Exploring the Spaghetti Problem

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Math 336 Spring 2018

June 4, 2018

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1 Introduction

Suppose a spaghetti of length 1 is dropped on the ground. It randomly breaks at two points. What is the probability the three segments form a triangle? We explore extensions of this question.

2 Questions

- What is the probability a triangle is formed?
- What is the probability an acute triangle is formed?
- Given that a triangle is formed, what is its expected area?
- What is its variance?
- Suppose there are n-1 breaks. What is the probability an n-gon can be formed?
- What is the expected area? What is its limiting value as n goes to infinity?
- What is the expected length of the k-th largest segment?

3 The classical question

Suppose two points are uniformly chosen on an interval. Divide the interval at these points. Then the 3 segments can form a triangle with probability 1/4.

<u>Proof:</u> (triangle)
Consider an equilateral triangle with altitude 1. We can associate breaking the spaghetti with randomly selecting a point p inside the triangle. Draw perpendicular lines from p to each side. Call their lengths a, b, c.

By the altitude theorem, $a + b + c = 1$. The symmetry of the triangle means the lengths are identically distributed. A triangle can be formed if and only if p lies in the shaded middle triangle, because otherwise the longest segment would have length greater than 1/2.

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Proof: (circle) [Lugo]

Cook the spaghetti and connect the ends to make a circle. Break it at three random points. We want to find the probability no arc of length 1/2 can contain all 3 points.

Imagine choosing the breaks in the following manner. Randomly select three pairs of antipodal points. Within each pair, randomly select one point to break at. An arc of length 1/2 can contain all 3 points if and only if they are consecutive. This occurs 6/8 times. Hence the complementary probability is 1/4.

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4 Variations

What is the probability of forming a triangle if we pick the first point uniformly at random, and pick the second point uniformly in (a) the larger interval (b) the right interval?

Claim: The answer to (a) is $2log(2) - 1$.

Proof: The longer length is uniformly distributed on $(\frac{1}{2}, 1)$. Call it l. Consider the second point to take values on $[0, l]$. We want to find the probability it is inside $[l - 1/2, 1/2]$. For fixed l, $p(l) = 1/l - 1$. We multiply by 2 to make this function a probability density. Therefore the desired probability is $P = 2\int_{1/2}^{1} 1/l - 1 dl = 2log(2) - 1$. ■

Claim: The answer to (b) is $log(2) - \frac{1}{2}$.

Proof: If the right interval is smaller, we cannot create a triangle. This occurs with probability 1/2. If the right interval is larger, we have the probability in (a). By the law of total probability, the answer is $P = 1/2 * 0 + 1/2 * (2log(2) - 1) = log(2) - 1/2$. ■

Open questions:

- Keep breaking until 3 segments can form a triangle. Let X be the number of breaks. What is the expected value of X? What is its variance?
- Break the spaghetti into n pieces. What is the probability there exist 3 pieces that can form a triangle? The probability any 3 pieces can form a triangle?

Suppose the spaghetti is broken into n pieces. What is the probability a triangle can be formed by (a) three of the pieces (b) any three of the pieces?

Claim: Suppose the spaghetti is broken into n pieces. Then the probability an n-gon can be formed is

$$1 - \frac{2}{2^{n-1}}$$

.

Proof: [Andrea] The probability space is the regular n-simplex with measure 1. The desired probability is obtained by slicing off the corners of the simplex at the midpoint of each edge. See the picture on page 2 for the 2-dimensional case. Full proof on page 555 of the American Mathematical Monthly, volume 113. ■

Open questions:

- What is the expected area of the n-gon? It's variance?
- What are their limits as n goes to infinity?

Claim: Suppose the stick is broken into n segments. Then the kth largest stick has expected length $\frac{\frac{1}{k}+\frac{1}{k+1}+...+\frac{1}{n}}{n}$.

Proof: [David p. 135] Put the lengths in order: $s_1 > s_2 > ... > s_n$. Set $s_i = x_i + ... + x_n$, where $x_n = s_n$ and $x_k = s_k - s_{k+1}$. Then $x_1 + 2x_2 + ... + nx_n = 1$. Set $y_k = kx_k$. Then $y_1 + ... + y_n = 1$. Each y_k is exchangeable, so $E(y_k) = 1/n$. Thus $E(x_k) = \frac{1}{kn}$. So $E(s_k) = E(x_k) + ... + E(x_n) = \frac{\frac{1}{k}+\frac{1}{k+1}+...+\frac{1}{n}}{n}$, as desired. ■

Open question: What is the variance of s_k?

Claim:

Proof:

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Claim: Consider the original question. Given that a triangle is formed, the expected area is $\pi/105$.

Proof: [Mathai p. 269] Let the spaghetti be $2s$ long for some fixed s. Let x and y be the points and a, b, c be the lengths, from left to right. By Heron's formula, the area is $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-x)(y-s)(s-y+x)}$. A triangle can be formed if and only if $y > s, x > y - s$, and $x < s$. Note that $x, y \in [0, 2s]$. Hence the measure of (x, y) such that a triangle can be formed is $s^2/2$.

Therefore the expected area is

$$E(A) = \frac{2}{2} \int_{y=s}^{2s} \int_{x=y-s}^{s} \sqrt{s(s-x)(y-s)(s-y+x)}dxdy = \frac{4\pi s^2}{105}$$

. Taking $s = 1/2$ gives $E(A) = \pi/105$ when the spaghetti has unit length.

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5 References

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