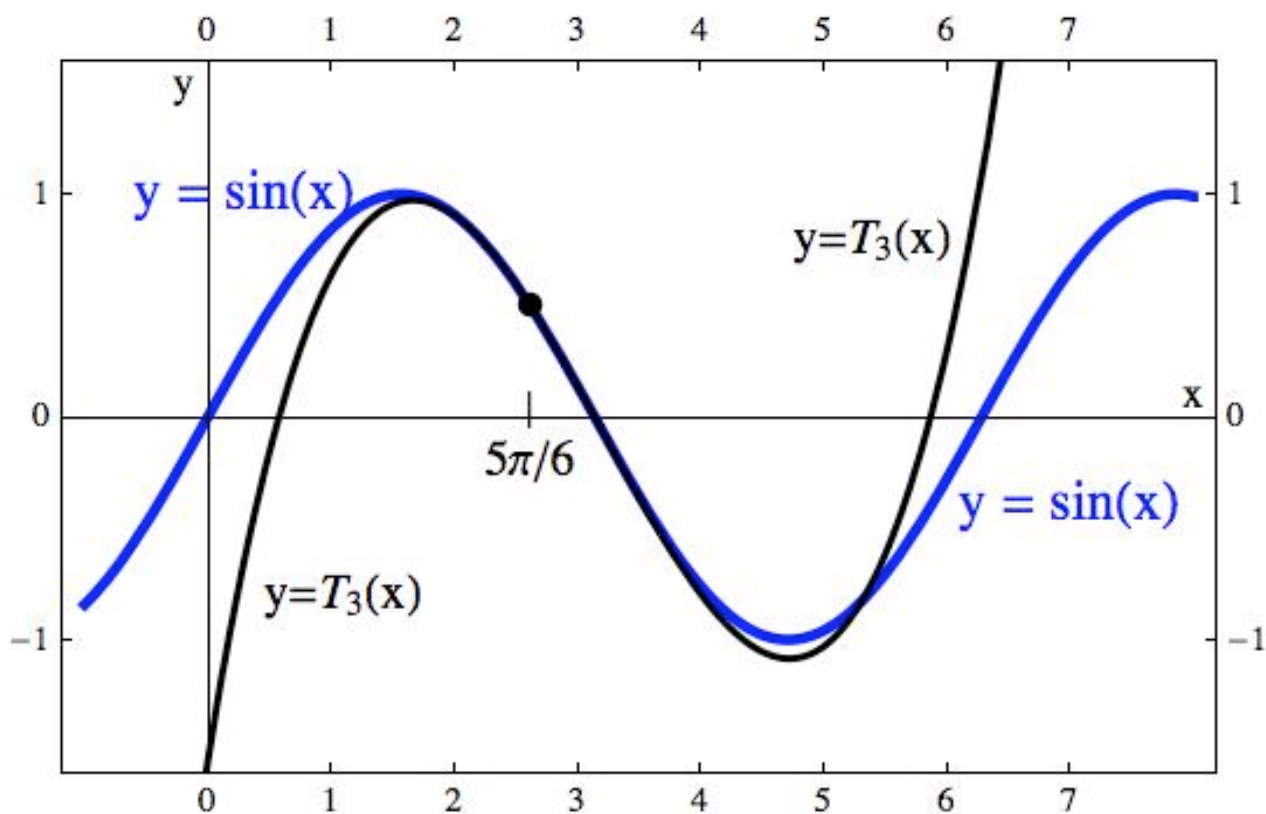


FIGURE 3. The Taylor polynomial $T_3(x)$ for $f(x) = \sin x$, centered at $x = 5\pi/6$.

In general, the Taylor polynomial of degree n for $f(x)$, centered at $x = a$, is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \dots \\ + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{(n-1)} + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $n!$ (read “ n factorial”) is the product of the first n positive integers:

$$n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n.$$

For convenience, we define

$$0! = 1;$$

then the formula for $T_n(x)$ can be written using summation notation:

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

(where $f^{(0)}$ just means the function f itself). By a similar computation to that for $T_2(x)$ or for $T_3(x)$, it can be shown that $T_n(x)$ satisfies the $n+1$ conditions

$$T_n(a) = f(a), \quad T_n'(a) = f'(a), \quad T_n''(a) = f''(a), \quad \dots, \quad T_n^{(n-1)}(a) = f^{(n-1)}(a), \quad T_n^{(n)}(a) = f^{(n)}(a).$$

In other words, $T_n(x)$ is the polynomial of degree n that has the same function value, first derivative value, second derivative value, etc., and n th derivative value at $x = a$ as the original function $f(x)$.

Example 1.3

Find the Taylor polynomial of degree n for $f(x) = \frac{1}{1-x}$, centered at $x = 0$.

	$f(x) = (1-x)^{-1},$	$f(0) = 1,$
	$f'(x) = -1 \cdot (1-x)^{-2} \cdot (-1) = 1 \cdot (1-x)^{-2},$	$f'(0) = 1,$
	$f''(x) = 2 \cdot 1 \cdot (1-x)^{-3},$	$f''(0) = 2!,$
	$f'''(x) = 3 \cdot 2 \cdot 1 \cdot (1-x)^{-4},$	$f'''(0) = 3!,$
	<i>etc.</i>	
	$f^{(n)}(x) = n! \cdot (1-x)^{-(n+1)},$	$f^{(n)}(0) = n!.$

The Taylor polynomial for degree n for $f(x) = \frac{1}{1-x}$, centered at $x = 0$, is

$$T_n(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3 + \dots + \frac{f^{(n)}(0)}{n!}(x-0)^n \\ = 1 + x + x^2 + x^3 + \dots + x^n.$$

1 Exercises

- Let $T_2(x) = 1 + x + \frac{x^2}{2}$ be the Taylor polynomial of degree two for $f(x) = e^x$, centered at $x = 0$ (see Example 1.1). Verify directly by taking their derivatives that $T_2(x)$ and $f(x)$ satisfy the three conditions $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2''(0) = f''(0)$.
- Let $T_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{5\pi}{6}\right) - \frac{1}{4}\left(x - \frac{5\pi}{6}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{5\pi}{6}\right)^3$ be the Taylor polynomial of degree three for $f(x) = \sin x$, centered at $x = \frac{5\pi}{6}$. (See Example 1.2). Verify directly by taking their derivatives that $T_3(x)$ and $f(x)$ satisfy the four conditions

$$T_3\left(\frac{5\pi}{6}\right) = f\left(\frac{5\pi}{6}\right), T_3'\left(\frac{5\pi}{6}\right) = f'\left(\frac{5\pi}{6}\right), T_3''\left(\frac{5\pi}{6}\right) = f''\left(\frac{5\pi}{6}\right), T_3'''\left(\frac{5\pi}{6}\right) = f'''\left(\frac{5\pi}{6}\right).$$
- Let $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$ be the Taylor polynomial of degree three for the function $f(x)$, centered at the point $x = a$.
 - Find $T_3'(x)$, $T_3''(x)$, and $T_3'''(x)$.
 - Evaluate $T_3'(x)$, $T_3''(x)$, and $T_3'''(x)$ at $x = a$ to verify that

$$T_3(a) = f(a), \quad T_3'(a) = f'(a), \quad T_3''(a) = f''(a), \quad T_3'''(a) = f'''(a).$$
- The function $f(x)$ is approximated near $x = 0$ by the second-degree Taylor polynomial $T_2(x) = 5 - 7x + 8x^2$. Find the value of a) $f(0)$, b) $f'(0)$, c) $f''(0)$, and d) $f'''(0)$.
- Suppose g is a function which has continuous derivatives, and that $g(0) = 3$, $g'(0) = 2$, $g''(0) = 1$, and $g'''(0) = -3$.
 - What is the Taylor polynomial of degree 2 for g , centered at $x = 0$?
 - What is the Taylor polynomial of degree 3 for g , centered at $x = 0$?
 - Use $T_2(x)$, and $T_3(x)$ to approximate $g(0.1)$.
- For the function $f(x) = \ln x$:
 - List the first four derivatives of $f(x)$.
 - What are the values of the first four derivatives of $f(x)$ evaluated at $x = 1$?
 - Write $T_4(x)$ as a polynomial in the "long form": $A + B(x-a) + C(x-a)^2 + \dots + E(x-a)^4$.
 - Optional Challenge question: Write $T_4(x)$ using summation notation.
 - Graph $f(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$, and $T_4(x)$ on the same graph, using the window: $-0.5 < x < 2$, and $-5 < y < 5$.
- For the function $f(x) = 2 + 3x + 4x^2 + 5x^3$:
 - Find $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, and $T_5(x)$, centered at $x = 0$.
 - What is the significance of $T_0(x)$?
 - What is $T_{103}(x)$?

8. For the function $f(x) = \ln x$:
- Find the equation of the line tangent to $f(x)$ at $x = 1$.
 - Find a function $T_1(x)$ that has the following two properties $T_1(1) = f(1)$ and $T_1'(1) = f'(1)$.
 - Graph $f(x)$ and $T_1(x)$ on the same graph.
9. For the function $f(x) = \ln x$:
- Find a function $T_2(x)$ that has the following three properties

$$T_2(1) = f(1), \quad T_2'(1) = f'(1) \quad \text{and} \quad T_2''(1) = f''(1)$$
 - Graph $f(x)$ and $T_2(x)$ on the same graph.
 - What is the COMPLETE special name given to $T_2(x)$?
 - What is the value of each of the following:
 $f(1), T_2(1),$
 $f(2), T_2(2),$
 and, finally, $f(0)$, and $T_2(0)$.
- 10.a) Find the Taylor polynomial of degree 4 for the function $f(x) = \ln x$, centered at $x = 2$.
 b) What is the difference between $T_4(2.2)$ and $f(2.2)$? (i.e. What is the “error?”)
- 11.a) Find the parabola that best approximates the unit circle $x^2 + y^2 = 1$ near the point $(0, 1)$.
 b) Use your answer to part (a) to estimate the y-coordinate of the point on the upper half of the unit circle with the x-coordinate equal to 0.1.
- 12.a) Find the Taylor polynomial of degree 4, centered at $x = 0$ for the function $f(x) = e^{x^2}$.
 b) Compare this result to the Taylor polynomial of degree 2 for the function $f(x) = e^x$, centered at $x = 0$. What do you notice?
 c) Use your observation in part b) to write out the Taylor polynomial of degree 10 for $f(x) = e^{x^2}$.
 d) What is the Taylor polynomial of degree 5 for the function $f(x) = e^{-2x}$, centered at $x=0$?
 e) Double-check these on your calculator.
13. Find the Taylor polynomial of the given degree for the given function, centered at the given point. (Leave your answers as sums of powers of $(x - a)$, as in Example 1.2.)
- $f(x) = \ln x$, degree 3, centered at $x = 2$.
 - $f(x) = \sqrt{1+x}$, degree 3, centered at $x = 0$.
 - $f(x) = \sin x$, degree 3, centered at $\pi/3$.
 - $f(x) = \cos x$, degree 4, centered at $x = \pi$.
 - $f(x) = \frac{1}{2-x}$, degree n, centered at $x = 0$.
14. Show how you can use the Taylor polynomial of degree 3 centered at $x = 0$, which says that $\sin x \approx x - \frac{x^3}{3!}$, to explain why $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

1 Exercise Solutions

$$\begin{aligned}
 1. \quad T_2(x) &= 1 + x + \frac{x^2}{2} \Rightarrow T_2(0) = 1 & f(x) &= e^x \Rightarrow f(0) = 1 \\
 T_2'(x) &= 1 + x \Rightarrow T_2'(0) = 1 & f'(x) &= e^x \Rightarrow f'(0) = 1 \\
 T_2''(x) &= 1 \Rightarrow T_2''(0) = 1 & f''(x) &= e^x \Rightarrow f''(0) = 1
 \end{aligned}$$

$$3. (a) \quad T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

$$T_3'(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2}(x-a)^2$$

$$T_3''(x) = f''(a) + f'''(a)(x-a)$$

$$T_3'''(x) = f'''(a)$$

$$(b) \quad T_3(a) = f(a) + f'(a)(a-a) + \frac{f''(a)}{2}(a-a)^2 + \frac{f'''(a)}{6}(a-a)^3 = f(a)$$

$$T_3'(a) = f'(a) + f''(a)(a-a) + \frac{f'''(a)}{2}(a-a)^2 = f'(a)$$

$$T_3''(a) = f''(a) + f'''(a)(a-a) = f''(a)$$

$$T_3'''(a) = f'''(a)$$

$$\begin{aligned}
 5. \quad (a) \quad T_2(x) &= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 \\
 &= 3 + 2x + \frac{1}{2}x^2
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad T_3(x) &= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\
 &= 3 + 2x + \frac{1}{2}x^2 - \frac{1}{2}x^3
 \end{aligned}$$

$$(c) \quad g(0.1) \approx T_2(0.1) = 3.205$$

$$(d) \quad g(0.1) \approx T_3(0.1) = 3.2045$$

7. (a) $T_0(x) = f(0) = 2$

$$T_1(x) = f(0) + \frac{f'(0)}{1!}(x-0) = 2 + 3x$$

$$T_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 2 + 3x + 4x^2$$

$$T_3(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = 2 + 3x + 4x^2 + 5x^3$$

$$T_4(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 =$$

$$= 2 + 3x + 4x^2 + 5x^3 + 0x^4 = 2 + 3x + 4x^2 + 5x^3$$

$$T_5(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5 =$$

$$= 2 + 3x + 4x^2 + 5x^3 + 0x^4 + 0x^5 = 2 + 3x + 4x^2 + 5x^3$$

(b) T_0 is the constant term in $f(x)$; the y-intercept in the graph.

(c) $T_{103}(x) = f(x) = 2 + 3x + 4x^2 + 5x^3$

9. (a)

$$f(x) = \ln x,$$

$$f(1) = 0,$$

$$f'(x) = \frac{1}{x}.$$

$$f'(1) = 1,$$

$$f''(x) = -\frac{1}{x^2},$$

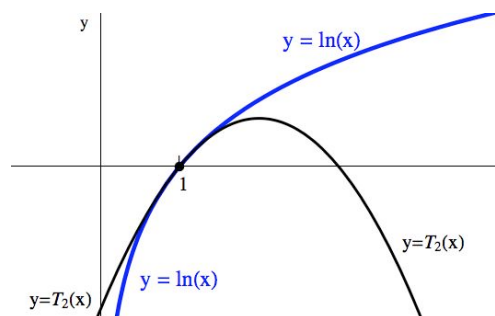
$$f''(1) = -1,$$

So,

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$= 0 + 1(x-1) + \frac{-1}{2}(x-1)^2$$

$$= x - 1 + \frac{-1}{2}(x-1)^2.$$



(c) The Taylor Polynomial of degree 2, centered at $x = 1$.

$$f(1) = 0$$

$$T_2(1) = 0$$

$$f(2) = \ln(2) \approx 0.6931$$

$$T_2(2) = 0.5$$

$$f(0) = \text{undefined}$$

$$T_2(0) = -3/2$$

11. (a) $y = f(x) = \sqrt{1-x^2}$ and a parabola is second degree, so we are looking for $T_2(x)$, centered at $x = 0$.

$$f(x) = \sqrt{1-x^2} \qquad f(0) = 1$$

$$f'(x) = \frac{-x}{\sqrt{1-x^2}} \qquad f'(0) = 0$$

$$f''(x) = -\frac{x^2}{(1-x^2)^{3/2}} - \frac{1}{\sqrt{1-x^2}} \qquad f''(0) = -1$$

$$\begin{aligned} T_2(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 \\ &= 1 + 0 + \frac{-1}{2}(x-0)^2 = 1 - \frac{x^2}{2}. \end{aligned}$$

$$(b) \quad T_2(0.1) = 1 - \frac{(0.1)^2}{2} = 0.995$$

$$13.(a) \quad T_3(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3$$

$$(b) \quad T_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

$$(c) \quad T_3(x) = \frac{1}{2}\sqrt{3} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3$$

$$(d) \quad T_4(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4$$

$$(e) \quad T_n(x) = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots + \frac{1}{2^{n+1}}x^n = \sum_{k=0}^n \frac{1}{2^{k+1}}x^k$$

2 Taylor's Inequality

In this section, we estimate the “remainder term”

$$R_n(x) = f(x) - T_n(x),$$

which is the difference between $f(x)$ and the Taylor polynomial $T_n(x)$ for $f(x)$, centered at $x = a$.

The size of $R_n(x)$ tells us how good an approximation $T_n(x)$ is to $f(x)$: the smaller $R_n(x)$ is, the closer $T_n(x)$ is to $f(x)$. To estimate the remainder term $R_n(x)$, we need a formula for $R_n(x)$. The formula we need is not provided here, but a consequence of this formula is

Taylor's Inequality.

Suppose that $f(x)$ has $n + 1$ continuous derivatives on an interval containing both a and x , and that $|f^{(n+1)}(t)| \leq M_{n+1}$ for all values of t between a and x . Then

$$|R_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |x - a|^{n+1},$$

where $R_n(x) = f(x) - T_n(x)$ is the remainder term and $T_n(x)$ is the Taylor polynomial of degree n for $f(x)$, centered at $x = a$.

The form of this upper bound for $|R_n(x)|$ is easy to remember; if we were to add one more term to $T_n(x)$ to get $T_{n+1}(x)$ (both centered at $x = a$), this last term would be

$$\frac{f^{(n+1)}(a)}{(n+1)!} |x - a|^{n+1},$$

if we were to take absolute values and replace $f^{(n+1)}(a)$ by M_{n+1} , we would get the expression for the upper bound for $|R_n(x)|$.

Caution: This mnemonic device for remembering the form of the upper bound for $|R_n(x)|$

does NOT mean that $R_n(x)$ equals $\frac{f^{(n+1)}(a)}{(n+1)!} (x - a)^{n+1}$. This exact value of $R_n(x)$ is

$$\frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1} \text{ for some value of } c \text{ between } a \text{ and } x.$$

Notice that when x is very close to a , $|x - a|$ is very small, so higher powers of $|x - a|$ are much smaller than lower powers. For example, when $|x - a| = 0.01$, then $|x - a|^2 = 0.0001$ and $|x - a|^3 = 0.000001$. The [absolute value of the] remainder term is bounded by a constant times $|x - a|^{n+1}$, which is one higher power of $|x - a|$ than the degree n . Among all polynomials of degree n , the Taylor polynomial $T_n(x)$ is the ONLY one that has this property; any other polynomial differs from $f(x)$ (for x very close to a) by at least a constant times $|x - a|^n$, which is bigger than a constant times $|x - a|^{n+1}$. This explains why we call $T_n(x)$ the polynomial of degree n that best fits $f(x)$ near $x = a$.

Example 2.1

The line tangent to the curve $y = \sin x$ at $(0, 0)$ is the line $y = x$. Use the Remainder Theorem to find an upper bound for $|R_1(0.1)|$.

Solution: The line $y = x$ is the Taylor polynomial $T_n(x)$ of degree one for the function $f(x) = \sin x$, centered at $x = 0$. Because $n = 1$, the difference between the curve and its tangent line is, $R_1(x) = f(x) - T_1(x)$, and Taylor's inequality takes the form

$$|\sin x - x| = |f(x) - T_1(x)| = |R_1(x)| \leq \frac{M_2}{2!} |x - a|^2.$$

In the above formula we need the value of M_2 , which is an upper bound of $|f''(t)|$ for t -values between $a = 0$ and $x = 0.1$. Since $f''(x) = -\sin x$, we have $|f''(t)| \leq 1$ for all t , so we can simply take $M_2 = 1$. This is a crude value for M_2 !

Substituting $M_2 = 1$, $a = 0$ and $x = 0.1$, we determine an upper bound for the difference between the curve and tangent line at $x = 0.1$:

$$|\sin 0.1 - 0.1| = |f(0.1) - T_1(0.1)| = |R_1(0.1)| \leq \frac{1}{2!} |0.1 - 0|^2 = 0.005.$$

In the above example we found an upper bound on the error at exactly one value of x ($x = 0.1$). In fact, this is an upper bound on the error for all x -values in the interval $-0.1 \leq x \leq 0.1$. This is true because of the manner in which M_2 is determined, along with the fact that for all x -values in this interval $|x - 0|^2 \leq |0.1 - 0|^2$. We explore this idea further in the next example.

Example 2.2

Give an upper bound formula for the difference between the function $f(x) = e^x$ and its Taylor polynomial $T_2(x) = 1 + x + \frac{x^2}{2}$ (centered at $x = 0$) for $0 \leq x \leq 0.1$ (see Figure 2).

Solution: Here, $a = 0$ and $x > a$ and $f'''(x) = e^x$ (which is positive and increasing for all x), so the maximum value of $|f'''(x)|$ on the interval $0 \leq x \leq 0.1$ is taken on at the right endpoint:

$$M_3 = \max_{0 \leq x \leq 0.1} |f'''(x)| = e^{0.1}.$$

Using the upper bound formula for $|R_n(x)|$ given above, we obtain for $0 \leq x \leq 0.1$

$$|f(x) - T_2(x)| = |R_2(x)| \leq \frac{M_3}{3!} |x - 0|^3 = \frac{e^{0.1} \cdot |x|^3}{6} \leq \frac{e^{0.1} \cdot 0.001}{6} < 0.0001842.$$

2 Exercises

1. a) Determine a better value for M_2 in Example 2.1.
b) What is the new error estimate?
2. For the Taylor polynomial of degree 4 for $f(x) = \ln x$, centered at $x = 2$:
 - a) What is the appropriate value of n to use in the formula $|R_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |x - a|^{n+1}$ to find the upper bound on the error one makes when using $T_4(2.2)$ to estimate $f(2.2)$?
 - b) What is the $(n + 1)^{\text{st}}$ derivative in this situation?
 - c) Is the $(n + 1)^{\text{st}}$ derivative an increasing or decreasing function?
 - d) Over what interval must the graph of the $(n + 1)^{\text{st}}$ derivative be examined in order to ascertain a value for M_{n+1} ?
 - e) Do you expect to find the “best” (i.e., the smallest) value for M_{n+1} on the left end, on the right or somewhere in the middle of the interval? (Recall that M_{n+1} must be at least as large as every value of the $(n + 1)^{\text{st}}$ derivative in the interval.)
3. Let $f(x) = \sqrt{1+x}$.
 - a) Find $|f^{(4)}(x)|$ and show that the function $g(x) = |f^{(4)}(x)|$ is decreasing for $x \geq 0$; conclude that

$$\max_{x \geq 0} |f^{(4)}(x)| = |f^{(4)}(0)|.$$
 - b) Use Taylor’s Inequality to give an upper bound for the absolute value of the difference $f(x) - T_3(x)$ between the function $f(x) = \sqrt{1+x}$ and its Taylor polynomial $T_3(x)$ of degree 3, centered at $x = 0$, for $0 \leq x \leq 0.1$.
 - c) You found $T_3(x)$ in Ex. 1.1#13b. Use your calculator to find both $f(0.1) = \sqrt{1.1}$ and $T_3(0.1)$ to as many decimal digits as your calculator shows. How does $|f(0.1) - T_3(0.1)|$ compare with your answer to part (b)?
4. For the Taylor polynomial of degree 4, centered at $x = 0$ for $f(x) = \sin x$:
 - a) According to Taylor’s Inequality, what is the largest possible difference between the Taylor polynomial and the function when estimating $f(0.75)$?
 - b) Would you expect $|R_4(0.75)|$ to be larger, smaller or the same if we instead centered our Taylor polynomial at $\pi/4$? Why?
 - c) What is the upper bound on the error when using the Taylor polynomial of degree 4, centered at $x = \pi/4$ to estimate $f(0.75)$?
5. For $g(x) = e^{2x}$
 - a) What is $T_2(x)$, centered at $x = 0$?
 - b) What is the maximum error we would expect in using $T_2(x)$ to estimate $g(0.1)$?
 - c) Is the error estimate larger, smaller or the same as Example 2.2’s error estimate for $T_2(0.1)$, centered at $x = 0$ for $f(x) = e^x$? What is it about the graphs of e^x and e^{2x} that explain this difference (or sameness)?

- d) Calculate the exact difference between $f(0.1)$ and its corresponding $T_2(0.1)$.
- e) Calculate the exact difference between $g(0.1)$ and its corresponding $T_2(0.1)$.
6. How many terms are necessary to guarantee the Taylor polynomial, centered at $x = 0$, is accurate to within 0.05 of the exact value when $x = 0.3$ for each of the following functions:
- a) $f(x) = e^x$ b) $f(x) = \sin x$ c) $f(x) = \cos x$
7. a) Draw a graph of the Taylor polynomial of degree three, centered at $x = 1$ for the function $f(x) = \ln x$ that illustrates the exact error in using $T_3(x)$ to estimate $\ln x$.
 b) What problem do we run into in trying to graph $|R_3(x)|$?
 optional: Discuss the pros and cons of some reasonable way around the problem to show a graph of $|R_3(x)|$.
8. This problem refines Example 2.1. Let $T_1(x)$ and $T_2(x)$ be the Taylor polynomials of degrees one and two for $f(x) = \sin x$, centered at $x = 0$.
- a) Show that $T_1(x) = T_2(x)$. (So in this case, the tangent line is also the Taylor polynomial of degree 2.)
- b) Use Taylor's Inequality with $n = 2$ to give an even better (i.e., smaller) upper bound on $|\sin x - x|$ for $|x| \leq 0.1$ than is given in Example 2.1.
9. This problem continues the previous problem. Throughout this problem, $f(x) = \sin x$, and all Taylor polynomials are centered at $x = 0$.
- a) Show that for n odd, $T_n(x) = T_{n+1}(x)$. (Hint: When n is odd, $f^{(n+1)}(0) = 0$.)
- b) When n is odd, find an upper bound on
- $$|\sin x - T_n(x)| = |\sin x - T_{n+1}(x)| = |R_n(x)| \text{ for } |x| \leq 0.1$$
- by using Taylor's Inequality for $n + 1$.
- c) What is the smallest value of n for which you can guarantee that
- $$|\sin x - T_n(x)| \leq 10^{-10}$$
- whenever $|x| \leq 0.1$.
- (Hint: By part (a), the smallest n will be odd. Use your answer to part (b) to set up an equation [actually, an inequality] for n . Do not try to solve this inequality for n directly because n appears in both a factorial and an exponent. Instead, try guess and check: guess $n = 1$, $n = 3$, etc., and see what your answer to part (b) becomes.)
- d) For the n you found in part (c), use your calculator to find both $\sin(0.1)$ (be sure your calculator is set on radians, not degrees) and $T_n(0.1)$ to as many decimal digits as your calculator shows. **[Remark:** Calculators use approximations similar to Taylor polynomials to compute trigonometric, exponential and logarithmic functions.]

2 Exercise Solutions

1. (a) $M_2 = 0.1$. (b) error estimate = 0.0005.
2. (a) $n = 4$. (b) $f^{(n+1)}(x) = f^{(5)}(x) = \frac{4!}{x^5} = \frac{24}{x^5}$ (c) decreasing (draw a graph)
 (d) M_{n+1} must be examined over the interval from a to x ; in this case $[2, 2.2]$
 (e) A graph shows that $f^{(n+1)}(x) = f^{(5)}(x) = \frac{24}{x^5}$ is largest on the left end (as is the case for any decreasing function).
3. (a) A graph shows that the largest value of $|f^{(4)}(x)| = \frac{15}{16(1+x)^{7/2}}$ occurs on the left end ($x = 0$), so its largest value is $15/16$.
 (b) $|R_3(x)| \leq \frac{M_{3+1}}{(3+1)!} |x-a|^{3+1} = \frac{15/16}{4!} |0.1-0|^4 = 3.90625 \times 10^{-6}$.
 (c) $\sqrt{1.1}_{calculator} \approx 1.048808848$. $T_3(0.1) = 1.04881250$, so $|T_3(0.1) - f(0.1)| = 3.6518299 \times 10^{-6}$, which is less than $|R_3(x)|$ calculated in part (b).
4. (a) The largest value we expect for any derivative of $\sin x$ is 1, so we choose $|M_{n+1}| = 1$.
 $|R_4(x)| \leq \frac{M_5}{5!} |x-0|^5 = \frac{1}{5!} |0.75-0|^5 = 0.0019775391$.
 (b) We would expect $|R_n(x)|$ to be smaller because 0.75 is closer to $\pi/4$ than to zero, while nothing else changes.
 (c) $|R_4(x)| \leq \frac{M_5}{5!} \left|x - \frac{\pi}{4}\right|^5 = \frac{1}{5!} \left|0.75 - \frac{\pi}{4}\right|^5 = 4.7 \times 10^{-10}$.
6. (a) $|R_n(0.3)| \leq \frac{M_{n+1}}{(n+1)!} |0.3-0|^{n+1} = \frac{e^{0.3}}{(n+1)!} |0.3|^{n+1} \leq 0.05 \Rightarrow n = 2$. NOTE: M is chosen at the right end since all derivatives are increasing functions.
 (b) $|R_n(0.3)| \leq \frac{M_{n+1}}{(n+1)!} |0.3-0|^{n+1} = \frac{1}{(n+1)!} |0.3|^{n+1} \leq 0.05 \Rightarrow n = 1$. The first-degree Taylor polynomial is already within 0.05 of any value $x < 0.3$ in $\sin x$!
 (c) Calculations are the same as in part (b).
- 8(a) $f(x) = \sin x$ $f(0) = 0$
 $f'(x) = \cos x$ $f'(0) = 1$
 $f''(x) = -\sin x$ $f''(0) = 0$
- $T_1(x) = f(0) + f'(0)(x-0) = x$
 $T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 = x$
- (b) $|R_2(x)| \leq \frac{M_3}{3!} |x-0|^3 = \frac{1}{3!} |0.1|^3 = 1.6 \times 10^{-4}$.

3 Infinite Series

Introduction

We are all familiar with the fact that the fraction $\frac{1}{3}$ is equivalent to the *infinite repeating decimal* .33333333333... .

$$\text{So we could write: } \frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots \quad [1]$$

The infinite addition of fractions on the right hand side of [1] is an example of what we call an “infinite series”. Here is the formal definition of an infinite series:

Definition: An ***infinite series*** is an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \dots \quad [2]$$

where $a_1, a_2, a_3, a_4, \dots$ is an infinite sequence of real numbers.

Using summation notation we can express the infinite series [2] as: $\sum_{k=1}^{\infty} a_k$.

Notation note: Sometimes it will be convenient to start the index k at the value 0, in which case the infinite series will be expressed as $\sum_{k=0}^{\infty} a_k$.

So far, all that we have done is given a name to what looks like the sum of an infinite number of terms, however we still need to make clear what exactly it means to find such a sum.

How do we add infinitely many terms?

Does every infinite series have a sum?

We consider three examples on the next page. After considering these examples we will make some formal definitions and answer the questions above.

Series #1. Consider the infinite series: $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

We begin by looking at the **partial sums** for this series.

$$\begin{aligned} 1 \\ 1 + \frac{1}{2} &= 1\frac{1}{2} \\ 1 + \frac{1}{2} + \frac{1}{4} &= 1\frac{3}{4} \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= 1\frac{7}{8} \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= 1\frac{15}{16} \end{aligned}$$

As you can see, the partial sums seem to be approaching the number 2. [Later in this section we will prove this rigorously, but for now, we can rely on our intuition]. In a situation like this where the partial sums are approaching a definite number, we say that the series is **converging** to the finite sum 2. We also say that the series is **convergent**.

Series # 2. It is easy to come up with a series that is not convergent. For example, consider the infinite series which adds the positive integers: $\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \dots$. Clearly, there is no finite sum for this series because as we add the terms, the partial sums:

$$\begin{aligned} 1 \\ 1 + 2 &= 3 \\ 1 + 2 + 3 &= 6 \\ 1 + 2 + 3 + 4 &= 10 \end{aligned}$$

are increasing without bound. When an infinite series does not have a finite sum we say it is **divergent**.

Series # 3. Finally let's consider the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$. This series is called the

Harmonic Series. Looking at the individual terms that we are adding we might guess that this series also converges. However, it turns out that the partial sums for this series are in fact unbounded. So this series does not have a sum. It is divergent. [See Exercise 7 where we prove that this series diverges].

The harmonic series is important because it represents a simple example of a series whose individual terms are approaching 0, but which nevertheless does not have a sum. This may seem counter intuitive but we have to remember that we are adding an *infinite number* of terms. So even if they are very small they can "add up." In the case of the harmonic series, if we add enough terms we can get the partial sums to exceed any finite number!

Before continuing let's summarize some of the terminology that we have developed so far.

Definition: For any infinite series $\sum_{k=1}^{\infty} a_k$, we define the **partial sums** of the series as:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\text{etc.} \end{aligned}$$

So in general the **n th partial sum** is $s_n = \sum_{k=1}^n a_k$. [Similarly, $s_n = \sum_{k=0}^n a_k$ for a series that starts with the index $k=0$. In this case the partial sums are $s_0 = a_0$, $s_1 = a_0 + a_1$, $s_2 = a_0 + a_1 + a_2$, and so on].

We are now ready to state how we can determine whether a series has a sum.

Definition: If the partial sums for a series converge to a finite number, which we will denote by **S** , we say that the series is **convergent** and write: $\sum_{k=1}^{\infty} a_k = S$. On the other hand, if the partial sums do not converge to any specific number, we say that the infinite series is **divergent** and therefore does not have a sum.

Using limit notation we could summarize the relationships stated in the definition above as follows.

$$\text{If } \lim_{n \rightarrow \infty} s_n = S \text{ then } \sum_{k=1}^{\infty} a_k = S$$

In words, “if the partial sums converge to a number **S** , then **S** is the sum of the series.”

[A thorough study of infinite series is beyond the scope of these materials. If you are interested in reading more on the general topic of infinite series see Sections 8.1 – 8.4 in *Calculus and Concepts* by Stewart.]

The remainder of this section will focus on one particular class of infinite series, the so-called “geometric series”.

Geometric Series

A geometric series is an infinite series that has the following special form:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots \quad (a \neq 0) \quad [3]$$

Here ***a*** is called the starting value, and ***r*** is the constant multiplier. Note that each term is obtained by multiplying the preceding one by ***r***.

The Sum of a Geometric Series

It can be readily proved (see Appendix A on page 23) that the geometric series [3] will converge to a finite sum provided $-1 < r < 1$, and that the sum is exactly equal to the

fraction $\frac{a}{1-r}$. On the other hand, if either $r \leq -1$ or $r \geq 1$, then the geometric series diverges.

So we will write:

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad \text{if } |r| < 1.$$

Note that $|r| < 1$ is just another way of writing $-1 < r < 1$. Caution: the formula for the sum

$\frac{a}{1-r}$ only works if $|r| < 1$. If $|r| \geq 1$ then the formula does not apply since there is no sum!

Before moving on we point out that the geometric series is very special. Most series do not have a simple sum formula like this. For general infinite series, even when we know that a sum exists, it is often difficult or impossible to find a closed form expression for the sum.

Example 3.1

Special case: $a = 1$. The geometric series when $a = 1$ is particularly simple in structure.

For $a = 1$, we have $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \dots$

The corresponding sum formula is $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ if $|r| < 1$.

If we use x in place of r then we can write:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \text{ provided } |x| < 1.$$

This is an important result with some interesting implications.

Here is a preview of what is coming up in the next section.

Consider the function $f(x) = \frac{1}{1-x}$. This function is defined for all real numbers except $x = 1$.

But notice that if we confine x to the interval $-1 < x < 1$ we can say that $f(x)$ equals the sum of a convergent infinite series.

That is, we can write: $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ provided $-1 < x < 1$.

This means that the partial sums

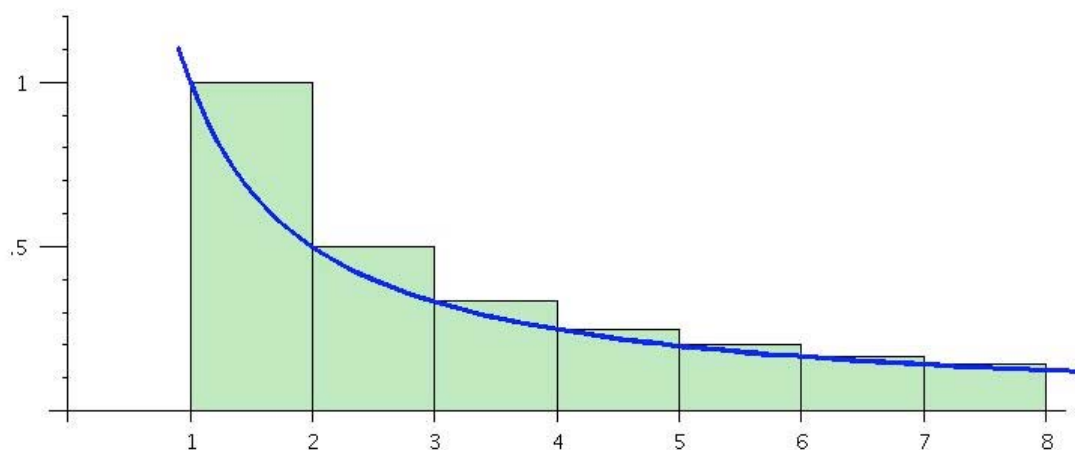
$$\begin{aligned} &1 \\ &1+x \\ &1+x+x^2 \\ &1+x+x^2+x^3 \\ &1+x+x^2+x^3+x^4 \\ &\text{etc.} \end{aligned}$$

converge to the function $y = \frac{1}{1-x}$. But these partial sums are the Taylor polynomials for this

function! As we will see in the next section the infinite series: $1 + x + x^2 + x^3 + \dots$ is therefore called the **Taylor Series** for this function.

3 Exercises

- Consider the infinite series $\sum_{k=0}^{\infty} (-1)^k$. Write down the values for the first four partial sums for this series (i.e. s_0, s_1, s_2, s_3). Find the sum of this series (if it exists). Justify your answer.
- The series $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ is a geometric series. What is its sum?
- Show that the infinite series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots$ is a geometric series by writing it in the form $\sum_{k=0}^{\infty} ar^k$. Calculate the sum.
- Find the sum of the series $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \dots$.
- Express the infinite repeating decimal $0.44444 \dots$ as a single fraction by first writing it as a geometric series.
- Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum:
 - $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$
 - $1 + 0.4 + 0.16 + 0.064 + \dots$
 - $\sum_{k=0}^{\infty} \frac{1}{(\sqrt{2})^k}$
 - $\sum_{k=0}^{\infty} \frac{\pi^k}{3^{k+1}}$ (hint: the first step is to put this into the form $\sum_{k=0}^{\infty} ar^k$)
- The series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ looks like it should converge but in fact diverges. Prove that this series cannot have a finite sum. Hint: Use the figure below to conclude that $s_n > \int_1^{n+1} \frac{1}{x} dx$. Then use this inequality to show that the partial sums cannot converge.



This series is called the **Harmonic Series**. It is the classic example of the fact that an infinite series may diverge even though the individual terms are converging to zero.

3 Exercise Solutions

1.

$$s_0 = (-1)^0 = 1$$

$$s_1 = (-1)^0 + (-1)^1 = 0$$

$$s_2 = (-1)^0 + (-1)^1 + (-1)^2 = 1$$

$$s_3 = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 = 0$$

Since $\lim_{n \rightarrow \infty} s_n$ does not exist, this series does not converge to a value.

3.

$$\begin{aligned} \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots \\ &= \frac{3}{10} \left[1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots \right] \\ &= \sum_{k=0}^{\infty} \frac{3}{10} \left(\frac{1}{10} \right)^k = \frac{3/10}{1 - 1/10} = \frac{1}{3} \end{aligned}$$

5.

$$\begin{aligned} 0.4444\dots &= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10000} + \dots \\ &= \sum_{k=0}^{\infty} \frac{4}{10} \left(\frac{1}{10} \right)^k = \frac{4/10}{1 - 1/10} = \frac{4}{9} \end{aligned}$$

7. In the drawing, the area inside the rectangles represents s_n .

Since each rectangle is above the function $1/x$, for all $x > 1$, the area under $f(x) = 1/x$ is smaller than the area inside the rectangles. (i.e. $s_n > \int_1^{n+1} \frac{1}{x} dx$)

Since we know $\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln(t) \rightarrow +\infty$ is unbounded, we conclude that the larger area (inside the rectangles) must also be unbounded.

Appendix A. Proof of the Sum Formula for a Geometric Series

Consider the geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$ ($a \neq 0$)

Case 1. If $r = 1$ then the series becomes $a + a + a + \dots$ which clearly does not have a finite sum so the series diverges.

Case 2. If $r \neq 1$ then we proceed as follows.

Consider the n th partial sum:

$$s_n = a + ar + ar^2 + ar^3 + \dots + ar^n \quad [\text{equation 1}]$$

Multiply both sides of equation 1 by r :

$$rs_n = ar + ar^2 + ar^3 + \dots + ar^{n+1} \quad [\text{equation 2}]$$

Subtract equation 2 from equation 1 and do a little algebra to get a formula for s_n :

$$s_n - rs_n = a - ar^{n+1}$$

$$(1 - r)s_n = a(1 - r^{n+1})$$

$$s_n = \frac{a}{1 - r}(1 - r^{n+1})$$

Now to find the sum of the geometric series, we must calculate the limit of the partial sums s_n as $n \rightarrow \infty$.

So we need to compute $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a}{1 - r}(1 - r^{n+1}) = \frac{a}{1 - r} \lim_{n \rightarrow \infty} (1 - r^{n+1})$

Now if $|r| < 1$ then $\lim_{n \rightarrow \infty} r^{n+1} = 0$. So, in that case, we find the sum to be $\lim_{n \rightarrow \infty} s_n = \frac{a}{1 - r}$.

On the other hand if $|r| > 1$, then $|r^{n+1}| \rightarrow \infty$ so $\lim_{n \rightarrow \infty} s_n$ does not exist.

Finally, if $r = -1$, then the partial sums s_n will oscillate between 0 and a and hence again $\lim_{n \rightarrow \infty} s_n$ does not exist.

4 Taylor Series

We now consider the limit of the Taylor polynomials $T_n(x)$ for a function $f(x)$, centered at $x = 0$, as the degree n goes to infinity.

Example 4.0

In Example 1.3, we showed that the Taylor polynomial $T_n(x)$ (centered at $x = 0$) for the function $f(x) = \frac{1}{1-x}$ is $T_n(x) = 1 + x + x^2 + x^3 + \dots + x^n$. For what values of x is it true that $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$?

Solution: For each fixed value of x ,

$$T_n(x) = 1 + x + x^2 + x^3 + \dots + x^n = \sum_{k=0}^n x^k$$

is the n th partial sum of the infinite geometric series

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots = \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n x^k \right).$$

By Example 3.1, this series does not converge if $|x| \geq 1$ and it converges to $\frac{1}{1-x}$ if $|x| < 1$.

We conclude that $T_n(x) \rightarrow f(x) = \frac{1}{1-x}$ as $n \rightarrow \infty$ if $|x| < 1$.

The infinite series

$$\lim_{n \rightarrow \infty} (T_n(x)) = \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n x^k \right) = 1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k$$

is called the Taylor series for $f(x) = \frac{1}{1-x}$, centered at $x = 0$.

Taylor Series

The Taylor series for a function $f(x)$, centered at $x = a$, is the infinite series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a)^1 + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Remark on centering at $x = 0$: In these notes, we will be looking mostly at Taylor series centered at $x = 0$. When we talk about the Taylor series for a function and do not say explicitly where it is centered, we will assume that it is centered at $x = 0$.

Example 4.0 shows that the Taylor series for $f(x) = \frac{1}{1-x}$ actually converges to $f(x)$ for $|x| < 1$:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for} \quad |x| < 1.$$

The Taylor series in Example 4.0 has a very special property. As we saw in Appendix A, there is a simple closed-form expression for $T_n(x)$,

$$T_n(x) = \frac{1 - x^{n+1}}{1 - x},$$

Which enables us to evaluate the limit $\lim_{n \rightarrow \infty} T_n(x)$ for $|x| < 1$ without difficulty.

For almost all other Taylor series, we cannot come up with a closed-form expression for $T_n(x)$ that is simple enough to enable us to evaluate the limit $\lim_{n \rightarrow \infty} T_n(x)$ easily. Instead, we can often use Taylor's Inequality to show that $T_n(x) \rightarrow f(x)$. Examples 4.2, 4.3, and 4.4 will illustrate how to do this.

We start with a useful limit.

Example 4.1

Show that for any fixed real number x , $\frac{x^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$.

Solution: Choose an integer $N \geq 2|x|$. Then $\frac{|x|}{N} \leq \frac{1}{2}$, so for $n \geq N$ (with both x and N being held fixed),

$$\begin{aligned} \frac{x^n}{n!} &= \frac{|x|^N}{N!} \cdot \frac{|x|}{N+1} \cdot \frac{|x|}{N+2} \cdot \frac{|x|}{N+3} \cdots \frac{|x|}{n} \\ &\leq \frac{|x|^N}{N!} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \frac{|x|^N}{N!} \cdot \left(\frac{1}{2}\right)^{n-N} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \end{aligned}$$

Example 4.2

Find the Taylor series for $f(x) = \sin x$ and show that it converges to $f(x) = \sin x$ for all x .

Solution:

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0, \\ f'(x) &= \cos x, & f'(0) &= 1, \\ f''(x) &= -\sin x, & f''(0) &= 0, \\ f'''(x) &= -\cos x, & f'''(0) &= -1, \\ f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0, \end{aligned}$$

and the values of $f^{(k)}(0)$ for $k = 0, 1, 2, 3, \dots$ repeat in cycles of length four: 0, 1, 0, -1, 0, 1, 0, -1, etc. The Taylor series for $f(x) = \sin x$ is

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + 0 + \frac{x^9}{9!} - \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}. \end{aligned}$$

For each fixed x , we want to show that this Taylor series converges to $f(x) = \sin x$. This is equivalent to saying that $T_n(x)$ (which is the n th partial sum of the Taylor series) converges to $f(x)$, which is equivalent to showing that $|f(x) - T_n(x)| = |R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$. Since the derivatives of $\sin x$ are either $\pm \cos x$ or $\pm \sin x$,

$$|f^{(n+1)}(x)| \leq 1$$

for all n and all x . So we can take $M_{n+1} = 1$ in Taylor's Inequality for all n and all x . Using the upper bound for $|R_n(x)|$ given in Taylor's Inequality, we obtain

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M_n}{(n+1)!} |x - 0|^{n+1} = \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0$$

as $n \rightarrow \infty$ by Example 4.1. We conclude that $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for each fixed x , so that the Taylor series for $f(x) = \sin x$ actually converges to $f(x) = \sin x$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sin x \quad \text{for all } x.$$

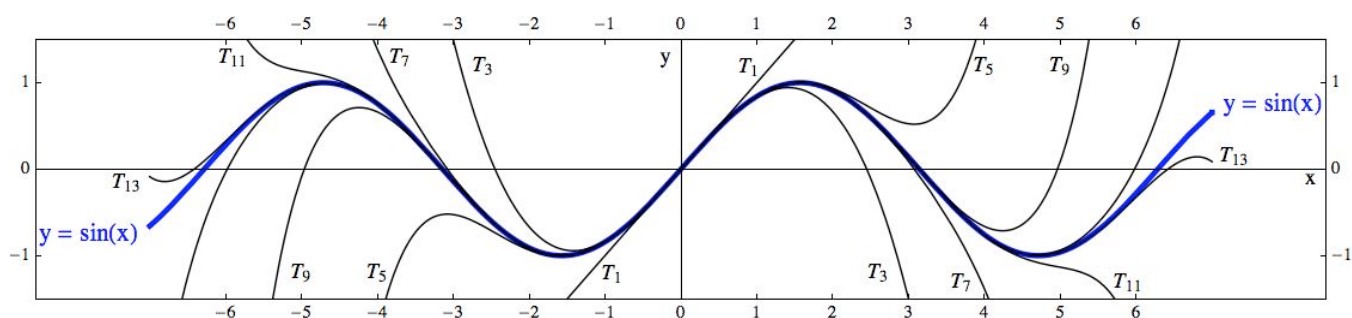


FIGURE 4 The Taylor polynomials $T_1(x)$, $T_3(x)$, $T_5(x)$, $T_7(x)$, $T_9(x)$, $T_{11}(x)$, $T_{13}(x)$, and $T_{15}(x)$ for $f(x) = \sin x$, centered at $x = 0$.

Figure 4 illustrates the convergence of the Taylor series for $f(x)$ (centered at $x = 0$) to $\sin x$. The partial sums of the Taylor series are the Taylor polynomials.

Example 4.3

Find the Taylor series for $f(x) = \cos x$ and show that it converges to $f(x) = \cos x$ for all x .

Solution:

$$\begin{array}{ll}
 f(x) = \cos x, & f(0) = 1, \\
 f'(x) = -\sin x, & f'(0) = 0, \\
 f''(x) = -\cos x, & f''(0) = -1, \\
 f'''(x) = \sin x, & f'''(0) = 0, \\
 f^{(4)}(x) = \cos x, & f^{(4)}(0) = 1,
 \end{array}$$

and the values of $f^{(k)}(0)$ for $k = 0, 1, 2, 3, \dots$ repeat in cycles of length four: 1, 0, -1, 0, 1, 0, -1, 0, etc. The Taylor series for $f(x) = \cos x$ is

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k &= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + 0 + \frac{x^8}{8!} - \dots \\
 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!}.
 \end{aligned}$$

For each fixed x , to show that the Taylor series for $f(x) = \cos x$ converges to $f(x) = \cos x$, we must show that $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$. Since the derivatives of $\cos x$ are either $\pm \cos x$ or $\pm \sin x$, we can take $M_{n+1} = 1$ for all n and all x . Using the upper bound for $|R_n(x)|$ given in Taylor's Inequality, we obtain

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |x - 0|^{n+1} = \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0$$

as $n \rightarrow \infty$ by Example 4.1. We conclude that $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for each fixed x , so the Taylor series for $f(x) = \cos x$ actually converges to $f(x) = \cos x$:

$$\sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \cos x \quad \text{for all } x.$$

The next example is a little more complicated than Examples 4.2 and 4.3. In Examples 4.2 and 4.3, we could use the value $M_{n+1} = 1$ for all n and all x . The value we use for M_{n+1} in Example 4.4 will have to depend on x .

Example 4.4

Find the Taylor series for $f(x) = e^x$ and show that it converges to $f(x) = e^x$ for all x .

Solution: Since $f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$, we have

$f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = e^0 = 1$, so the Taylor polynomial of degree n is

$$\begin{aligned} T_n(x) &= f(0) + f^{(1)}(0)(x-0)^1 + \frac{f^{(2)}(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \dots \\ &\quad + \frac{f^{(n-1)}(0)}{(n-1)!}(x-0)^{n-1} + \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}, \end{aligned}$$

and the Taylor series for $f(x) = e^x$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

The $(n+1)^{\text{st}}$ derivative of $f(x) = e^x$ is again $f^{(n+1)}(x) = e^x$.

For each fixed $x \geq 0$, the maximum value of $|f^{(n+1)}(t)| = e^t$ on the interval $0 \leq t \leq x$ is taken on at the right endpoint: $M_{n+1} = \max_{0 \leq t \leq x} |f^{(n+1)}(t)| = e^x$. Using the upper bound for $|R_n(x)|$ given in Taylor's Inequality, we obtain

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |x-0|^{n+1} = \frac{e^x x^{n+1}}{(n+1)!} \rightarrow 0$$

as $n \rightarrow \infty$ by Example 4.1 (because here, x is held fixed, so e^x is also held fixed as $n \rightarrow \infty$). We conclude that $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for $x \geq 0$.

For each fixed $x < 0$, the maximum value of $|f^{(n+1)}(t)| = e^t$ on the interval $x \leq t \leq 0$ is taken on at the right endpoint: $M_{n+1} = \max_{x \leq t \leq 0} |f^{(n+1)}(t)| = e^0 = 1$. Using the upper bound for $|R_n(x)|$ given in Taylor's Inequality, we obtain

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |x-0|^{n+1} = \frac{x^{n+1}}{(n+1)!} \rightarrow 0$$

as $n \rightarrow \infty$ by Example 4.1. We conclude that $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for $x < 0$, too.

So the Taylor series for $f(x) = e^x$ actually converges to $f(x) = e^x$:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x \quad \text{for all } x.$$

Summary to this Point

Taylor Polynomials

Let f be a function with derivatives of order k , for $k = 1, 2, 3, \dots, N$ throughout some interval containing “ a ” as an interior point. Then for $1 \leq n \leq N$, the Taylor polynomial of order n , generated by f at $x = a$ is

$$T_n(x) = f(a) + f^{(1)}(a)(x-a)^1 + \frac{f^{(2)}(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

In the special case where $a = 0$, the Taylor polynomial is:

$$T_n(x) = f(0) + f^{(1)}(0)x^1 + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k.$$

Taylor Series

Let f be a function with derivatives of all orders throughout some interval containing “ a ” as an interior point.

Then the Taylor series generated by f at $x = a$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f^{(1)}(a)(x-a)^1 + \frac{f^{(2)}(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

In the special case where $a = 0$, the Taylor series is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x^1 + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

The Taylor series centered at $x = 0$ is sometimes also called the **MacLaurin** series.

CAUTION: Just because we can compute the Taylor series for a function using the above formula, does not in itself guarantee that the series actually converges to the function. So, in general, we cannot equate a function with its Taylor series. We have to establish this equality in each case before we can assert it.

That is why we did not write $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ above.

Radius of Convergence

As we have seen in Example 4.0, Taylor series do not always converge for all x . Figure 5

illustrates the convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ to $f(x) = \frac{1}{1-x}$ for $|x| < 1$, and the divergence for $|x| \geq 1$. The partial sums of the Taylor series are the Taylor polynomials.

Figure 5 shows only $T_1(x)$, $T_4(x)$, $T_7(x)$, $T_{10}(x)$, and $T_{13}(x)$ to avoid crowding.

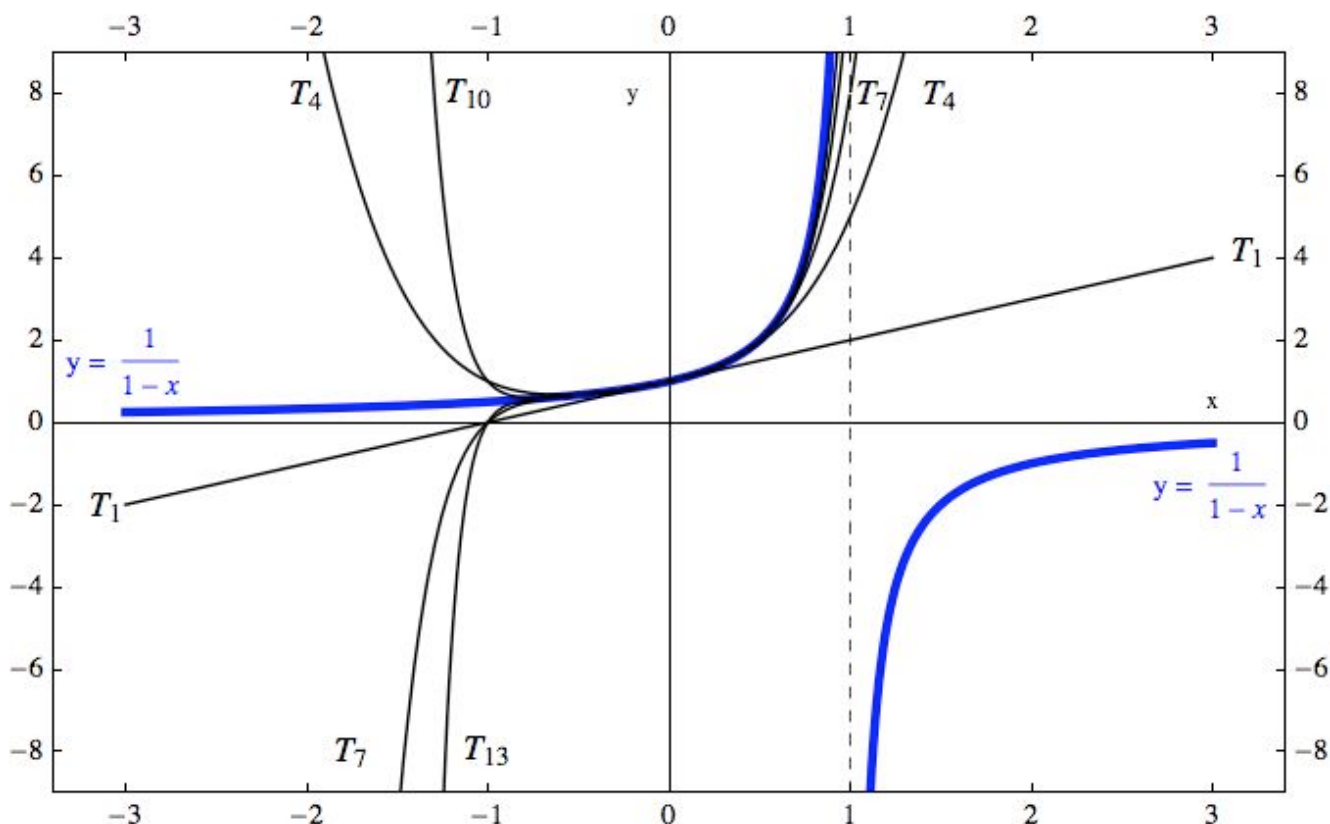


FIGURE 5. The Taylor polynomials $T_1(x)$, $T_4(x)$, $T_7(x)$, $T_{10}(x)$, and $T_{13}(x)$ for

$$f(x) = \frac{1}{1-x}, \text{ centered at } x = 0.$$

The vertical asymptote at $x = 1$ clearly poses a problem for the convergence of this Taylor series for $x \geq 1$. Notice, however, that there is also a problem in convergence for $x \leq -1$, even though $f(x)$ doesn't have a vertical asymptote at $x = -1$.

As we will see below, it is impossible for a Taylor series centered at $x = 0$ to diverge for $x > 1$ and also converge for $x < -1$. So in some sense it is true that the vertical asymptote at $x = 1$ not only causes divergence of this Taylor series for $x > 1$, it also causes divergence of this Taylor series for $x < -1$.

The behavior that we saw in Example 4.0 (the series converges for $|x| < 1$ and diverges for $|x| > 1$) is surprisingly general for all Taylor series:

Fact: Radius of Convergence

For the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ for a function $f(x)$, centered at $x = a$, there are only three possibilities:

- (i) The series converges only at $x = a$. (In this case, we say $R = 0$.)
- (ii) The series converges absolutely for all x . (In this case, we say $R = \infty$.)
- (iii) There is a positive number R for which the series converges absolutely whenever $|x - a| < R$ and diverges whenever $|x - a| > R$.

The number R (which may be 0 or infinity) is called the **radius of convergence** for the Taylor series for $f(x)$, centered at $x = a$.

Remark: Notice that there is no conclusion about convergence when $|x - a| = R$, i.e. when $x = a - R$ or $x = a + R$. It is possible that the series could converge at both, one, or neither of these points. We will not study what happens when $|x - a| = R$ in these notes.

Example 4.5

Example 4.0 shows that the radius of convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ is $R = 1$. Examples 4.2, 4.3, and 4.4 show that the radius of convergence of the Taylor series for each of (a) $\sin x$, (b) $\cos x$, and (c) e^x is $R = \infty$.

4 Exercises

1. Consider the Taylor series for $f(x) = \sin(-x)$.
 - a) Express the series in summation notation.
 - b) Find an expression for $|R_3(x)|$.
 - c) Keeping in mind the proof (Example 4.1) that $\frac{x^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$, show that $\lim_{n \rightarrow \infty} |R_n(x)| = 0$ for all (fixed) values of x .
2. Find the Taylor series for $f(x) = \cos\left(\frac{x}{2}\right)$ and show that it converges to $\cos\left(\frac{x}{2}\right)$ for all values of x .
3. Find the Taylor series for $f(x) = e^{x+2}$ and show that it converges to e^{x+2} for all values of x .
4. For what values of x does the Taylor series for $f(x)$ converge to $f(x) = \frac{1}{1-2x}$?

4 Exercise Solutions

$$\begin{array}{ll}
1(a) & f(x) = \sin(-x) & f(0) = 0 \\
& f'(x) = -\cos(-x) & f'(0) = -1 \\
& f''(x) = -\sin(-x) & f''(0) = 0 \\
& f'''(x) = \cos(-x) & f'''(0) = 1 \\
& \vdots & \vdots
\end{array}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x-0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
&= \frac{f^{(0)}(0)}{0!} x^0 + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots \\
&= 0 - 1x + 0 + \frac{1}{3!} x^3 + 0 + \dots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!}.
\end{aligned}$$

$$(b) \quad |R_3(x)| \leq \frac{M_4}{4!} |x-0|^4 = \frac{1}{4!} |x|^4.$$

$$(c) \quad \lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} \frac{M_{n+1}}{(n+1)!} |x-0|^{n+1} = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0, \text{ since } x \text{ is fixed and per Example 4.1.}$$

$$\begin{array}{ll}
3. & f(x) = e^{x+2} & f(0) = e^2 \\
& f'(x) = e^{x+2} & f'(0) = e^2 \\
& f''(x) = e^{x+2} & f''(0) = e^2 \\
& \vdots & \vdots
\end{array}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k &= \frac{f^{(0)}(0)}{0!} x^0 + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots \\
&= e^2 + e^2 x + \frac{e^2}{2} x^2 + \dots = \sum_{k=0}^{\infty} \frac{e^2 x^k}{k!}.
\end{aligned}$$

$$\lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} \frac{M_{n+1}}{(n+1)!} |x-0|^{n+1} = \lim_{n \rightarrow \infty} \frac{e^2 |x|^{n+1}}{(n+1)!} \rightarrow e^2 \cdot 0 = 0, \text{ for all (fixed) values of } x.$$

Common Taylor Series & Radii of Convergence

Here are a few series worth remembering... [Yes—that means you should memorize them!]

Function	Taylor series	Restrictions	Radius of Convergence
$\frac{1}{1-u}$	$1 + u + u^2 + u^3 + u^4 + \dots$ $= \sum_{k=0}^{\infty} u^k$	$ u < 1$	$R = 1$
$\frac{1}{1+u}$	$1 - u + u^2 - u^3 + u^4 - \dots$ $= \sum_{k=0}^{\infty} (-1)^k u^k$	$ u < 1$	$R = 1$
$\cos u$	$1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \frac{u^8}{8!} - \dots$ $= \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k}}{(2k)!}$	none	$R = \infty$
$\sin u$	$u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \dots$ $= \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+1}}{(2k+1)!}$	none	$R = \infty$
e^u	$1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \dots$ $= \sum_{k=0}^{\infty} \frac{u^k}{k!}$	none	$R = \infty$

5 Operations with Taylor Series

We conclude our discussion of Taylor series by introducing some operations that can be done with Taylor series. The first of these operations is simple substitution.

Example 5.1

Find the Taylor series for $f(x) = \frac{1}{1+4x^2}$, find its radius of convergence, and show that it converges to $f(x) = \frac{1}{1+4x^2}$ whenever it converges.

Solution: By Example 4.0, the Taylor series for $g(u) = \frac{1}{1-u}$ is

$$1 + u + u^2 + u^3 + \dots$$

Since $f(x) = g(-4x^2)$, we substitute $-4x^2$ in for u in the Taylor series for $g(u)$ to obtain the Taylor series for $f(x)$:

$$1 + (-4x^2) + (-4x^2)^2 + (-4x^2)^3 + \dots = \sum_{j=0}^{\infty} (-4x^2)^j = \sum_{j=0}^{\infty} (-4)^j x^{2j}.$$

To find the limit, note that the Taylor series for $f(x) = \frac{1}{1+4x^2}$ is a geometric series with $r = -4x^2$. By Example 3.1, this series is convergent for $|r| < 1$ and divergent for $|r| \geq 1$.

Since $|r| < 1 \Leftrightarrow |-4x^2| < 1 \Leftrightarrow |x^2| < \frac{1}{4} \Leftrightarrow |x| < \frac{1}{2}$,

Example 3.1 implies that

$$\begin{aligned} \sum_{j=0}^{\infty} (-4)^j x^{2j} &= \sum_{j=0}^{\infty} (-4x^2)^j = 1 + (-4x^2) + (-4x^2)^2 + (-4x^2)^3 + \dots \\ &= \frac{1}{x - (-4x^2)} = \frac{1}{x + 4x^2} = f(x) \text{ for } |x| < \frac{1}{2}, \end{aligned}$$

and that this series is divergent for $|x| \geq \frac{1}{2}$. So the radius of convergence is $R = \frac{1}{2}$.

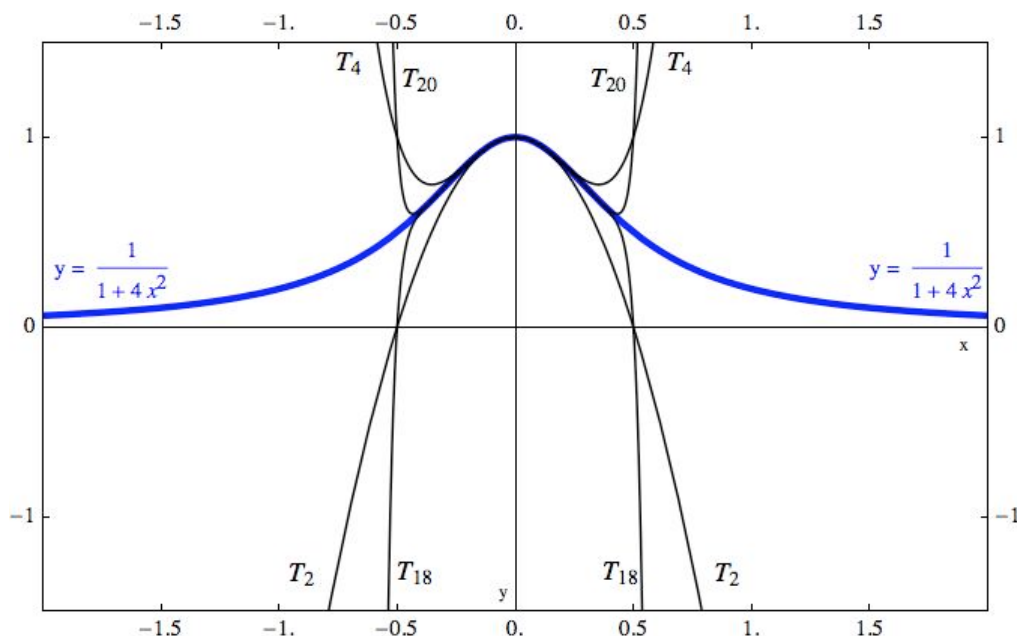


FIGURE 6. The Taylor polynomials $T_2(x)$, $T_4(x)$, $T_{18}(x)$, and $T_{20}(x)$ for

$$f(x) = \frac{1}{1+4x^2}, \text{ centered at } x = 0.$$

Figure 6 illustrates the convergence of the Taylor series for $f(x) = \frac{1}{1+4x^2}$ (centered at $x = 0$)

to $f(x) = \frac{1}{1+4x^2}$ for $|x| < \frac{1}{2}$, and the divergence for $|x| \geq \frac{1}{2}$. The partial sums of the Taylor series are the Taylor polynomials. Figure 6 shows only $T_2(x)$, $T_4(x)$, $T_{18}(x)$, and $T_{20}(x)$ to avoid crowding.

At first glance, the radius of convergence of the function in Figure 6 is mysterious. There are no apparent vertical asymptotes to cause problems, but the radius of convergence is still only

$R = \frac{1}{2}$. It turns out, however, that there are “vertical asymptotes” (called poles) if we allow x to be a complex number; this topic is discussed in detail in more advanced courses (Math 427 at the UW for example).

Term by term differentiation and integration

We can find derivatives and integrals of functions that are represented by Taylor series by differentiating or integrating each term in the Taylor series.

Example 5.2

Use Taylor series to compute $\frac{d}{dx}(e^x)$.

Solution:

$$\begin{aligned}\frac{d}{dx}(e^x) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \\ &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ &= e^x\end{aligned}$$

Example 5.3

Use Taylor series to compute $\frac{d}{dx}(\sin x)$.

Solution:

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \cos x\end{aligned}$$

Example 5.4

Find the Taylor series for $\ln(1+x)$.

Solution: We could compute the Taylor series directly by taking derivatives and evaluating them at $x = 0$, as we did in Examples 4.1, 4.2, 4.3, and 4.4. Alternatively, we

can get the Taylor series for $\frac{1}{1+x}$ by substituting $-x$ in for u in $\frac{1}{1-u} = 1 + u + u^2 + \dots$ to

get the Taylor series $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$. We can then integrate this series term by term to get

$$\begin{aligned}\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - \dots \\ \ln(1+x) &= \int \frac{1}{1+x} dx = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right) + C\end{aligned}$$

To evaluate the constant C , substitute in $x = 0$ to get $C = 0$. We get

$$\ln(1+x) = \int \frac{1}{1+x} dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}.$$

One useful fact is that

- the Taylor series (centered at $x = a$) for a function $f(x)$,
- the Taylor series (centered at $x = a$) for its derivative $f'(x)$ (obtained from term-by-term differentiation of the Taylor series for $f(x)$), and
- the Taylor series (centered at $x = a$) for its indefinite integral $\int f(x) dx$ (obtained from term-by-term integration of the Taylor series for $f(x)$)

all have the same radius of convergence. For example, the Taylor series for $\frac{1}{1+x}$ has radius of convergence $R = 1$, so the Taylor series for $\ln(1+x)$ that we obtained in Example 5.4 also has radius of convergence $R = 1$.

Example 5.5

Find the antiderivative $F(x)$ of $f(x) = e^{-x^2}$ which satisfies $F(0) = 0$.

Solution: Substitute in $-x^2$ for u in the Taylor series

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \dots$$

to get

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{j!}.$$

Integrate term by term (and set the constant $C = 0$ to make $F(0) = 0$) to obtain

$$F(x) = x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} + \dots = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1) \cdot j!}.$$

Example 4.4 shows that the Taylor series for e^u converges to e^u for all u , so (by substituting in $-x^2$ for u as in Example 5.5) the Taylor series for e^{-x^2} that we obtained in Example 5.5 converges to e^{-x^2} for all x , and thus the radius of convergence of the Taylor series for e^{-x^2} is $R = \infty$. We conclude that the radius of convergence of the Taylor series for the function $F(x)$ in Example 5.5 is also $R = \infty$. We could view this series as the formula which defines $F(x)$. The error function $\operatorname{erf}(x)$, used widely in statistics, is given by the formula

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} F(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

5 Exercises

Use the series on the previous page to find the Taylor series [centered at $x = 0$] for the following AND find the new radius of convergence:

Simple substitution...

1. $\cos(2x)$

2. $\frac{1}{1+x^2}$

3. e^{x^2}

4. $\frac{1}{1-6x}$

Multiplying both sides by the same object...

5. $\frac{x^3}{1-x}$

6. $\frac{8x}{x+1}$

7. $x^5 \sin x$

8. $\frac{2}{x-1}$ [Careful!]

Differentiating both sides...

9. $\frac{1}{(1-x)^2}$ $\left[\text{Hint: } \frac{1}{(1-x)^2} = \frac{d}{dx} \left[\frac{1}{1-x} \right] \right]$

10. Find the derivative of $\sin x$, by differentiating the Taylor series for $\sin x$.

11. Find the derivative of $\cos x$, by differentiating the Taylor series for $\cos x$.

12. Find the derivative of e^x , by differentiating the Taylor series for e^x .

Integrating both sides...

13. Using Taylor series, find the antiderivative, $F(x)$, of $\sin x$, which satisfies $F(0) = 0$.

14. Using Taylor series, find the antiderivative, $F(x)$, of $\cos x$, which satisfies $F(0) = 1$.

15. Using Taylor series, find the antiderivative, $F(x)$, of e^x , which satisfies $F(0) = 1$.

Mixing them up...

16. $\frac{1}{1+4x^2}$

17. $\ln(1+x)$

18. $\frac{1}{4+x^2}$

19. $\arctan x$

20. Find a Taylor Series that represents the antiderivative, $F(x)$, of e^{-x^2} satisfying $F(0) = 0$.

[Important question: Which integration techniques would be used to help us find $\int e^{-x^2} dx$?]

21. Let $f(x) = \frac{1}{2-x}$. In problem #13(e) on Exercise 1.1, you found the Taylor polynomial $T_n(x)$ of degree n for this function $f(x)$, centered at $x = 0$; this Taylor polynomial $T_n(x)$ is the n th partial sum of the Taylor series for $f(x)$. In this problem, you will get the Taylor series for $f(x)$ in a different way.

a) Write $f(x)$ as follows:

$$\frac{1}{2-x} = \frac{1}{2\left(1-\left(\frac{x}{2}\right)\right)} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{x}{2}\right)}.$$

Now substitute $x/2$ in for u in $\frac{1}{1-u} = 1 + u + u^2 + \dots$ to find the Taylor series for $\frac{1}{1-\left(\frac{x}{2}\right)}$,

and then multiply by $\frac{1}{2}$ to get the Taylor series for $f(x)$.

b) Find the radius of convergence R for the Taylor series for $f(x) = \frac{1}{2-x}$, and show that

this series converges to $f(x)$ for $|x| < R$ and diverges for $|x| > R$.

(Hint: Example 5.1 is similar.)

Find the Taylor series and radius of convergence for the following functions (include the general term).

22. a) $f(x) = \frac{1}{1+x}$

b) $f(x) = \frac{-1}{(1+x)^2}$

23. a) $f(x) = \ln(1-x)$. Hint: What is $\int \frac{1}{1-x} dx$?

b) $f(x) = \ln(1+x)$

c) $f(x) = \ln(1+x^2)$

d) $f(x) = \ln(1-3x)$

24. a) $f(x) = e^{x+2}$

b) $f(x) = e^{-2x}$

c) $f(x) = e^{x^2}$

Find the Taylor series (ROC not needed).

25. a) $f(x) = \sqrt{1+x}$

b) $f(x) = (1+x)^{3/2}$

c) $f(x) = \frac{1}{\sqrt{1+x}}$

26. Consider the functions $y = e^{-x^2}$ and $y = \frac{1}{(1+x^2)}$.

a) Write the Taylor expansions for the two functions about $x = 0$. What is similar about the two series? What is different?

b) Looking at the series, which function do you predict will be greater over the interval $(-1, 1)$? Graph both and see.

c) Are these functions even or odd? How might you see this by looking at the series expansions?

d) By looking at the coefficients, explain why it is reasonable that the series for $y = e^{-x^2}$

converges for all values of x , but the series for $y = \frac{1}{(1+x^2)}$ converges only on the

interval $(-1, 1)$.

27. The hyperbolic sine function $\sinh(x)$ (pronounced “sinh” like “cinch”) and the hyperbolic cosine function $\cosh(x)$ (pronounce “cosh” like the first syllable in “kosher”) are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

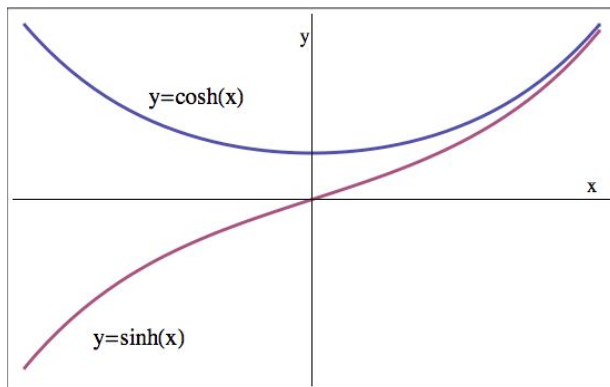
- Find the Taylor series for $f(x) = \sinh x$ by evaluating $f(0)$, $f'(0)$, $f''(0)$, etc. (See Hint below.)
- Find the Taylor series for $f(x) = \cosh x$ by evaluating $f(0)$, $f'(0)$, $f''(0)$, etc. (See Hint below.)
- Show that the Taylor series for $\cosh x$ converges to $\cosh x$ for $x \geq 0$.
(Hint: Example 4.4 for $x \geq 0$ is similar.)

Hint: Here are some other useful facts about the functions $\sinh x$ and $\cosh x$ (again, you are NOT asked to show these statements as part of this problem.)

i) $\frac{d}{dx}(\sinh x) = \cosh x$

ii) $\frac{d}{dx}(\cosh x) = \sinh x$

- $\cosh x > 0$ for all x
(so $\sinh x$ is always increasing)
- $\sinh x > 0$ for $x > 0$
(so $\cosh x$ is increasing for $x > 0$)
- $\sinh x < 0$ for $x < 0$
(so $\cosh x$ is decreasing for $x < 0$)



Note: The Taylor series for $\cosh x$ also converges to $\cosh x$ for $x < 0$, and the Taylor series for $\sinh x$ converges to $\sinh x$ for all x , but you are not asked to show these statements here.

- Substitute $-x$ in for u in the Taylor series $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \dots$ to get the Taylor series for e^{-x} .
- Find the Taylor series for $\sinh(x)$ by subtracting the Taylor series for e^{-x} from the Taylor series for e^x and then dividing by 2. (Answer same as part (a)?)

28. The hyperbolic sine and cosine are differentiable and satisfy the conditions $\cosh(0) = 1$ and $\sinh(0) = 0$, and

$$\frac{d}{dx}(\cosh x) = \sinh x, \quad \text{and} \quad \frac{d}{dx}(\sinh x) = \cosh x.$$

- Using only this information, find the Taylor polynomial of degree $n = 8$ about $x = 0$ for $f(x) = \cosh x$.
- Estimate the value of $\cosh 1$.
- Take the derivative of part a) to find a Taylor polynomial of degree $n = 7$, centered at $x = 0$ for $g(x) = \sinh x$.

5 Exercise Solutions

$$21 (a) \quad \frac{1}{2-x} = \sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}.$$

(b) By Example 3.1, the Geometric Series $\sum_{k=0}^{\infty} au^k$ converges for $|u| < 1$ and diverges for $|u| \geq 1$.

Since $u = \frac{x}{2}$, $|u| < 1 \Leftrightarrow \left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$, shows that $\sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}$ converges for $|x| < 2$ and diverges for $|x| \geq 2$. The radius of convergence, $R = 2$.

$$22 (a) \quad \text{Since } f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } |x| < 1 \text{ by Example 3.1,}$$

$$\text{then } f(u) = \frac{1}{1+u} = \sum_{k=0}^{\infty} u^k \text{ for } |u| < 1, \text{ where } u = -x.$$

$$\text{So, } f(-x) = \frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k \text{ for } |-x| < 1, \text{ hence } R = 1.$$

$$\begin{aligned} (b) \quad \text{Since } f(x) &= \frac{-1}{(1+x)^2} = \frac{d}{dx} \left(\frac{1}{1+x} \right), \\ &= \frac{d}{dx} \left(\sum_{k=0}^{\infty} (-1)^k \cdot (x)^k \right) \text{ for } |-x| < 1 \\ &= \sum_{k=1}^{\infty} (-1)^k \cdot k(x)^{k-1} \text{ for } |x| < 1. \end{aligned}$$

Note that k begins at 1 instead of zero since the derivative removes the 0th-degree (the constant) term.

24. a) We did this one already in Exercises 4, #3.

$$(b) \quad \text{Since } f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ for all } x \text{ by Example 4.4,}$$

$$\text{then } f(u) = e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} \text{ for all } u, \text{ where } u = -2x.$$

$$\text{So, } f(-2x) = e^{-2x} = \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!} \text{ for all } (-2x) \text{ (i.e. for all } x), \text{ hence } R = \text{infinity.}$$

$$(c) \quad \text{Since } f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ for all } x \text{ by Example 4.4,}$$

$$\text{then } f(u) = e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} \text{ for all } u, \text{ where } u = x^2.$$

$$\text{So, } f(x^2) = e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \text{ for all } (x^2) \text{ (i.e. for all } x), \text{ hence } R = \text{infinity.}$$

26 (a) Since $f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all x by Example 4.4,

then $f(u) = e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}$ for all u , where $u = -x^2$.

So, $f(-x^2) = e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k}$ for all $(-x^2)$ (i.e. for all x), hence $R = \text{infinity}$.

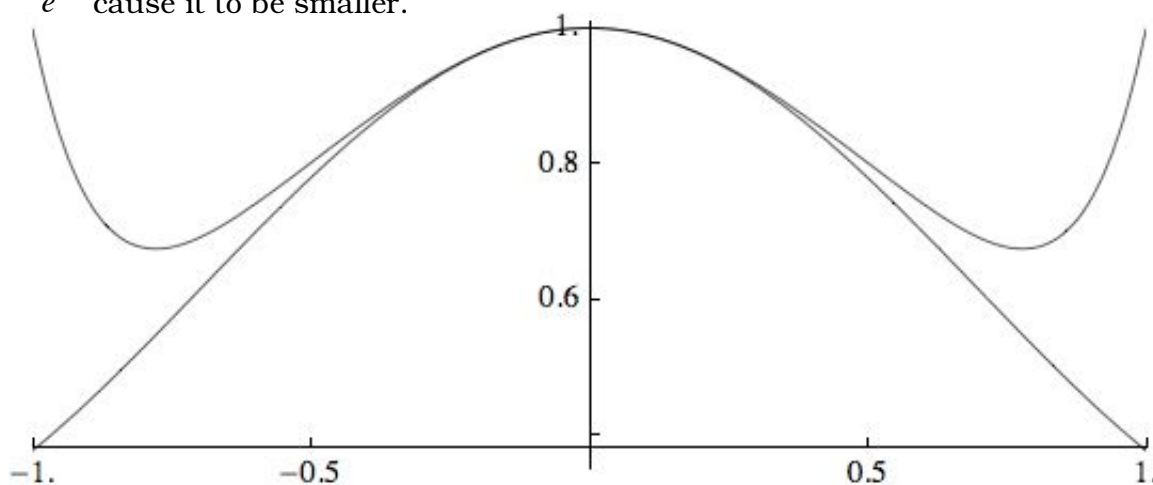
Since $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$ by Example 4.4,

then $f(u) = \frac{1}{1+u} = \sum_{k=0}^{\infty} u^k$ for $|u| < 1$, where $u = -x^2$.

So, $f(-x^2) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$ for $|-x^2| < 1$. Since $|x| < 1$, $R = 1$.

Both series have all even powers of x , and their terms have matching signs. Each terms of one series differ from the other only by a factor of $k!$.

(b) $\frac{1}{(1+x^2)} \geq e^{-x^2}$ over the interval $(-1, 1)$ because the denominators in the Taylor Series for e^{-x^2} cause it to be smaller.



(c) They are both even, as can be seen from the powers of x in their Taylor series.

(d) As we saw in Example 5.5, when we substitute for x in the Taylor series, its radius on convergence can be found with the same substitution, so in the case of $y = e^{-x^2}$, $|-x^2|$ can be any real number. So the radius of convergence is still infinite.

For $y = \frac{1}{(1+x^2)}$, $|-x^2| < 1$ so x is still restricted to being between -1 and 1 . Hence $R = 1$.