The Uniformization Theorem

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The Koebe uniformization theorem is a generalization of the Riemann mapping Theorem. It says that a simply connected Riemann surface is conformally equivalent to either the unit disk $\mathbb{D}$, the plane $\mathbb{C}$, or the sphere $\mathbb{C}^*$. We will give a proof that illustrates the power of the Perron method. As is standard, the hyperbolic case ($\mathbb{D}$) is proved by constructing Green's function. The novel part here is that the non-hyperbolic cases are treated in a very similar manner by constructing the *Dipole* Green's function.

A Riemann surface is a connected Hausdorff space W, together with a collection of open subsets $U_\alpha \subset W$ and functions $z_\alpha : U_\alpha \to \mathbb{C}$ such that

(i) $W = \cup U_\alpha$

(ii) z_α is a homeomorphism of U_α onto the unit disk $\mathbb{D}$, and

(iii) if $U_\alpha \cap U_\beta \neq \emptyset$ then $z_\beta \circ z_\alpha^{-1}$ is analytic on $z_\alpha(U_\alpha \cap U_\beta)$.

The functions z_α are called coordinate functions, and the sets U_α are called coordinate disks. We can extend our collection of coordinate functions and disks to form a base for the topology on W by setting $U_{\alpha,r} = z_\alpha^{-1}(r\mathbb{D})$ and $z_{\alpha,r} = \frac{1}{r} z_\alpha|_{U_{\alpha,r}}$, for $r < 1$. We can also compose each z_α with a Möbius transformation of the disk onto itself so that we can assume that for each coordinate disk U_α and each $p_0 \in U_\alpha$ there is a coordinate function z_α with $z_\alpha(p_0) = 0$. For the purposes of the discussion below, we will assume that our collection $\{z_\alpha, U_\alpha\}$ includes all such maps, except that we will further assume that each U_α has compact closure in W, by restricting our collection.

A Riemann surface W is called simply connected if every closed curve in W is homotopic to a point. A function $f : W \to \mathbb{C}$ is called analytic if for every coordinate function z_α, the function $f \circ z_\alpha^{-1}$ is analytic on $\mathbb{D}$. Harmonic, subharmonic, and meromorphic functions on W are defined in a similar way. Note that by (ii) and (iii) the functions $z_\beta \circ z_\alpha^{-1}$ are one-to-one analytic maps. This means that any theorem about plane domains whose proof depends solely on the local behavior of functions is also a valid theorem about Riemann surfaces. For example, the monodromy theorem (see e.g. Ahlfors[A1]) holds for Riemann surfaces, and so does its corollary that every harmonic function on a simply connected Riemann surface has the form $u = \text{Re} f$ for some analytic function f defined on W. The Perron process also works on a Riemann surface. A Riemann surface W is pathwise connected since the set of points than can be connected to p_0 is open for each $p_0 \in W$.

Suppose W is a Riemann surface. Fix $p_0 \in W$ and let $z : U \to \mathbb{D}$ be a coordinate function such that $z(p_0) = 0$. Let $\mathcal{F}_{p_0}$ be the collection of subharmonic functions on $W \setminus p_0$ satisfying

$$(a) \quad v = 0 \text{ on } W \setminus K, \text{ for some compact } K \subset W \text{ with } K \neq W, \text{ and}$$

$$(b) \quad \limsup_{p \to p_0} \big(v(p) + \log |z(p)|\big) < \infty$$

Set

$$g_W(p, p_0) = \sup\{v(p) : v \in \mathcal{F}_{p_0}\}. \tag{1}$$

Condition (b) does not depend on the choice of the coordinate function z_α, provided $z_\alpha(p_0) = 0$. The collection $\mathcal{F}_{p_0}$ is a Perron family, so one of the following two cases holds by Harnack's Theorem:

Case 1: $g_W(p, p_0)$ is harmonic in $W \setminus \{p_0\}$, or

Case 2: $g_W(p, p_0) = +\infty$ for all $p \in W \setminus \{p_0\}$.

In the first case, $g_W(p, p_0)$ is called Green's function with pole (or logarithmic singularity) at p_0. In the second case we say that Green's function does not exist.

In this note we give an essentially self contained proof of the following result.

The Uniformization Theorem (Koebe[1907]). *Suppose W is a simply connected Riemann surface. If Green's function exists for W, then there is a one-to-one analytic map of W onto $\mathbb{D}$. If W is compact, then there is a one-to-one analytic map of W onto $\mathbb{C}^*$. If W is not compact and if Green's function does not exist for W, then there is a one-to-one analytic map of W onto $\mathbb{C}$.*

First we will prove some facts about Green's function on Riemann surfaces, then we will prove the Uniformization Theorem in Case 1, when Green's function exists. Then we give a similar proof of the Uniformization Theorem in Case 2, using the "Dipole" Green's function, assuming it exists. Finally we show that the Dipole Green's function exists on every Riemann surface.

Lemma 1. *Suppose $p_0 \in W$ and suppose $z : U \to \mathbb{D}$ is a coordinate function such that $z(p_0) = 0$. If $g_W(p, p_0)$ exists then,*

$$g_W(p, p_0) > 0 \text{ for } p \in W \setminus \{p_0\}, \text{ and} \tag{2}$$

$$g_W(p, p_0) + \log|z(p)| \text{ extends to be harmonic in } U. \tag{3}$$

Proof. The function

$$v_0(p) = \begin{cases} -\log|z(p)| & \text{for } p \in U \\ 0 & \text{for } p \in W \setminus U \end{cases}$$

is in $\mathcal{F}_{p_0}$. Hence $g_W(p, p_0) \geq 0$ and $g_W(p, p_0) > 0$ if $p \in U$. By the maximum principle in $W \setminus \{p_0\}$ (2) holds.

If $v \in \mathcal{F}_{p_0}$ and $\varepsilon > 0$, then $v + (1+\varepsilon)\log|z|$ extends to be subharmonic in U, and equal to $-\infty$ at p_0. Thus

$$\sup_U \big(v + (1+\varepsilon)\log|z|\big) = \sup_{\partial U} v \leq \sup_{\partial U} g_W < \infty.$$

Taking the supremum over $v \in \mathcal{F}_{p_0}$ and sending $\varepsilon \to 0$, we obtain

$$g_W + \log|z| \leq \sup_{\partial U} g_W < \infty$$

in $U \setminus \{p_0\}$. We also have that

$$g_W + \log|z| \geq v_0 + \log|z| \geq 0 \tag{4}$$

for $p \in U \setminus \{p_0\}$. Thus p_0 is a removable singularity for the harmonic function $g_W + \log|z|$ and (3) holds. $\square$

The Green's function for the unit disk $\mathbb{D}$ is given by

$$g_{\mathbb{D}}(z, a) = \log\left|\frac{1 - \overline{a}z}{z - a}\right|.$$

Indeed by the maximum principle, each candidate subharmonic function v in the Perron family $\mathcal{F}_a$ is bounded by $(1+\varepsilon)g_{\mathbb{D}}(z, a)$, for $\varepsilon > 0$. Moreover $\max(g_{\mathbb{D}}(z, a) - \varepsilon, 0) \in \mathcal{F}_a$, when $\varepsilon > 0$.

The next lemma gives a large collection of Riemann surfaces for which Green's function exists.

Lemma 2. *Suppose W_0 is a Riemann surface and suppose U_0 is a coordinate disk whose closure is compact in W_0. Set $W = W_0 \setminus \overline{U_0}$. Then $g_W(p, p_0)$ exists for all $p, p_0 \in W$ with $p \neq p_0$.*

Proof. Fix $p_0 \in W$ and let $U \subset\subset W$ be a coordinate disk with coordinate function $z : U \to \mathbb{D}$ and $z(p_0) = 0$. To prove that g_W exists, we show that the family $\mathcal{F}_{p_0}$ is bounded above. Fix r, with $0 < r < 1$, and set $rU = \{p \in W : |z(p)| < r\}$. If $v \in \mathcal{F}_{p_0}$ and $\varepsilon > 0$ then by the maximum principle

$$v(p) + (1+\varepsilon)\log|z(p)| \leq \max_{q \in \partial U}\big(v(q) + (1+\varepsilon)\log|z(q)|\big) = \max_{q \in \partial U} v(q),$$

for all $p \in U$. As before, letting $\varepsilon \to 0$, we obtain

$$\max_{p \in \partial rU} v(p) + \log r \leq \max_{p \in \partial U} v(p). \tag{5}$$

Let $\omega(p) = \omega(p, \partial rU, W \setminus \overline{rU})$ be the Perron solution to the Dirichlet problem on $W \setminus \overline{rU} = W_0 \setminus \{\overline{U_0} \cup \overline{rU}\}$ for the boundary data which equals 1 on ∂rU and equals 0 on ∂U_0. In other words, we let $\mathcal{F}$ denote the collection of functions u which are subharmonic on $W \setminus rU$ with $u = 0$ on $W \setminus K$ for some compact set K, depending on u, and such that

$$\limsup_{p \to \zeta} u(p) \leq 1$$

for $\zeta \in \partial rU$. Then

$$\omega(p) = \sup\{u(p) : u \in \mathcal{F}\}.$$

By the Perron process, ω is harmonic in $W \setminus U$. Regularity for the Dirichlet problem is a local question. Indeed we can define a local barrier at each point of the boundary of rU and at each point of the boundary of U_0. Thus the harmonic function ω extends to be continuous at each point of ∂U_0 and each point of ∂rU so that $\omega(p) = 0$ for $p \in \partial U_0$ and

$\omega(p) = 1$ for $p \in \partial rU$. This implies ω is not constant and $0 < \omega(p) < 1$ for $p \in W \setminus rU$. Returning to our function $v \in \mathcal{F}_{p_0}$, by the maximum principle we have that

$$v(p) \leq \left(\max_{\partial rU} v\right)\omega(p)$$

for $p \in W \setminus rU$ since $v = 0$ off a compact subset of W. So

$$\max_{\partial U} v \leq \left(\max_{\partial rU} v\right)\max_{p\in\partial U}\omega(p) \leq \left(\max_{\partial rU} v\right)(1-\delta) \tag{6}$$

for some $\delta > 0$. Adding inequalities (5) and (6) yields

$$\delta \max_{\partial rU} v \leq \log\frac{1}{r}$$

for all $v \in \mathcal{F}$, with δ is independent of v. This implies that Case 2 does not hold and hence Green's function exists. □

The next Lemma relates Green's function on a Riemann surface to Green's function on its universal cover.

Lemma 3. *Suppose W is a Riemann surface for which g_W exists. Let W^* be a simply connected universal covering surface of W and let $\pi : W^* \to W$ be the universal covering map. Then g_{W^*} exists and satisfies*

$$g_W(\pi(p^*), \pi(p_0^*)) = \sum_{q^*:\pi(q^*)=\pi(p_0^*)} g_{W^*}(p^*, q^*). \tag{7}$$

We interpret the infinite sum on the right side of (7) to be the supremum of all sums over finitely many q^*.

Proof. Suppose $q_1^*, \ldots, q_n^*$ are distinct points in W^* with $\pi(q_j^*) = p_0 = \pi(p_0^*)$. Suppose $v_j \in \mathcal{F}_{q_j^*}$, the Perron family for the construction of $g_{W^*}(\cdot, q_j^*)$. So $v_j = 0$ off K_j^*, a compact subset of W^* and

$$\limsup_{p^*\to q_j^*}\bigl(v(p^*) + \log|z\circ\pi(p^*)| < \infty,$$

where z is a coordinate chart on W with $z(p_0) = 0$. Recall that $g_W(p, p_0) + \log|z(p)|$ extends to be finite and continuous at p_0, and hence

$$\lim_{p^* \to q_j^*} g_W(\pi(p^*), p_0) + \log|z(\pi(p^*))|$$

exists and is finite where $\pi(q_j^*) = p_0$. Thus for $\varepsilon > 0$,

$$\left(\sum_{j=1}^{n} v_j(p^*)\right) - (1+\varepsilon) g_W(\pi(p^*), p_0)$$

extends to be subharmonic and equal to $-\infty$ at q_j^*, for $j = 1, \ldots, n$, and less than or equal to 0 off $\cup_j K_j^*$. By the maximum principle, it is bounded above by 0 and letting $\varepsilon \to 0$ and taking the supremum over all such v we conclude that $g_{W^*}(p^*, q_j^*)$ exists and

$$\sum_{j=1}^{n} g_{W^*}(p^*, q_j^*) \leq g_W(\pi(p^*), p_0).$$

Taking the supremum over all such finite sums we have

$$S(p^*) \equiv \sum_{q^*; \pi(q^*) = p_0} g_{W^*}(p^*, q^*) \leq g_W(\pi(p^*), p_0).$$

Moreover, as a supremum of finite sums of positive harmonic functions,

$$S(p^*) + \log|z(\pi(p^*))|$$

is harmonic in a neighborhood of each q_j^* by Harnack's Theorem.

Now take $v \in \mathcal{F}_{p_0}$, the Perron family used to construct $g_W(p, p_0)$. Let U^* be a coordinate disk containing p_0^* such that $z \circ \pi$ is a coordinate function mapping U^* onto $\mathbb{D}$. Then by the maximum principle

$$v(\pi(p^*)) - (1+\varepsilon) S(p^*) \leq 0$$

for $p^* \in U^*$ and $\varepsilon > 0$. Taking the supremum over all such v and letting $\varepsilon \to 0$, we obtain

$$g_W(\pi(p^*), p_0) \leq S(p^*)$$

and thus (7) holds. $\square$

What we have proved so far works for any Riemann surface. We will now consider simply connected Riemann surfaces and prove the Uniformization Theorem in Case 1.

Theorem 4. *If W is a simply connected Riemann surface then the following are equivalent:*

$$g_W(p,p_0) \text{ exists for some } p_0 \in W \tag{8}$$

$$g_W(p,p_0) \text{ exists for all } p_0 \in W, \tag{9}$$

$$\text{There is a one-to-one analytic map } \varphi \text{ from } W \text{ onto } \mathbb{D}. \tag{10}$$

Moreover if g_W exists, then

$$g_W(p_1,p_0) = g_W(p_0,p_1), \tag{11}$$

and $g_W(p,p_0) = -\log|\varphi(p)|$, where $\varphi(p_0) = 0$.

Proof. Suppose there is a one-to-one analytic map φ of W onto $\mathbb{D}$ and let $p_0 \in W$. By composing φ with a Möbius transformation, we can assume that $\varphi(p_0) = 0$. If $v \in \mathcal{F}_{p_0}$ and if $\varepsilon > 0$, then by (b)

$$v + (1+\varepsilon)\log|\varphi|$$

is subharmonic in W and equal to $-\infty$ at p_0. By the maximum principle, since $v = 0$ off a compact subset $K \subset W$,

$$v + (1+\varepsilon)\log|\varphi| \le 0$$

on W. Taking the supremum over all such v and letting $\varepsilon \to 0$, shows that $g_W(p,p_0) < \infty$ and therefore (9) holds. Clearly (9) implies (8).

Now suppose (8) holds. By (3) there is an analytic function f defined on a coordinate disk U containing p_0 so that

$$\mathrm{Re} f(p) = g_W(p,p_0) + \log|z(p)|$$

for $p \in U$. Hence the function

$$\varphi(p) = ze^{-f(p)}$$

is analytic in U and satisfies $|\varphi(p)| = e^{-g_W(p,p_0)}$ and $\varphi(p_0) = 0$. On any coordinate disk U_α with $p_0 \notin U_\alpha$, $g_W(p,p_0)$ is the real part of an analytic function. Thus by the monodromy theorem, there is a function φ, analytic on W, such that

$$|\varphi(p)| = e^{-g_W(p,p_0)} < 1.$$

We claim that φ is one-to-one. If $\varphi(p) = \varphi(p_0) = 0$, then clearly $p = p_0$. Let $p_1 \in W$, with $p_1 \neq p_0$. Then by (2), $|\varphi(p_1)| < 1$ and

$$\varphi_1 \equiv \frac{\varphi - \varphi(p_1)}{1 - \overline{\varphi(p_1)}\varphi}$$

is analytic on W and $|\varphi_1| < 1$. If $v \in \mathcal{F}_{p_1}$, then by the maximum principle

$$v + (1+\varepsilon)\log|\varphi_1| \leq 0.$$

Taking the supremum over all such v and sending $\varepsilon \to 0$, we see that $g_W(p,p_1)$ exists and that

$$g_W(p,p_1) + \log|\varphi_1| \leq 0. \tag{12}$$

Setting $p = p_0$ in (12) gives

$$g_W(p_0,p_1) \leq -\log|\varphi_1(p_0)| = -\log|\varphi(p_1)| = g_W(p_1,p_0).$$

Switching the roles of p_0 and p_1 gives (11). Moreover equality holds in (12) at $p = p_0$ so that $g_W(p,p_1) = -\log|\varphi_1(p)|$ for all $p \in W \setminus \{p_1\}$. Now if $\varphi(p_2) = \varphi(p_1)$, then by the definition $\varphi_1(p_2) = 0$ and thus $g_W(p_2,p_1) = \infty$ and $p_2 = p_1$. Therefore φ is one-o-one. The image $\varphi(W) \subset \mathbb{D}$ is simply connected, so if $\varphi(W) \neq \mathbb{D}$ then by the Riemann Mapping Theorem we can find a one-to-one analytic map ψ of $\varphi(W)$ onto $\mathbb{D}$ with $\psi(0) = 0$. The map $\psi \circ \varphi$ is then a one-to-one analytic map of W onto $\mathbb{D}$, with $\psi \circ \varphi(p_0) = 0$, proving (10). □

We now consider Riemann surfaces which do not have a Green's function. As seen above, Green's function for the disk with pole at 0 is given by $G = -\log|z|$. There is no Green's function on the sphere or the plane, but this same function G plays a similar role. Instead of one pole, or logarithmic singularity, G has two poles on the sphere, with opposite signs. We will call it a Dipole Green's function. The next lemma says that a Dipole Green's function exists for every Riemann surface. For a simply connected Riemann surface without Green's function, the Dipole Green's function will be used to construct a conformal map to the sphere or the plane in much the same way as Green's function was used in Case 1.

Lemma 5. *Suppose W is a Riemann surface and for $j = 1, 2$, suppose that $z_j : U_j \to \mathbb{D}$ are coordinate functions with coordinate disks U_j satisfying $\overline{U_1} \cap \overline{U_2} = \emptyset$, and $z_j(p_j) = 0$. Then there is a function $G(p) \equiv G(p, p_1, p_2)$, harmonic in $p \in W \setminus \{p_1, p_2\}$ such that*

$$G + \log|z_1| \text{ extends to be harmonic in } U_1, \tag{13}$$

$$G - \log|z_2| \text{ extends to be harmonic in } U_2, \tag{14}$$

and

$$\sup_{p \in W \setminus \{U_1 \cup U_2\}} |G(p)| < \infty. \tag{15}$$

Before proving Lemma 5, we will use it to prove the Uniformization Theorem in Case 2, since the proof is similar to the proof in Case 1.

Proof of the Uniformization Theorem in Case 2. By Theorem 4, we may suppose that $g_W(p, p_1)$ does not exist for all $p, p_1 \in W$. By the monodromy theorem, as in the proof of Theorem 4, there is a meromorphic function φ defined on W such that

$$|\varphi(p)| = e^{-G(p,p_1,p_2)}.$$

Note that φ has a simple zero at p_1, a simple pole at p_2 and no other zeros or poles.

Let us prove φ is one-to-one. Take $p_0 \in W \setminus \{p_1, p_2\}$, let φ_1 be the meromorphic function on W such that

$$|\varphi_1(p)| = e^{-G(p,p_0,p_2)}$$

and consider the function

$$H(p) = \frac{\varphi(p) - \varphi(p_0)}{\varphi_1(p)}.$$

Then H is analytic on W because its poles at p_2 cancel and because φ_1 has a simple zero at p_0. By (15) and the analyticity of H, $|H|$ is bounded on W. But if $v \in \mathcal{F}_{p_1}$, the Perron family used to construct $g_W(p, p_1)$, and if $\varepsilon > 0$, then by the maximum principle

$$v(p) + (1+\varepsilon)\log\left|\frac{H(p) - H(p_1)}{2\sup_W |H|}\right| \leq 0.$$

Since $g_W(p, p_1)$ does not exist, $\sup\{v(p) : v \in \mathcal{F}_{p_1}\} \equiv +\infty$, and therefore

$$H(p) \equiv H(p_1) = -\varphi(p_0)/\varphi_1(p_1) \neq 0, \infty.$$

From the definition of H, if $\varphi_1(p)$ is finite and not zero, then $\varphi(p) \neq \varphi(p_0)$ since H is a non-zero constant. If $\varphi_1(p) = 0$ then $p = p_0$ from the definition of φ_1. Finally φ_1 has a pole only at p_2. But φ also has a pole at p_2, and only at p_2, and thus $\varphi(p_2) \neq \varphi(p_0)$. We have proved that $\varphi(p) = \varphi(p_0)$ only if $p = p_0$. Since p_0 is arbitrary, this proves that φ is one-to-one.

We have shown that φ is a one-to-one analytic map from W to a simply connected region $\varphi(W) \subset \mathbb{C}^*$. If $\mathbb{C}^* \setminus \varphi(W)$ contains more than one point, then by the Riemann mapping Theorem, there is a one-to-one analytic map of $\varphi(W)$, and hence of W, onto $\mathbb{D}$. Since we assumed that g_W does not exist, this contradicts Theorem 4. Thus $\mathbb{C}^* \setminus \varphi(W)$ contains at most one point, and the last two statements of Theorem 6 are now obvious. □

One consequence of the proof of the Uniformization Theorem in Case 1 is the symmetry property of Green's function in Corollary 6. This property will be used in the proof of Lemma 5.

Corollary 6. *Suppose W is a Riemann surface for which Green's function $g_W(p,q)$ exists, for some $p, q \in W$, with $p \neq q$. Then $g_W(p,q)$ exists for all $p, q \in W$ with $p \neq q$ and*

$$g_W(p,q) = g_W(q,p). \tag{16}$$

Proof. By Lemma 3 and Theorem 4, we may suppose that $\mathbb{D}$ is a universal cover of W. In this case $g_{\mathbb{D}}(a,b) = -\log|(a-b)/(1-\bar{b}a)|$. Note that if τ is a Möbius transformation of the disk onto the disk, then $g_{\mathbb{D}}(a,\tau(b)) = g_{\mathbb{D}}(\tau^{-1}(a),b)$ for all $a,b\in\mathbb{D}$ with $a\neq b$. Let $\mathcal{G}$ denote the group of deck transformations, a group of Möbius transformations τ satisfying $\pi\circ\tau=\pi$ and if $\pi(q^*)=\pi(p^*)$ then there is a $\tau\in\mathcal{G}$ such that $\tau(p^*)=q^*$. Suppose $g(p,p_0)$ exists for some $p_0\in W$ and all $p\neq p_0$. Choose $p_0^*\in\mathbb{D}$ so that $\pi(p_0^*)=p_0$. By Lemma 3 for $p* \neq p_0^*$

$$
\begin{aligned}
g_W(\pi(p^*),\pi(p_0^*)) &= \sum_{\tau\in\mathcal{G}} -\log\left|\frac{p^*-\tau(p_0^*)}{1-\overline{\tau(p_0^*)}p^*}\right| \\
&= \sum_{\tau\in\mathcal{G}} -\log\left|\frac{\tau^{-1}(p^*)-p_0^*}{1-\overline{\tau^{-1}(p^*)}p_0^*}\right|. \qquad (17)
\end{aligned}
$$

Fix $p^*\in\mathbb{D}$. Each term in the sum

$$S(q^*) = \sum_{\tau\in\mathcal{G}} -\log\left|\frac{\tau^{-1}(p^*)-q^*}{1-\overline{\tau^{-1}(p^*)}q^*}\right|$$

is a positive harmonic function of $q^*\in\mathbb{D}\setminus\tau^{-1}(p^*)$. Since the sum of positive harmonic functions converges at $q^*=p_0^*$, the function S is harmonic in $\mathbb{D}\setminus\{\tau^{-1}(p^*):\tau\in\mathcal{G}\}$, by Harnack's Theorem. If $v\in\mathcal{F}_p$, the Perron family for $g_W(q,p)$ where $p=\pi(p^*)$, then by the maximum principle $v\leq(1+\varepsilon)S\circ\pi^{-1}$ for all $\varepsilon>0$, and letting $\varepsilon\to0$ and taking the supremum over all $v\in\mathcal{F}_p$ we conclude that $g_W(q,p)$ exists for all $q\neq p$. Equation (16) then follows from Lemma 3 and (17). □

We now complete the proof of the Uniformization Theorem by proving Lemma 5.

Proof of Lemma 5. Suppose z_0 is a coordinate function with coordinate chart U_0 such that $\overline{U_0}\cap\overline{U_j}=\emptyset$ for $j=1,2$. Let p_0 be the point in W such that $z_0(p_0)=0$. Set $tU_0=\{p\in W:|z_0(p)|<t\}$ and set $W_t=W\setminus tU_0$. By Lemma 2 and Theorem 4, $g_{W_t}(p,p_1)$ exists for all $p,p_1\in W_t$ with $p\neq p_1$. Fix r, $0<r<1$, and set $rU_1=\{p\in W:|z_1(p)|<r\}$. By the maximum principle

$$g_{W_t}(p,p_1)\leq M_1(t)\equiv\max_{q\in\partial rU_1} g_{W_t}(q,p_1), \qquad (18)$$

for all $p \in W_t \setminus rU_1$, because the same bound holds for all candidates in the Perron family defining g_{W_t}. By (5)

$$M_1(t) \le \max_{p \in \partial U_1} g_{W_t}(p, p_1) + \log \frac{1}{r}. \tag{19}$$

By (18), $u_t(p) \equiv M_1(t) - g_{W_t}(p, p_1)$ is a positive harmonic function in $W_t \setminus rU_1$ and by (19) there exists $q \in \partial U_1$ with $u_t(q) \le \log \frac{1}{r}$. Riemann surfaces are pathwise connected so that if K is a compact subset of $W_1 \setminus rU_1$ containing $\{p_2\} \cup \partial rU_1$, then by Harnack's inequality there is a constant $C < \infty$ depending on K and r but not on t, so that for all $p \in K$

$$0 \le u_t(p) \le C,$$

and

$$|g_{W_t}(p, p_1) - g_{W_t}(p_2, p_1)| = |u_t(p_2) - u_t(p)| \le 2C.$$

Likewise, if K' is a compact subset of $W_1 \setminus \{|z_2| < r\}$ containing $\{p_1\} \cup \partial rU_2$, there is a constant $C < \infty$ so that

$$|g_{W_t}(p, p_2) - g_{W_t}(p_1, p_2)| \le C,$$

for all $p \in K'$.

By Corollary 6, $g_{W_t}(p_1, p_2) = g_{W_t}(p_2, p_1)$ and so the function

$$\begin{aligned} G_t(p, p_1, p_2) &\equiv g_{W_t}(p, p_1) - g_{W_t}(p, p_2) \\ &= (g_{W_t}(p, p_1) - g_{W_t}(p_2, p_1)) - (g_{W_t}(p, p_2) - g_{W_t}(p_1, p_2)) \end{aligned}$$

is harmonic in $W_t \setminus \{p_1, p_2\}$ and satisfies

$$|G_t(p, p_1, p_2)| \le C,$$

for all $p \in K \cap K'$ and some finite C independent of t. We may suppose, for instance, that $K \cap K'$ contains $\partial U_1 \cup \partial U_2$. If $v \in \mathcal{F}_{p_1}$, the Perron family for $g_{W_t}(p, p_1)$, then since $v = 0$ off a compact subset of W_t and since $g_{W_t} > 0$, by the maximum principle

$$\begin{aligned} \sup_{W_t \setminus U_1} v(p) - g_{W_t}(p, p_2) &\le \max(0, \sup_{\partial U_1} v(p) - g_{W_t}(p, p_2)) \\ &\le \max(0, \sup_{\partial U_1} g_{W_t}(p, p_1) - g_{W_t}(p, p_2)) \le C, \end{aligned}$$

and taking the supremum over all such v yields

$$\sup_{p \in W_t \setminus U_1} G(p, p_1, p_2) \leq C.$$

Similarly

$$\inf_{p \in W_t \setminus U_2} G_t(p, p_1, p_2) = - \sup_{p \in W_t \setminus U_2} -G_t(p, p_1, p_2) \geq -C,$$

and

$$|G_t(p, p_1, p_2)| \leq C$$

for all $p \in W_t \setminus \{U_1 \cup U_2\}$. The function $G_t + \log |z_1|$ extends to be harmonic in U_1, so by the maximum principle, we have that

$$\sup_{U_1} |G_t + \log |z_1|| = \sup_{\partial U_1} |G_t + \log |z_1|| = \sup_{\partial U_1} |G_t| \leq C.$$

Similarly

$$\sup_{U_2} |G_t - \log |z_2|| = \sup_{\partial U_2} |G_t - \log |z_2|| = \sup_{\partial U_2} |G_t| \leq C.$$

By normal families, there exists a sequence $t_n \to 0$ so that G_{t_n} converges uniformly on compact subsets of $W \setminus \{p_0, p_1, p_2\}$ to a function $G(p, p_1, p_2)$ satisifying (13), (14), and (15). The function $G(p, p_1, p_2)$ extends to be harmonic at p_0 because it is bounded in a punctured neighborhood of p_0. $\square$

Since each of the spaces $\mathbb{D}, \mathbb{C}$, and $\mathbb{C}^*$ is second countable, we have the following consequence.

Corollary 7. *Every simply connected Riemann surface satisfies the second axiom of countability.*

Comments:

1. It is possible to avoid the use of the Riemann Mapping Theorem (RMT) and then RMT is a consequence of Theorem 6. In doing so, we use ideas from other proofs of that theorem.

There are two places RMT is used. The first is at the end of the proof in Case I, where we obtain a conformal map φ of W onto a possibly proper subset of $\mathbb{D}$. Here is how to prove that φ is onto:

If there exists $a \in \mathbb{D} \setminus \varphi(W)$, then $(z-a)/(1-\overline{a}z)$ is a non-vanishing function on the simply connected region $\varphi(W)$ and hence we can define an analytic square root

$$\tau(z) = \sqrt{\frac{z-a}{1-\overline{a}z}}$$

for $z \in \varphi(W)$. Set

$$\sigma(z) = \frac{\tau(z) - \tau(0)}{1 - \overline{\tau(0)}\tau(z)}.$$

Then $\sigma \circ \varphi$ is an analytic function mapping W into $\mathbb{D}$. Moreover σ^{-1} is an analytic function defined on all of $\mathbb{D}$, mapping $\mathbb{D}$ into $\mathbb{D}$ with $\sigma^{-1}(0) = 0$ and it is not a rotation. By Schwarz's lemma $|\sigma^{-1}(z)| < |z|$ and so

$$|\sigma \circ \varphi(z)| > |\varphi(z)|. \tag{20}$$

By the argument used to establish (9) and (11), $g_W + \log|\sigma \circ \varphi| \leq 0$. But $g_W(p, p_0) = -\log|\varphi(p)|$, so that $\log|\sigma \circ \varphi| \leq \log|\varphi|$, contradicting (20). □

The idea to apply Schwarz's lemma in the context of the proof of the Riemann Mapping Theorem, I learned from a student, Will Johnson.

The second place where RMT is used is at the conclusion of the proof in Case 2, where there is no Green's function. We proved that there is a conformal map φ of W onto a subset of $\mathbb{C}^*$. If $\varphi(W)$ omitted at least two points in $\mathbb{C}^*$, then we used the RMT to prove there is a conformal map onto the disk and thus contradict Theorem 4. To avoid RMT, we can simply use a part of its proof. If there are at least two points $a, b \in \mathbb{C}^* \setminus \varphi(W)$ then

$$\psi(z) = \frac{z-a}{z-b}$$

is a nonvanishing analytic function on the simply connected region $\varphi(W)$, and hence we can define an analytic square root $\sigma(z) = \sqrt{\psi(z)}$. It is not hard to check that ψ is one-to-one, and if $\psi(\varphi(W))$ covers a neighborhood of z_1, $z_1 \neq 0$, then $\psi(\varphi(W))$ omits a neighborhood of $-z_1$. We can then apply a linear fractional transformation σ so that $\sigma(\psi(\varphi(W))) \subset \mathbb{D}$. If $g_{\mathbb{D}}$ is Green's function on $\mathbb{D}$ then the Perron family $\mathcal{F}_{p_0}$ for constructing Green's function on W is bounded by $g_{\mathbb{D}}(f(p), f(p_0))$ where $f = \sigma \circ \psi \circ \varphi$. Thus Green's function exists, contradicting our assumption.

2. It would be nice to avoid the use of universal covering surfaces. One need only prove that Green's function is symmetric for $W \setminus U$ where W is simply connected and U is a coordinate disk. One can do the construction of the covering space in this special case for example. If it is desired to prove the more general Uniformation Theorem for all Riemann surfaces, then maybe constructing the universal cover in this special case would be good motivation for later doing it in general. This is the approach I generally use for my class. An alternative that I've used is to just do all the topology of covering spaces and deck transformations before beginning the uniformization theorem. This will highlight the importance of simply connected surfaces before proving the theorem.

3. In the proof, it is enough to consider only subharmonic functions that are continuous. At several places in the proof we have allowed $-\infty$ as an isolated value for a subharmonic function. In each such place, the maximim principle can be applied instead to the region with a puncture at such a point.

4. By the last part of the proof of Lemma 3, a converse to Lemma 3 holds: If g_{W^*} exists and if the sum in (7) is finite, then g_W exists. However, there are Riemann surfaces for which $W^* = \mathbb{D}$, so that g_{W^*} exists, yet the sum in (7) is infinite. The Fuchsian group is then said to be of Divergence type. In this case g_W does not exist. For example, $W = \mathbb{C} \setminus \{0, 1\}$.

5. The symmetry of Green's function $g(a, b) = g(b, a)$ can also be proved via Green's Theorem on Riemann surfaces. Then Lemma 3 and Corollary 6 can be eliminated. But that approach seems to involve more work to include all the details.

6. [A1] Ahlfors, Complex Analysis, has a proof of the Monodromy Theorem that easily extends to Riemann surfaces. There is also a very general version (more general than is needed here) for Riemann surfaces in Ahlfors, Conformal Invariants.