THIS AIN'T NO MEAGER THEOREM

DYLAN WILSON

1. Preliminaries

1.1. **Introduction.** The explicit purpose of this article is to illustrate the far-reaching strength of the Baire Category Theorem. However, there is, as always, a more subtle purpose: to illustrate the elegant manner in which different areas of mathematics find themselves unreasonably connected with each other, in ways we cannot always fathom. The inspiration for this article came when the author discovered an interesting application of the theorem in the question of convergence of Fourier Series (cf. Theorem 6, section 2). The first section of this article is meant to familiarize the reader with Baire spaces, and present the main theorem. The second section is based largely off of examples and theorems from Dugundji and Folland [(6), (4)], the third is pulled from a paper by Blass and Irwin (2), and the fourth is an outline of a paper by Bagemihl and Seidel (1). Throughout, the author has made additions, small improvements, and slight generalizations of results, but the bulk of the paper remains the work of much more talented mathematicians, and the author would like to thank them for their wonderful work.

At the heart of it, the Baire category theorem is a guarantee about the size of certain topological spaces. On the surface, it looks like a useless and intuitive observation, but, with only a bare (Baire?) minimum of cleverness, we can use this tool to prove that certain classes of objects must exist. the trick is to show that the set of all objects that do not have the desired property is "small, in a sense that will be made precise, and this will imply that the set cannot occupy the whole space. Indeed, the notion of size prescribed in the theorem is wholly new, independent of concepts of measure and slightly removed from the ideas of density within a set. Without further delay, then, we proceed to the development of the main theorem.

1.2. **Spaces and categories.** The following definitions will be useful in the statement of the theorem.

Definition. A *topological space* is a set, X, together with a family of subsets, $\mathcal{T}$, called the *topology on* X, that satisfies the following properties.

i. $X, \varnothing \in \mathcal{T}$.

ii. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.

iii. For any family $\{A_\sigma\}_{\sigma\in\Sigma}$ of sets in $\mathcal{T}$, $\bigcup_{\sigma\in\Sigma} A_\sigma \in \mathcal{T}$.

We call the members of $\mathcal{T}$ the *open sets* in X, and define the *closed sets* of X to be the complements of the open sets. Given any $A \subset X$, $\overline{A}$, the *closure* of A, is the intersection of all the closed sets containing A, and Int A, the *interior* of A, is the union of all the open subsets of A. When the topology is understand, we will denote the topological space simply by X, otherwise we will use the standard $(X, \mathcal{T})$.

Definition. A set, A, is *dense* in a topological space X if $\overline{A} = X$. It is *somewhere dense* if Int $\overline{A} \neq \varnothing$, and it is *nowhere dense* if Int $\overline{A} = \varnothing$. (Note that the order of operations here is important, the rationals are dense in the reals but $\overline{\text{Int } \mathbb{Q}} = \varnothing$). A metric space with a countable, dense subset is called *separable.*

Definition. A *metric space* is a set M together with a metric $d : M \times M \to \mathbb{R}$ that satisfies

i. $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.

ii. $d(x, y) \leq d(x, z) + d(z, y)$.

Open and closed sets are defined as in Euclidean space, relative to this metric, in terms of open balls. It is easily seen that every metric space is a topological space, though not all topologies can be generated by a metric. A metric space is called *complete* if every Cauchy sequence converges, and a topological space is called *topologically complete* if it is homeomorphic to a complete metric space.

Examples.

- The family of subsets, $\mathcal{H}$, of $\mathbb{R}$ defined as all finite intersections and arbitrary unions of half-open intervals of the form $[a, b)$ is a topology on $\mathbb{R}$. However, $(\mathbb{R}, \mathcal{H})$ is not metrizable. It is an otherwise very nice space; it has a countable dense subset, every pair of disjoint closed sets (including points) can be separated by disjoint neighborhoods and also by functions. In other words, it is what's called separable and a perfectly normal Hausdorff space.
- Every subset of a topological space naturally inherits a topology; a set is open in the subset if it is the intersection of an open set in the larger space with the subset. So, given the standard topology on $\mathbb{C}$, the set $S^1 \subset \mathbb{C}$ is a topological space. Similarly, we can see that it is a complete metric space if we restrict the metric function to S^1.
- The set $\mathbb{Z}^{\aleph_0}$ of infinite sequences of integers is a complete metric space under the metric $\rho(x, y) = 2^{-n}$ where n is the first number with $x)n \neq y_n$. This set is also a group under componentwise addition, and this is not just a trivial observation. Often topological spaces can be endowed with other structures resulting, occasionally, in an intertwining between the different structures therein. Indeed, we will come back to this example in a later section, and see that its topological structure gives rise to some algebraic properties that might have otherwise gone unnoticed.

We are now ready to state and prove the Baire category theorem, which comes as a direct result of the definition of a complete metric space. The reader would do well to remember the elementary nature of this result in the sequel, as it only makes the vast applications of this theorem that much more impressive.

Theorem 1 (Baire Category Theorem). *Let X be a complete metric space. Then* (a) *the countable intersection of open, dense sets is dense, and* (b) *X is not a countable union of nowhere dense sets.*

Proof. Let $\{A_n\}$ be a sequence of open dense sets in X and $A = \bigcap_0^\infty A_n$. We note that an equivalent formulation of a dense set is that it intersects every open set. With this in mind, we choose $W \subset X$ nonempty and open, and show that $W \cap A \neq \varnothing$. The idea behind the proof is to construct a Cuahcy sequence which converges to a point that must be in all of the partial intersections, and thus in the infinite intersection. Since A_0 is open and dense, $W \cap A_0$ is nonempty and open, so we can choose a point, x_0, and a radius, $r_0 \in (0,1)$, so that $B(x_0, r_0) \subset W \cap A_0$. In general, we note that, having chosen x_i and r_i for $i < n$, $A_n \cap B(x_{n-1}, r_{n-1})$, because A_n is dense. Furthermore, this intersection is open and nonempty, so we can choose $x_n \in A_n \cap B(x_{n-1}, r_{n-1})$ and $r_n \in (0, 2^{-n})$ so that $\overline{B(x_n, r_n)} \subset A_n \cap B(x_{n-1}, r_{n-1})$. But now we have our Cauchy sequence because, for $n, m > N$, $x_n, x_m \in B(x_N, r_N)$ and $r_n \to 0$. This implies that $x = \lim x_n$ exists, and, by construction

$$x \in \overline{B(x_n, r_n)} \subset A_n \cap B(x_0, r_0) \subset A_n \cap W$$

for every n, so we are done. The second part follows quite easily, because if $\{E_n\}$ is a sequence of nowhere dense sets, then $\{\overline{E_n}^c\}$ is a sequence of open dense sets. But then $\bigcap \overline{E_n}^c \neq \varnothing$, so $\bigcup E_n \subset \bigcup \overline{E_n} \neq X$. □

This inspires the following definitions.

Definition. A subset, M, of a topological space is *meager* (or of the *first category*) if it is the countable union of nowhere dense sets. If a set is not meager it is of the *second category.* The complement of a meager set is called *residual.*

Definition. A topological space, B, is a *Baire space* if it is not the union of any countable collection of nowhere dense sets (so it is of the second category in itself).

So the Baire category theorem says that every complete metric space is a Baire space. This is, however, not a necessary condition for a topological space to be a Baire space. There are spaces which are not even metrizable that are Baire spaces. (Indeed, the other main statement of Baire's theorem is for locally compact Hausdorff spaces, and yet there are Baire spaces which are neither). It should be quite evident that the Baire category theorem should have a lot to say about purely topological questions, and these will be the first few applications shown here.

Examples.

- Assign an integer, n, to each real number. Let $A_n = \{x \in \mathbb{R} : x \mapsto n\}$, then $\mathbb{R} = \bigcup A_n$, so one of the A_n's must be somewhere dense.
- A countable union of nowhere dense sets can still be dense. For example $\mathbb{Q} = \bigcup \{r_n\}$, where r_n is an enumeration of the rationals, but the rationals are dense in the reals. Indeed this is true for any separable metric space with no isolated points.
- The product of any family of Baire spaces is a Baire space.

- A countable intersection of open sets is called a G_δ set (in the terminology of Borel sets). In a complete metric space, any countable intersection of dense G_δ sets is dense. Because $G_\delta = \bigcap A_n$ and $G_\delta \subset A_n$, so the A_n's are open and dense, and a countable intersection of a countable intersection is countable. This holds, in fact, for any finite iteration of this process.
- Let r_n be an enumeration of the rationals, and let $a_{n,m} = \frac{1}{2^{n+m}}$. Then $R = \bigcap_m \bigcup_n (r_n - a_{n,m}, r_n + a_{n,m})$ is certainly residual, because it's complement is a countable union of meager sets (use DeMorgan's laws), which is meager. On the other hand, the Lebesgue measure of this set is 0, because, by upper continuity of the measure, $\mu(R) = \lim_{m\to\infty} \mu(\bigcup(r_n - a_{n,m}, r_n + a_{n,m}) = 0$.

Proposition 1. *If a complete metric space, X, has no isolated points, then it is uncountable.*

Proof. For suppose that it were countable. Every singleton in X is closed with empty interior, because it is nonisolated, so every singleton is nowhere dense in X. But then $X = \bigcup_{x\in X}\{x\}$, so that X is meager in itself. But this contradicts Baire's theorem. □

Corollary. $\mathbb{R}$ *and the Cantor set are uncountable.*

Proposition 2. *For any Baire space, B, if $\{f_\alpha\}$ is a family of continuous, real-valued functions, and $M(y) = \sup_\alpha\{f_\alpha(y)\}$ is finite for all y, then M is uniformly bounded on some open set.*

Proof. Consider $A_n = \{y : M(y) \leq n\}$, for $n \in \mathbb{N}$. Each A_n is closed, and every y in B must be in one of the A_n's by assumption. Since A_n is a countable, closed covering of B, one of the A_n's must contain an open set, by Baire's theorem, which is what we wanted to show. □

2. Applications in Real and Functional Analysis

2.1. **Subsets of $C(I)$.** Mathematicians in the early days of analysis used to underestimate the depth of the universe they had created. But then, just as the concept of what a "function" is began to develop, pathological and counter-intuitive results started popping up everywhere. There seems to be a history of this throughout mathematics, where we discover that the nice objects we are used to are not the only ones out there. In fact, this result is usually paired with, "most of the things out there are not nice." Ever since the Pythagorean fantasy of the commensurable was shattered by $\sqrt{2}$, we have been constantly surprising ourselves with the size of the set of objects we can never hope to grasp.

Examples.

- As we saw in the first section, $\mathbb{R}$ is uncountable, whilst $\mathbb{Q}$ is countable. So, just to stick it to Pythagoras, "most" numbers are irrational.
- In fact, we can say even more than this. A number, like $\sqrt{2}$, is called *algebraic* if it is the solution to a polynomial equation with rational coefficients. But alas, notice that, if we denote $p(x) = a_n x^n + \cdots + a_0$, then

$$\{p(x) : |n| + |a_n| + \cdots + |a_0| < k\}, \quad k \in \mathbb{N}$$

is countable for every k. As a consequence, the union of all of these sets is countable- but surely this encompasses all of the polynomials with rational coefficients. And so, we find that "most" numbers are transcendental- not a solution to any rational polynomial.
- To take an example from set theory, we know that the "set of all sets" is too large to be a set! Indeed, if such a set, call it U, existed, then we could consider a subset of U defined by $A = \{x \in U : x \text{ is not a member of itself}\}$. But then we ask, "Is A in A?" Well, if it is not in A, then, by construction, it must be. On the other hand, if it is in A, then it can't be! That being said, every set that any working mathematician would ever possibly need, or could even imagine, can be found in a couple of iterations of the power set past the ω'th iteration of the power set on $\varnothing$. (So, $P(P(\cdots P(P(\varnothing))\cdots))$, where there are about $\omega + 12$ P's). And that is a, relatively, small set.
- There are nonmeasurable sets.

In this section, we give a few of the classical results in this spirit- though a constructivist would probably cringe at these proofs.

Proposition 3. *The set of everywhere continuous, nowhere differentiable functions forms a residual set in $C(I)$.*

Proof. We construct continuous functions with infinite right derivatives at every point. Define
$$A_n = \left\{ f \in C(I) : \exists x \in \left[0, 1 - \frac{1}{n}\right] \text{ so that, for all } h \in \left(0, \frac{1}{n}\right), \left|\frac{f(x+h) - f(x)}{h}\right| \leq n \right\}$$
for every n. Certainly, if a function is differentiable at a point, it must belong to one of the A_n. So we wish to show that there are continuous functions outside of the union of all of these sets. Of course, it suffices to show each is closed with no interior points. I claim that if $\{f_m\}$ is a uniformly convergent sequence of functions in A_n, then the limit is also in A_n. Indeed, define $M(m,h) = \min_{x \in [0,1-(1/n)]} \left|\frac{f_m(x+h) - f_m(x)}{h}\right|$. This function is continuous, by the uniform convergence of the functions, and we know that $\lim_{m\to\infty} \limsup_{h\to 0} M(n,h) \leq n$. By some elementary considerations, and relying heavily on the uniform convergence of the function, we can interchange the limiting operations and preserve the inequality, getting $\limsup_{h\to 0} M(\infty, h) \leq n$, so the limit of the sequence is in A_n. This means that A_n is closed. It remains to show that the interior of A_n is empty. But take any function, $f \in A_n$, and any $\epsilon > 0$. I can uniformly approximate this function by a continuous function made up of finitely many line segments each with slope $\pm 2n$. Indeed, picture an ϵ band around the graph of the continuous function (or, even better, a smooth subband therein), and, beginning at the left, bottom endpoint, draw a line of slope $2n$ until it intersects the top band. Continue by drawing a line beginning at that new point of slope $-2n$ and terminating at the bottom band, etc. This process cannot continue indefinitely due to the compactness of the interval. The function is certainly continuous, and it is in A_n^c. But ϵ was arbitrary, so no point of A_n is an interior point. The theorem then follows. □

Constructive example. Let $f(x) = \sum_0^\infty 2^{-n} \cos(20^n \pi x)$. Notice that $\sum |2^{-n} \cos(20^n \pi x)| \leq \sum |2^{-n}| < \infty$, so, by the Weierstrass M-test, we have uniform and absolute convergence. Furthermore, f is continuous because its partial sums are. It can be shown, however, that this function is nowhere differentiable (cf. (8), for one).

So the reader has certainly rid themselves of any hopes to guarantee smoothness in continuous functions. But what about guaranteeing smoothness in... smooth functions? If given a smooth function, can we guarantee that this function is representable, at least somewhere, as the sum of its Taylor series? The answer, sadly, is no. But first, let's make sure that we're working on a Baire space.

C^∞ is a topological space, and we can attach to it the following metric:

$$d(f,g) = \sum \min\left[2^{-n}, \|f^{(n)} - g^{(n)}\|\right]$$

It follows from completeness of I and standard theorems in analysis that this is a complete metric, and we now use it to prove our result.

Proposition 4. *There exists a residual set of $f \in C^\infty(I)$ that are nowhere analytic.*

Proof. By the Cauchy-Hadamard formula for the radius of convergence of a power series, if f is analytic at a, it must be the case that

$$\sup\{\sqrt[k]{|f^{(k)}(a)/k!|}\} < \infty$$

So we let

$$T(a;c) = \{f \in C^\infty : \forall k, |f^{(k)}(a)| \leq k!c^k\}$$

If f is analytic at a, it must lie in one of these $T(a,c)$. But now notice that any function that is analytic at a point is analytic at a neighborhood of a point, so if a function is not analytic on all of the rationals, it is not analytic anywhere. With this in mind we note that we can express all the analytic functions as a subset of $\bigcup\{T(a,c) : a \in \mathbb{Q}, c \in \mathbb{N}\}$. All we have to do now, thanks to Baire, is show that each of these is nowhere dense. Notice that each is closed, because $T(a,c) = \bigcap_k \{f \in C^\infty : |f^{(k)}(a)| \leq c^k k!\}$ and and intersection of closed sets is closed. The proof that each set also has empty interior is due to H. Salzamann and K. Zeller. Given $f \in T(a,c)$, and any $B(f, 2\epsilon)$, choose an n so that $\sum_n^\infty 2^{-i} < \epsilon$ (which we can do because the tail must go to zero). Then select a $b > 2$ so that $\epsilon b^n > (2n!)c^{2n}$. Define a function

$$s(x) = f(x) + \epsilon b^{-n} \cos b(x-a)$$

then s is inifnitely differentiable. Furthermore, $\sup\{|f^{(k)}(x) - s^{(k)}(x)|, x \in I\} =$ $\|f^{(k)} - s^{(k)}\| \leq \epsilon b^{k-n} < \epsilon 2^{k-n}$, for every $k < n$, so we know that $s \in B(f, 2\epsilon)$. On the other hand, $|s^{(2n)}(a) - f^{(2n)}(a)| = \epsilon b^n > (2n)!c^{2n}$, so $s \notin T(a,c)$. □

2.2. **Banach spaces.**

Definition. A *norm* on a vector space X over a field K is a function $\|\cdot\| : X \to \mathbb{R}^+$ that satisfies

i. $\|x+y\| \leq \|x\| + \|y\|$, $x, y \in X$.

ii. $\|\alpha x\| = |\alpha|\|x\|$, $x \in X$ and $\alpha \in K$.

iii. $\|x\| = 0$ iff $x = 0$.

Notice that every normed linear space has with it a metric defined as $d(x,y) = \|x-y\|$. It is easy to see that this satisfies all the properties of a metric. When it so happens that the space is complete with respect to the metric induced by its norm, we call the space a *Banach space.*

Definition. A linear map $T : \mathcal{X} \to \mathcal{Y}$ between two normed linear spaces is called *bounded* if there is some $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for every $x \in \mathcal{X}$. Notice that this is much different from the normal notion of boundedness- as it should be, because the only linear map bounded in the usual sense would be the zero map. We denote the space of all bounded linear maps between $\mathcal{X}$ and $\mathcal{Y}$ by $L(\mathcal{X}, \mathcal{Y})$, and we define a norm on this space as follows

$$\|\Lambda\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\}$$

The reader can verify that this does, in fact, satisfy the properties of a norm. Also, as a matter of convention, when $\mathcal{Y} = \mathbb{R}$ or $\mathbb{C}$, we call the map a *linear functional.* Also, the reader may verify that the following are equivalent for linear maps on Banach spaces; T: T is bounded, T is continuous, T is continuous at a point.

Definition. A map $f : X \to Y$ between two topological spaces is called *open* if the image of any open set is open.

Definition. A map $T : \mathcal{X} \to \mathcal{Y}$ between is called *closed* if the graph of f, $\Gamma(f) = \{(x,y) : y = f(x)\}$ is a closed subspace of $\mathcal{X} \times \mathcal{Y}$.

Examples.

- All Euclidean spaces are Banach. So is the set of continuous functions on $\mathbb{R}$ with the uniform norm, and so is the set of analytic functions on $\mathbb{C}$ with the uniform metric.
- Given any continuous function $K : [a,b] \times [c,d] \to \mathbb{R}$, the linear operator defined by $(Lf)(y) = \int_a^b K(x,y)f(x)\,dx$ is bounded and continuous on the domain $C([a,b])$.
- Given any Banach space of functions, the evaluation operator $\pi_x(f) = f(x)$ is bounded and continuous.

Theorem 2 (The Open Mapping Theorem). *Every surjective, bounded linear map between Banach spaces is open.*

Proof. Denote by $B_k(0)$ the open ball of radius $k \in \mathbb{N}$ around 0 in $\mathcal{X}$. Certainly it is true that $\mathcal{X} = \bigcup kB_k(0)$. Then, since T is surjective,

$$\mathcal{Y} = T(\mathcal{X} = T(\bigcup B_k(0) = \bigcup kT(B_1(0)).$$

Now, $\mathcal{Y}$ is complete, so one of these sets must be somewhere dense. But if one of these sets is somehwere dense, then certainly $\overline{kT(B_1(0))} = k\overline{T(B_1(0))}$, so that $T(B_1(0))$ is somewhere dense. Indeed, by the same logic, $T(A_n)$ is somewhere dense for all $A_n = \{x \in$

$\mathcal{X} : \|x\| < 2^{-n}$. Now we note the following result about the A_n's. If $\{y \in \mathcal{Y} : \|y - y_0\| < \eta\} \subset \overline{T(A_1)}$, where y_0 and $\eta > 0$ are fixed, then $\{y \in \mathcal{Y} : \|y\| < \eta\} \subset \overline{T(A_1)} - y_0$. Indeed,

$$\{y \in \mathcal{Y} : \|y\| < \eta\} \subset 2\overline{T(A_1)} = \overline{T(A_0)}.$$

But since T is linear, and scalar multiplication gives homeomorphic sets, we must have that $C_n = \{y \in \mathcal{Y} : \|y\| < \eta 2^{-n}\} \subset \overline{T(A_n)}$, for every n. I now claim that $C_1 = \{y \in \mathcal{Y} : \|y\| < \eta/2\} \subset T(A_0)$. Pick any $y \in C$, we approximate the desired preimage by partial sums. We know that $y \in \overline{T(A_1)}$, so there is an $x_1 \in A_1$ with $\|y - T(x_1)\| < \eta/4$. But now $y - T(x_1)$ lies in C_2. Continuing recursively, given that $y - \sum_1^{n-1} T(x_k) \in C_n \subset \overline{T(A_n)}$, we pick an $x_n \in A_n$ so that $\|y - \sum_1^n T(x_k)\| < \frac{\eta}{2^{n+1}}$. Since $x_n \in A_n$, for every n, we see that $\|x_n\| \leq 2^{-n}$, so $\sum x_n$ converges to some point x. Moreover, by continuity of T, we have that $T(x) = T(\sum x_n) = \sum T(x_n) = y$ (because their difference goes to 0 like $2^{-(n+1)}$. But now that we have $C_1 \subset A_0$, we are done, because T commutes with translations and dilations. □

Since open bijections are isomorphisms, the following is an immediate consequence:

Corollary. *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T \in L(\mathcal{X}, \mathcal{Y})$ is bijective, then T is an isomorphism.*

As a quick application of this result outside the area of functional analysis, we have another negative result for those who had hoped for a universal comparison test for absolutely convergent series.

Proposition 5. *There is no slowest rate of decay of terms in a convergent series: that is, there is no sequence $\{a_m\}$ of positive real numbers such that $\sum a_n|c_n| < \infty$ iff $\{c_n\}$ is bounded.*

Proof. Denote by $B(\mathbb{N})$ the set of all bounded, complex-valued function on $\mathbb{N}$, and the set of all absolutely summable sequences by $L^1(\mu)$ (where μ is the counting measure). Define a linear operator, $T : B(\mathbb{N} \to L^1(\mu)$ by $Tf(n) = a_n f(n)$. Notice that this operator is bounded by $\sup a_n$. The set, A, of all f with $f(n) = 0$ on all but a set of measure zero (finitely many n) is dense in $L^1(\mu)$ by basic measure theoretic considerations. On the other hand, when $B(\mathbb{N})$ is given the uniform metric, this set is not dense. But, notice that T is bijective, so it should be an isomorphism which preserves density, so we have a contradiction. □

The remaining three functional analysis theorems are complete the standard and classical applications of the Baire category theorem, and they are quite powerful.

Theorem 3 (The Closed Graph Theorem). *Every closed linear map between Banach spaces is bounded.*

Proof. Let π_1 and π_2 project $\Gamma(T)$ onto $\mathcal{X}$ and $\mathcal{Y}$ respectively. Both are bounded and linear. Now we observe that $\mathcal{X} \times \mathcal{Y}$ is complete because both $\mathcal{X}$ and $\mathcal{Y}$ are. This implies that $\Gamma(T)$ is complete because closed subspaces of complete spaces are complete. Since projection functions are bijections, their inverses are bounded by the above corollary. Consequently, $T = \pi_2 \circ \pi_1^{-1}$ is bounded. □

Theorem 4 (The Uniform Boundedness Principle). *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $\mathcal{A}$ is a subset of $L(\mathcal{X}, \mathcal{Y})$. If $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ for all x in some nonmeager subset of $\mathcal{X}$, then $\sup_{T \in \mathcal{A}} \|T\| < \infty$.*

Proof. We describe a family of subsets of $\mathcal{X}$,

$$D_n = \bigcap_{T \in \mathcal{A}} \{x \in \mathcal{X} : \|Tx\| \leq n\}$$

Certainly each of the D_n's are closed. But, by hypothesis, one of these D_n must contain a closed ball $\overline{B(x_0, r)}$, and, consequently $\overline{B(0, r)} \subset E_{2n}$, because if $\|x\| \leq r$, then $x - x_0 \in E_n$ so

$$\|Tx\| \leq \|T(x - x_0)\| + \|Tx_0\| \leq 2n$$

Now, using the definition of the norm on an operator, we have that, for $T \in \mathcal{A}$, $\sup_{T \in \mathcal{A}} \|T\| \leq 2n/r$, because $\|Tx\| \leq 2n$ when $T \in \mathcal{A}$ and $\|x\| \leq r$. □

Theorem 5 (The Principle of Condensation of Singularities). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\{T_{jk}\} \subset L(\mathcal{X}, \mathcal{Y})$. Suppose that, for every k, there is some $x \in \mathcal{X}$ such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$. Then there is a residual set of x's so that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$ for all k.*

Proof. Suppose that the set of all x such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} < \infty$ for all k was nonmeager. Then, for fixed k, $\sup_j \|T_{jk}\| < \infty$, by the uniform boundedness principle. But this cannot be, because there is some x where the supremum is infinite, by hypothesis. Therefore the prescribed set must be meager, which means that its complement is, by definition, residual. □

We turn, now, to an interesting question regarding Fourier series. In *Théorie analytique de la chaleur*, where Fourier originally introduced his theory of trigonometric series, he boldly stated that "every function may be represented this way." Apparently the mathematical community took that as a challenge and, in response, spent the next century trying to figure out how to prove that statement. Of course, the biggest was problem was... how do we state that statement in the first place? There was no formal concept of function, or limits, or integrals. Fourier's nonchalant challenge to the world was, perhaps, the inspiration behind much of modern analysis. Riemann invented his theory of integration to study certain trigonometric series, and Lebesgue did the same- without which, measure theory would not be here. Even our notions of the infinite, and many developments in set theory, were originally developed by Cantor while he was... studying the convergence of Fourier series! Carleson's proof that the Fourier series for every L^2 function converges, almost everywhere, was the holy grail of convergence results. And yet, people used to think that the Fourier series' of continuous functions must converge, at least pointwise, *everywhere*. We now show that this is not true; indeed, the set of all functions for which this is true is "small." Déjà vu?

Theorem 6. *There is a dense, G_δ set, $F \subset C(S^1)$ with the following property: For every $f \in F$, the set of points where the Fourer series of f diverges is a dense G_δ subset of $\mathbb{R}$.*

Proof. We denote the nth partial sum of the Fourier series of f at a point x by

$$s_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,dt$$

where D_n is the Dirichlet kernel, defined as

$$D_n(t) = \sum_{-n}^{n} e^{ikt}$$

Now, let's fix x, and consider the family of linear functionals $\Theta_n : C(S^1) \to \mathbb{R}$, defined as $\Theta_n(f) = s_n(f;x)$. Since the supremum norm $\|f\|_\infty$on $C(S^1)$ gives rise to a complete metric space, we know that $C(S^1)$ is a Banach space. Furthermore

$$\|\Theta_n\| = \sup\{\|\Theta_n(f)\| : \|f\| = 1\} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|dt$$

Indeed, we can show that equality holds here. We can pick a Lesbesgue integrable function, $g(x-t)$, that is 1 when $D_n(t)$ is positive, and -1 when $D_n(t)$ is negative. This is certainly the pointwise limit of a sequence of continuous functions, $\{f_j\}$. We then use the dominated convergence theorem to find

$$\lim \Theta_n(f_j) = \lim \frac{1}{2\pi}\int_{-\pi}^{\pi} f_j(x-t)(t)D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt$$

So that the $\|\Theta\| = (1/2\pi)\int |D_n(t)|\,dt$. Now, we wish to show that the right hand side diverges. We use the classical representation of the Dirichlet kernel, which the reader may verify.

$$D_n(t) = \frac{\sin{(n+1/2)t}}{\sin{(t/2)}}$$

Now then, with a quick nod to the so-called "Jordan Lemma," we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt = \frac{1}{\pi}\int_{0}^{\pi} \left|\frac{\sin{(n+1/2)t}}{\sin{(t/2)}}\right|\,dt$$

$$\geq \frac{2}{\pi}\int_0^{\pi(n+1/2)} t^{-1}|\sin t|\,dt > \frac{2}{\pi}\sum_1^n \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin t|\,dt = \frac{4}{\pi^2}\sum \frac{1}{n} \to \infty$$

From this it follows that $\sup_n\{\|\Theta_n\|\} = \infty$, so, applying the contrapositive of the uniform boundedness principle, we have that; there is a residual set F_x, corresponding to each x on the circle, so that every $f \in F_x$ has a divergent Fourier series at x. Applying Baire's theorem one more time, taking the intersection of F_{x_i} for a countable, dense subset x_i, we get a set $F \subset C(S^1)$ with all the desired properties. □

3. Applications to the Baer-Specker Group

In the first section, we introduced the set $\Pi = \mathbb{Z}^{\aleph_0}$ of all sequences of integers. This is a group under componentwise addition, called the Baer-Specker group. At the same time, it is a complete metric space. We remind the reader of the metric here:

$$\rho(x,y) = 2^{-n} \quad \text{where } n \text{ is the first number with } x_n \neq y_n$$

Notice that if $\{x^k\}$ is Cauchy, then $\{x_n^k\}$ is eventually constant for each n. Call the final value y_n, and now note that $x^k \to y$, because I can always choose k large enough so that the first N terms agree. Before going further, we define some group theoretic terms, and a few more topological ones; the reader is not assumed to know anything beyond the definition of a group, and even that is included here again for reference.

Definition. A *group* is a set, G, together with a binary operation, $*$, satisfying the following properties: (i) there is an identity, e, (ii) the group operation is associative, (iii) every element has an inverse. When the group operation is commutative, the group is called *abelian.* A *subgroup* is a subset of G that is closed under the group operation. A subgroup, S, of an abelian group, A, is called *pure* if all of the nth roots of elements in A are also in S.

Definition. Given any subgroup, H, in G, we define a *left coset* of H to be $g * H = \{g * h : g \in G, h \in H\}$, and a *right coset* similarly as $H * g$. In abelian groups, these collide and we can speak simply of cosets. The reader can verify that any two left (resp. right) cosets are either identical or disjoint.

Definition. A *homomorphism* between a group $(G, *)$ and a group $(H, +)$ is a map, $h : G \to H$ with the property that $h(x * y) = h(x) + h(y)$. A bijective homomorphism is called an *isomorphism.*

Definition. A *Borel set* is a set obtainable from open sets by repeated formations of complements and countable unions. An *analytic set* is the image, under a continuous function, of a Borel set in a complete separable metric space.

Examples.

- The set of all roots of unity is a group under complex multiplication. Any specific set of nth roots of unity is a subgroup of this group.
- G_δ sets are Borel.
- The set of all 2×2 matrices with determinant 1 forms a group under multiplication, known as the general linear group.

We proceed to some results. First note that the structure of the basic neighborhoods of 0 is a subgroup

$$V_k = \{x \in \Pi : x_n = 0, n < k\}$$

and all other k-neighborhoods of points are simply cosets of this subgroup. Now, to connect the topology to the group structure, we will need to know that the homomorphisms of Π are continuous with respect to the given topology. However, a stronger result can be stated, and was proven by Specker in the 1950's:

Theorem 7 (Specker's Theorem). *Every homomorphism $h : \Pi \to \mathbb{Z}$ has the form $h(x) = \sum_1^n a_i x_i$, for some finite n and coefficients $a_i \in \mathbb{Z}$.*

Now notice that a function into a product space (the cartesian product of topological spaces) is continuous if and only if all of its components are. But surely, due to the finiteness of the above sum, homomorphisms into $\mathbb{Z}$ are all continuous. The following corollary is then immediate.

Corollary. *Every homomorphism $h : \Pi \to \Pi$ is continuous.*

With this result, we show that since Π is large in a categorical sense, it must also be large in a group theoretic sense.

Theorem 8. *The group* Π *is not a union of a countable chain or proper subgroups,* $P_0 \subset P_1 \subset \cdots \subset \Pi$*, each isomorphic to* Π*.*

Proof. Suppose such a chain did exist, with a union equal to all of Π. By Baire's theorem, one of these subgroups must be somewhere dense, call it P_m. Therefore its closure contains some open set, $a + V_k$, for some $a \in \Pi$. We see that $V_k \subset P_m$. To that end, fix $z \in V_k$, and map $x \mapsto x + z$, which is a topological automorphism of Π. Then, since $(a+V_k) \setminus P_m$ is meager, so is the homeomorphic image. Now, again applying Baire's theorem, $(a+V_k) \setminus P_m$ and $z + ((a+V_k) \setminus P_m)$ cannot cover all of $a+V_k$. Therefore, there is some $x \in a + V_k$ that belongs to neither of these neighborhoods, and $x \in P_m$. Since V_k is also a subgroup, we have that $x - z \in P_m$, and since P_m is a subgroup, $z \in P_m$ as desired. Now then, I claim that one of the P_n's covers all of Π. Indeed, some P_l contains finitely many unit vectors, $e_0, ..., e_{k-1}$ (where $e_i = \{\delta_{ij}\}_j$). Then certainly, P_n, where $n = \max m, l$, contains all of these unit vectors and everything in the unit ball. But, using finite group operations, every memeber of Π can be produced from these objects, so $P_n = \Pi$, which is a contradiction. □

The proof of this theorem did not rely at all on the fact that each P_n was isomorphic to Π except insofar as to show that each one was analytic. So this corollary follows easily:

Corollary. Π *is not the union of a chain of countably many analytic, proper subgroups.*

This is quite a strong statement, restricting the ways that we may represent Π in some manner by chains of subgroups. It means that Π is quite large in several senses of its structure. We turn, now, to a proper subgroup of Π, which we will call D; the set of all $x \in \Pi$ such that, for each positive integer q, all but finitely many terms of x are divisible by q. We can actually represent this subgroup as a Borel set in Π like so;

$$D = \bigcap_q \bigcup_k \bigcap_{n \geq k} \{x \in \Pi : q \text{ divides } x_n\}$$

Now, this subgroup is not large in the sense of Π, for I can represent it as the union of a chain of proper subgroups, each isomorphic to D. Consider

$$D_k = \{x \in D : x_n \text{ is even for all } n \geq k\}$$

Notice something quite nice that comes as a result of this: we know that D is not a complete metric space. Indeed, if it were, then the proof of Theorem 8 would have gone through almost verbatim. But the reason that D is not complete comes from purely *algebraic* considerations. It is here where you can begin to see the deep bond between algebraic and topological structures that live on the same space, and the beginnings of a beautiful field known as algebraic topology. We now proceed to guarantee what little we can about the relationship between chains of subgroups and D- we must add, however, the must stronger assumption that the subgroups are pure.

Theorem 9. *The group* D *is not the union of a countable chain of pure subgroups, each isomorphic to* D*.*

We leave the proof of this fact to the reader; though nontrivial, the proof is not difficult, but takes a bit of space. (See (2) for a proof).

4. Applications to Classical Function Theory

4.1. **Short detour to Hausdorff spaces.** Topological spaces, as discussed in the opening section, provide a rich and varied set of spaces to study. At times, however, it is too rich. There are spaces where limits are not unique, and which can be quite ugly in all manner of ways. Hausdorff spaces are a bit calmer, by ensuring that there are sufficiently many open sets, and second-countable Hausdorff spaces are even calmer, ensuring that there are not too many open sets. When dealing with functions and analysis, it is most natural to work on nice spaces like these. In fact, certain types of Hausdorff spaces are nice enough to be Baire spaces, as we prove below.

Definition. A topological space, $\mathcal{H}$, is *Hausdorff* if, given any two distinct points, $x, y \in \mathcal{H}$, there is a neighborhood of x, U_x, and a neighborhood of y, U_y, such that $U_x \cap U_y = \varnothing$.

Definition. A topological space, X, is *compact* if every open cover of X admits a finite subcover. A space is locally compact if every point has a compact neighborhood.

Definition. A *basis*, $\{B_\alpha$, for a topological space, X, is a subset of open sets with the property that every open set is a union of members of $\{B_\alpha$. If a topological space has a countable basis, it is called *second-countable.*

Definition. A family of sets, $\mathcal{F}$, in a space has the finite intersection property if $\bigcap_{F\in B} F \neq \varnothing$, for finite $B \subset \mathcal{F}$.

Proposition 6. *A space is compact iff every closed family of subsests, $\{T_\alpha\}$, with the finite intersection property satisfies $\bigcap_\alpha T_\alpha \neq \varnothing$.*

Proposition 7. *Every point in an open subset, U, of a locally compact Hausdorff space has a compact neighborhood, N_x, with $N_x \subset U$.*

Theorem 1′ (Baire Category Theorem). *Every locally compact Hausdorff space, $\mathcal{H}$, is a Baire space.*

Proof. Suppose I have a countable collection of dense, open sets $\{A_n\}$. Given an arbitrary open set $G \subset \mathcal{H}$, I wish to show that it intersects the intersection of all of the A_n's. $A_0 \cap G$ is open, so take $x_0 \in A_0 \cap G$. Now, since $\mathcal{H}$ is locally compact and Hausdorff, there is a compact neighborhood, N_{x_0}, of x_0 so that $N_{x_0} \subset A_0 \cap G$. Having defined $N_{x_{n-1}}$ and x_{n-1}, we define N_{x_n} as follows: $A_n \cap$ Int $N_{x_{n-1}}$ is nonempty and open, so there is an $x_n \in A_n \cap$ Int $N_{x_{n-1}}$, and a compact neighborhood, N_{x_n} so that $N_{x_n} \subset A_n \cap$ Int $N_{x_{n-1}}$. The family of sets $\{N_{x_n}\}$ consists of compact sets, all contained inside one large compact set, N_{x_0}. Furthermore, all finite intersections are nonempty, therefore, $\bigcap N_{x_n} \neq \varnothing$, so there is an $x \in G \cap \bigcap_1^N A_n$ for all n, which is what we wanted. □

Theorem 10 (Baire Variation). *Let X be a non-empty, complete metric space. Let $\{\beta_n\}$ be a countable family of subsets of X, and let $\mathcal{N} = (\bigcup \beta_n)^c$. Suppose that the following conditions are satisfied: (1) For every β_n, if A is dense in an open set G, and $A \subset \beta_n$, then $G \subset \beta_n$; and (1) For every β_n, every nonempty open set in X intersects β_n^c. Then there is a residual set, $R \subset \mathcal{N} \subset X$.*

Proof. I claim that each of the β_n is nowhere dense. For suppose it were dense in some open set, then, by the first hypothesis, that open set would be entirely contained in β_n, which contradicts the second hypothesis. By Baire's theorem, $\bigcup \beta_n \neq X$, and the complement of that union is residual. □

Theorem 11. *Let $\mathcal{H}_1$ be Hausdorff, let $\mathcal{H}_2$ be Hausdorff and second-countable, and let X be a complete metric space, all non-empty. Let $H : x \to 2^{\mathcal{H}_1}$, and denote by H_G the union of the H_g for $g \in G$. Suppose that* (3) *if the set $D \subset X$ is dense in some nonempty open set G, then H_D is dense in H_G. Furthermore, let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be continuous such that* (4) *if G is a non-empty open subset of X, then $f(H_G)$ is dense in $\mathcal{H}_2$. Then we can conclude that there is a residual set, $R \subset X$, so that, for every $r \in R$, $f(H_r)$ is dense in $\mathcal{H}_2$.*

Proof. Let $\{B_n\}$ be a basis for $\mathcal{H}_2$. Define $\beta_n = \{x \in X : f(H_x) \cap B_n = \varnothing\}$. Notice that (1) follows from the continuity of f. indeed, if $f(H_x) \cap B_n = \varnothing$ for every $x \in D \subset G$ in a dense subset of G, then we must have $f(H_g) \cap B_n = \varnothing$ for all $g \in G$, otherwise, by the openness of B_n, we could find an open set around g that's not contained in β_n, contradicting the density of D. Then (4) implies (2), so the conclusion follows by Theorem 10. □

Theorem 12. *Let $\mathcal{H}$ be Hausdorff and second-countable, and X be a complete metric space, both non-empty. Let $\{B_n\}$ be a basis of $\mathcal{H}$, and $\{f_k\}$ a sequence of continuous functions each of which maps X into $\mathcal{H}$. Suppose, further, that* (5) *for every non-empty open set $G \subset X$, and every B_n, there is a k so that $f_k(G) \cap B_n \neq \varnothing$. Then there is a residual set, $R \subset X$, so that, for every $r \in R$, $\{f_k(r)\}$ is dense in $\mathcal{H}$.*

Proof. We are set up to use Theorem 11. Let f be the identity and $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. For every $x \in X$, let H_s consist of the points $f_k(s)$ for all k. (3) is a consequence of the continuity of each f_k. To prove (4), let G and U be non-empty oepn subsets of X and $\mathcal{H}$, respectively. Now, $H_G = \bigcup f_k(G)$, and U is the union of some collection of basis sets, B_n, (4) then follows from (5). The theorem then follows from the preceding one. □

Theorem 13 (Diophantine Approximation). *Let $\{t_m\}$ be an increasing sequence of positive numbers tending to infinity. Then there is a residual set, $R \subset \mathbb{R}$, such that, for every $r \in R$, the set $\{t_m r + j\}$, $j \in \mathbb{N}$, is dense in $\mathbb{R}$.*

Proof. Let (a, b) be any open interval. Since $t_m \to \infty$, I can choose an m so that $t_m(b - a) > 1$, and, consequently, every real number can be written in the form $t_m x + j$ for some $x \in (a, b)$, just by shifting over by the appropriate integer. If we now identify $\mathcal{H}$ and X with $\mathbb{R}$, in the notation of Theorem 12, and then let $\{B_n\}$ consist of open intervals with rational endpoints, and $\{f_k\}$ to be any arrangement of the sequence $\{t_m x + j\}$, as a function of the real variable x, then the theorem follows from the preceding one. □

4.2. **Cluster Sets and Meromorphic Functions.**

Definition. Given a function, $f : D \to R$, the *cluster set* of the function at a point, $z_0 \in \overline{D}$, is the set of all points, $w \in R$ such that there is some sequence of points $z_n \to z_0$ with $f(z_n) \to w$. A cluster set is *degenerate* if it is a singleton, *subtotal* if it does not contain all of R, and *total* if it is all of R. Clearly, every cluster set is closed.

Definition. A *Riemann surface*, $\mathcal{R}$, is a second-countable Hausdorff space with the property that every point has a neighborhood that is homeomorphic to an open set in $\mathbb{C}$.

Definition. A *quotient space* of a topological space, T, and a subspace, S, denoted by T/S, is the space obtained by equating all the points in S.

Examples.

- Define $f(re^{i\theta})$ on the unit disk as being 1 whenever r and θ are rational, and -1 otherwise. Then the cluster set at any point on the circumference is $\{1, -2\}$.
- Let $f(z) = e^{i/|z|}$, then the cluster set at 0 is the unit circle.
- Let $f(x) = \sin 1/x$, then the cluster set at 0 is the interval $[-1, 1]$.
- The Riemann sphere, $\mathbb{C} \cup \infty$ is, appropriately, a Riemann surface. At every point, consider the hemisphere determined by that point. Clearly it is homeomorphic to a disk in $\mathbb{C}$. The Riemann sphere is a complete, separable metric space with respect to the spherical metric, and it is compact so it is second-countable and Hausdorff.
- If you take the unit square, $I^2 = [0, 1]^2$, and identify $(x, 0) \sim (x, 1)$, then you get a cylinder as the quotient space. If you then identify $(1, y) \sim (0, y)$, you get a torus. Tori are also Riemann surfaces.

The study of cluster sets began with Painlevé in 1895, when he introduced the notion as a way of describing the behavior of an analytic function near a singularity. He did this before the concept of measure was introduced, and so, was not able to gain the kind of deep results that he had, no doubt, hoped for. Later, the ideas of cluster sets were employed by Fatou and Carathéodory in the early 1900's to prove some uniqueness theorems and apply them to the study of differential equations. The study of cluster sets is now firmly in the camp of classical analysis, and has probably been absent from most graduate students' curriculum since the late 1940's. Of course, that doesn't mean the results weren't interesting, and we present a few of them here as an illustration of their connection with Baire's category theorem. The next two theorems will be stated for reference, without proof, but proofs can be found in (3)

Fatou's Theorem on Radial Limits. *If f is analytic and bounded on the unit disk, then the radial limits $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exist for all points $e^{i\theta}$ except, perhaps, for a set of measure zero.*

F. and M. Riesz's Uniqueness Theorem. *If f is analytic and bounded and if the set of points $e^{i\theta}$ with radial limits equal to 0 is of positive measure, then f is identically zero in the disk.*

Theorem 14. *Let f be meromorphic on the unit disk and map onto some Riemann surface, $\mathcal{R}$. Suppose that the cluster set of f at every point on the unit circle is $\mathcal{R}$. Then there is some residual set, $R \subset S^1$, so that the radial cluster set at every point $r \in R$ is $\mathcal{R}$.*

Proof. For every element, $s = e^{i\theta}$, in S^1, let H_s be the radius terminating in that point. Certainly, if $D \subset S^1$ is dense in some open set $G \subset S^1$, then H_D is dense in H_G. Furthermore, given any open set $G \subset S^1$, $f(H_G)$ is all of $\mathcal{R}$, because a cluster set is invariant under restriction to a smaller open set. Thus, identifying $\mathcal{H}_1$ with the unit disk, $\mathcal{H}_2$ with $\mathcal{R}$ and the complete metric space with S^1, the theorem follows from Theorem 11. □

Theorem 15. *Let f be meromorphic on the unit disk and let its range be some domain $\Omega \subset \mathbb{C}_\infty$. Suppose that the cluster set of f at every point of S^1 is $\overline{\Omega}$. Then there is a residual set, $R \subset S^1$, such that the radial cluster set at every point of R is $\overline{\Omega}$.*

Proof. This follows from the preceding theorem if we identify the boundary of Ω with a single point and consider the quotient space $\overline{\Omega}/\partial\Omega$. This is then a Riemann surface and the above applies. □

Theorem 17. *Suppose one of the following is satisfied: (a) f is meromorphic on the unit disk and has no radial limits on a set of positive measure in S^1; or (b) f is a non-constant meromorphic function on the unit disk and its radial cluster set on every point in a set of positive measure on S^1 contains a fixed constant c, finite or infinite. Then there is some residual subset of S^1, at each point of which, the radial cluster set of f is the whole plane.*

Proof. In view of Theorem 6, it suffices to show that the cluster set at every point of S^1 is the whole complex plane. For the sake of contradiction, suppose that there is some point $z_0 \in S^1$ with a subtotal cluster set. Since cluster sets are closed, this means that there is some neighborhood, U_a of a finite point, a, which does not intersect $f(U_{z_0}$, where U_{z_0} is some neighborhood of z_0. Now, this means that $g(z) = \frac{1}{f(z)-a}$ is bounded and analytic on U_{z_0}, and we can see that, by Fatou's theorem, $f(z)$ must posses radial limits in points on a set of positive measure in $S^1 \cap U_{z_0}$. This contradicts (a), it remains to contradict (b). But notice that, for (b), $g(z) \to 1/(c-a)$ along every radius terminating in a set of positive measure, and, by an obvious extension of the theorem of F. and M. Riesz, g must be constant. But then f is constant, contradicting the hypothesis. □

Notice that we used very few properties of the unit disk in proving all of these theorems. Indeed, suppose we replaced the unit disk, in each of these theorems, by a fundamental parallelogram. It is easy to see that all of these theorems still hold if we consider a radius to be a line emanating from the center (the point equidistant from every side) and terminating in a point on the side. Certainly the boundary of the parallelogram still forms a complete metric space, and so all of the above theorems hold unchanged. In particular, this implies the existence, for every non-constant elliptic function, of a residual set of radii, on each of which the cluster set of the function is the whole plane. Now we turn to a uniqueness theorem following from Theorem 7(b).

Corollary. *If f is meromorphic on the unit disk and its radial cluster set contains some constant c, finite or infinite, on a residual set of positive measure, then $f(z) \equiv c$.*

Proof. For suppose not, then f satisfies condition (b) above, so every radial cluster set on some residual subset of S^1 contains the whole complex plane. But this cannot hold if there is another residual set with only subtotal radial cluster sets and of positive measure. □

At this point we might wonder how much we can get away with here. How large can the set be where the meromorphic function has a total radial cluster set? The answer is, surprisingly, however large or small we wish. The proof of the following theorem uses standard techniques in cluster set analysis, but would take us beyond the scope of this paper. See (1) and (3) for more on this.

Theorem 18. *Given any number $\mu \in [0, 2\pi]$, there is some meromorphic function, f, on the unit disk such that the set R of points on S^1 where the radial cluster set is total is of measure μ. In the case $\mu = 0$, the R may be finite, countable, or uncountable.*

5. Concluding Remarks

Hopefully the reader has been as pleasantly surprised as the author by the wide reach of the Baire category theorem, and, indeed, mathematics in general. There has been another theme, however, shown throughout this paper, and that is the motif of negative results, alluded to in section 2. We hope that, instead of seeing these pathological examples as a fault to mathematics, the reader views these counter-intuitive results as a testament to the beautiful field that we have created. Indeed, is it not astounding that a structure, built by concepts completely understood by mankind, can give rise to objects more complex than we can fathom? We have stumbled upon a Pandora's box, and the author would not be at all surprised if the set of all mathematics we could ever conceive is of first category in the set of all mathematics there actually is (not that this is at all well-defined). If the reader is interested in further applications and connections to the Baire Category Theorem, we note here two interesting facts. One is that there is another characterization of meager sets in any topological space based on the existence, or non-existence, of winning strategies for players of a "Banach-Mazur game." This ties Baire to yet another area of mathematics. As a final frontier in mathematical logic, we find that the Baire category theorem is equivalent to a weak form of the Axiom of choice called the "Axiom of Dependent choice, which states that: Given any nonempty set X and any complete binary relation, R, there is a sequence $\{x_n\} \in X$ so that $x_n \, R \, x_{n+1}$ for every $n \in \mathbb{N}$. We invite the reader to explore the aspects of choice in this axiom, and look back upon the proof of Baire's theorem to see where choice is used.

References

[1] Bagemihl, F. and W. Seidel, *A General Principle Involving Baire Category with Application to Function Theory and other Fields.* The Institute for Advanced Study; August 17, 1953

[2] Blass, Andreas and John Irwin *Baer Meets Baire: Applications of Category Arguments and Descriptive Set Theory to $\mathbb{Z}^{\aleph_0}$*, 1991

[3] Collingwood, E.F. and A.J. Lohwater *The Theory of Cluster Sets* Cambridge Tracts in Mathematics and Physics, 1966

[4] Dugundji, James *Topology* Allyn & Bacon, 1964

[5] Folland, Gerald *Advanced Calculus* Prentice-Hall, 2002

[6] Folland, Gerald *Real Analysis: Modern Techniques and Applications*, John and Wiley Sons, Inc., 1999

[7] Gemignani, Michael C., *Elementary Topology* Dover Publications, 1967

[8] Rudin, Walter *Principles of Mathematical Analysis*, McGraw Hill, 1976