WORKSHEET 1, MATH 505

DUE WEDNESDAY, FEBRUARY 10, 2010

1. Frobenius normal form

Throughout this section k will be field. Make a note of one significant different with the Jordan canonical form: k is NOT assumed to be algebraically closed.

Lemma 1.1. *Let $A = k[t]$, and let M be a cyclic torsion A-module (hence, M is finite dimensional as a k vector space). Let $q(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0$ be the minimal polynomial of t considered as a linear operator on M. Prove that M has a basis with respect to which the matrix for t has the following form:*

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \cdots & \cdots & \cdots & \ldots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix} \tag{1}$$

Proof. Exercise. □

Theorem 1.2. *Let M be a finitely generated torsion $k[t]$-module. Prove that there exist non-constant monic polynomials $q_1(x), \ldots, q_m(x)$, determined uniquely, such that $q_1(x) \mid q_2(x) \mid \ldots \mid q_m(x)$, and*

$$M \simeq A/(q_1(x)) \oplus A/(q_2(x)) \oplus \cdots \oplus A/(q_m(x)).$$

Proof. Exercise. □

Definition 1.3. The polynomials $q_1, \ldots, q_m$ are called the **invariant factors** of M.

Definition 1.4. Let V be a finite dimensional vector space over k, and let $\mathcal{L} : V \to V$ be a k-linear operator. Consider a ring homomorphism

$$\phi : k[t] \to \mathrm{End}_k(V)$$

defined by sending t to $\mathcal{L}$ and extending (k)-linearly. Let $\mathrm{Ker}\,\phi$ be the kernel ideal, and let $q_{\mathcal{L}}(t)$ be a monic polynomial in $k[t]$ which generates $\mathrm{Ker}\,\phi$. Then $q_{\mathcal{L}}(t)$ is the **minimal polynomial** of $\mathcal{L}$.

Note that $q_{\mathcal{L}}(t)$ exists since $k[t]$ is a PID and is unique since we assume it is monic.

Theorem 1.5. *Let V be a finite dimensional k-vector space, and let $\mathcal{L} : V \to V$ be a k-linear transformation of V. Let $\chi_{\mathcal{L}}(t)$ be the characteristic polynomial of $\mathcal{L}$.*

(1) *There exist uniquely determined non-constant monic polynomials $q_1(x), \ldots, q_m(x)$ such that*
(2) (a) *$q_i(x) \mid q_{i+1}(x)$ for $1 \le i \le m-1$.*
    (b) *$q_m(x)$ is the minimal polynomial of $\mathcal{L}$.*
    (c) *$\chi_{\mathcal{L}}(t) = \epsilon q_1(x) q_2(x) \ldots q_m(x)$ for $\epsilon \in k$.*
(3) *There exists a basis of V such that the matrix of $\mathcal{L}$ with respect to this basis is a block matrix with m blocks $A_1, \ldots, A_m$, and each block A_i is in the Frobenius normal form for q_i.*

Proof. Exercise. □

2. Constructive approach to the Structure theorem

In this section we describe general strategy on how to find a decomposition of a finitely generated abelian group into a direct sum of cyclic subgroups. This strategy can be formalized to give a different, more constructive, proof of the structure theorem.

General strategy: Let M be a finite dimensional A-module, where A is a PID. Then M fits in a short exact sequence

$$0 \longrightarrow R \xrightarrow{T} F \longrightarrow M \longrightarrow 0$$

where both $F \simeq A^n$ and $R \simeq A^m$ are free modules. The embedding $R \to F$ is given by a matrix T of size $n \times m$ and rank m.

Claim: There exists a change of bases for both R and F, described by the matrices P and Q respectively such that QTP, the matrix of the embedding $R \to F$ in the new bases, is diagonal and has the form

$$\begin{pmatrix} d_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_\ell & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $d_1 \mid d_2 \mid \dots \mid d_\ell$ are the invariant factors of M, and the number of zero rows (if any) is the rank of M. Then,

$$M \simeq A/(d_1) \oplus A/(d_2) \oplus \dots \oplus A/(d_\ell) \oplus A^{n-\ell}$$

You can consult, for example, [Dummit and Foote], §12.1, exercises 16-21, for further explanation.

Problem 2.1. *Suppose A is an abelian group generated by $\{x_1, x_2, x_3\}$ subject to the relations*

$$x_1 + 2x_2 - x_3 = 0$$
$$5x_1 - 3x_2 + 2x_3 = 0$$

Find a decomposition of A as a direct sum of cyclic groups.