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Traps for reflected Brownian motion

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Abstract. Consider an open set $D \subset \mathbb{R}^d$, $d \geq 2$, and a closed ball $B \subset D$. Let $\mathbb{E}^x T_B$ denote the expectation of the hitting time of B for reflected Brownian motion in D starting from $x \in D$. We say that D is a trap domain if $\sup_x \mathbb{E}^x T_B = \infty$. A domain D is not a trap domain if and only if the reflecting Brownian motion in D is uniformly ergodic. We fully characterize the simply connected planar trap domains using a geometric condition and give a number of (less complete) results for $d > 2$.

Mathematics Subject Classifications (2000): 60J45, 35P05, 60G17

1. Introduction

In this section, we will limit ourselves to an informal statement of the problem and a brief review of our results. See Section 2 for rigorous statements of the theorems and Section 3 for the proofs.

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be an open connected set with a finite volume and let X be the normally reflected Brownian motion (RBM) on $\overline{D}$ constructed using Dirichlet form methods (see section 2 for details). Note that X is well-defined for every starting point in D and for $x \in D$ we let $\mathbb{P}^x$ denote the distribution of X_t starting from $X_0 = x$, with corresponding expectation $\mathbb{E}^x$. Let $B \subset D$ be a closed ball with non-zero radius and denote by $T_B = \inf\{t \geq 0 : X_t \in B\}$ the first hitting time of B by X. If $\mathbb{E}^x T_B$ is very large for some x then RBM starting from x appears to be trapped near the boundary of D. We will say that $D \subset \mathbb{R}^d$, $d \geq 2$, is a *trap domain* if

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$$\sup_{x \in D} \mathbb{E}^x T_B = \infty, \tag{1.1}$$

and otherwise D is called a *non-trap domain*. The definition of a trap domain does not depend on the choice of B (see Lemma 3.3 in the last section). At this point, the reader might like to consult Proposition 2.13, Proposition 2.11, and Figure 3.2 below for some simple examples of trap and non-trap domains.

Our article is mainly devoted to the following problem.

Problem 1.1. *Find necessary and sufficient geometric conditions for D to be a trap domain.*

The notion of a trap domain is closely related to the notion of Markov chain ergodicity (see [MT], Part III). We will make this remark more precise in the next proposition. Let $\|\mu\|_{TV}$ denote the total variation norm of a measure μ. When $\mu = f(x)\,dx$, then $\|\mu\|_{TV} = \int |f(x)|\,dx$. Let Π_D denote the uniform probability measure in D.

Proposition 1.2. *Let $D \subset \mathbb{R}^d$ be a connected open set with finite volume. Then the following are equivalent.*

(i) D is non-trap.
(ii) $\lim_{t\to\infty} \sup_{x\in D} \|\mathbb{P}^x(X_t \in \cdot) - \Pi_D\|_{TV} = 0$.
(iii) There are positive constants c_1 and c_2 such that $\sup_{x\in D} \|\mathbb{P}^x(X_t \in \cdot) - \Pi_D\|_{TV} \le c_1 e^{-c_2 t}$.

Properties (ii) and (iii) are called the uniform ergodicity of reflecting Brownian motion in D. The above equivalence is proved for discrete time Markov chains in Theorem 16.0.2 (ii) and (vi) of [MT].

It will be convenient to express Problem 1.1 in purely analytic terms. Let $G(x, y)$ be defined on $(D \setminus B) \times (D \setminus B)$ by

$$\int_{(D\setminus B)\cap A} G(x, y)dy = \mathbb{E}^x \int_0^{T_B} \mathbf{1}_{\{X_t \in A\}} dt, \qquad A \subset \overline{D},$$

where dy denotes d-dimensional Lebesgue measure. In other words, $G(x, y)$ is the Green function for the domain $D \setminus B$ with the (zero) Neumann boundary conditions on ∂D (in the distributional sense) and (zero) Dirichlet boundary conditions on ∂B. The existence of $G(x, y)$ follows from a result in [Fu] saying that there exists a strictly positive function $p_t(x, y)$ on $(0, \infty) \times D \times D$ such that for every $x \in D$ and $A \subset \overline{D}$,

$$\mathbb{P}^x(X_t \in A) = \int_{D\cap A} p_t(x, y)dy,$$

(see Section 3.1 below for details). We will call $p_t(x, y)$ the Neumann heat kernel on D. From a technical point of view, it is easier to define the Green function with the specified boundary conditions than the corresponding RBM. The condition

$$\sup_{x\in D\setminus B} \int_{D\setminus B} G(x, y)dy = \infty, \tag{1.2}$$

is equivalent to (1.1) but avoids some thorny questions related to the construction of RBM. For example, if D is a simply connected planar domain, $G(x, y)$ can be constructed in an elementary way using a conformal mapping and the reflection principle (see (3.2)). Problem 1.1 can be expressed as

Problem 1.3. *Find necessary and sufficient geometric conditions for D so that (1.2) holds.*

We will give a complete solution to Problems 1.1 and 1.3 in the case of finitely connected planar domains. This result will allow us to analyze explicitly several examples, it will provide clues to finding trap domains among non-finitely connected and higher dimensional domains, and it will indicate technical difficulties that one is likely to encounter while dealing with domains in $\mathbb{R}^d$, $d > 2$.

Problems 1.1 and 1.3 are closely related to some other potential theoretic questions. Recall that $p_t(x, y)$ denotes the heat kernel for D with the Neumann boundary conditions. We will say that the *parabolic Harnack principle* (PHP in short) holds in D if for some $t_0 > 0$, $c_1 = c_1(D, t_0) < \infty$,

$$p_t(x, y) \le c_1\, p_t(v, z) \quad \text{for all } t \ge t_0 \text{ and } v, x, y, z \in D. \tag{1.3}$$

It is not hard to show that (1.3) is equivalent to the existence of $c_2 < \infty$ and $c_3 > 0$ such that for some $t_1 > 0$ and all $t \ge t_1$,

$$\sup_{x,y\in D} \left| p_t(x, y) - \frac{1}{\mathrm{Vol}(D)} \right| \le c_2 e^{-c_3 t}. \tag{1.4}$$

It is well known that, for a domain with finite volume, a uniform bound for the transition densities of the reflected Brownian motion, such as (1.3) or (1.4), implies that the 1-resolvent of the Neumann Laplacian is compact (see the proof of Theorem 2.5(i)).

Proposition 1.4. *Let $D \subset \mathbb{R}^d$ be a connected open set with finite volume.*

(i) Conditions (1.3) and (1.4) are equivalent.
(ii) If the parabolic Harnack principle holds in D then D is a non-trap domain.
(iii) There exists a non-trap domain where the 1-resolvent of the Neumann Laplacian is not compact and, therefore, the parabolic Harnack principle does not hold.

Among other results, our second most complete theorem is concerned with J_α domains, a class of domains that may have thin and long channels or bottlenecks (the parameter α indicates their shape). We will define J_α domains as in Maz'ja [M], and then prove that J_α domains satisfy the parabolic Harnack principle for $\alpha < 1$. We will also show that the result is sharp by constructing a trap domain in J_1.

However, the result on J_α domains is somewhat misleading in its completeness. There are natural classes of J_1 domains and even non-J_α domains that are not trap domains. We will define twisted starlike domains and prove that they are not trap domains. This class of domains includes the usual starlike domains. A generic example of a twisted starlike domain (but not necessarily a starlike domain) is a domain whose boundary is locally the graph of a function. Next, we will analyze a modified von Koch domain to compare our results on simply connected planar domains and J_α domains.

The techniques developed in this paper allow us to determine the relationship between various classes of "irregular" or "non-smooth" domains, characterized by the following properties or their lack of: intrinsic ultracontractivity, compactness of Neumann resolvent, the parabolic Harnack principle and the "trap" property. In particular we answer a question posed by Davies and Simon in [DS1, p.372]. These results will be discussed in a separate paper [BC1]. The idea of trap domains might also be useful in extending some results of [BHM] to a larger class of domains.

2. Main results

It is elementary to see that bounded domains with smooth boundaries are not trap domains so Problem 1.1 is meaningful only if D has a rough boundary.

2.1. Simply connected planar domains

This subsection will use complex analytic notation and concepts. Consult [P] for the definitions of prime ends, harmonic measure, etc.

Suppose D is a simply connected open subset of the complex plane $\mathbb{C}$, $z_0 \in D$ is a fixed base point, and ζ is a prime end in D. Consider a collection $\{\gamma_n\}_{n\geq 1}$ of non-intersecting cross cuts of D such that γ_{n+1} separates γ_n from ζ and γ_n's tend to ζ. Suppose further that σ is a curve in D connecting z_0 to ζ such that $\sigma \cap \gamma_n$ is a single point z_n, for each n. This system of curves divides D into subregions: let Ω_n denote the component of $D \setminus \gamma_n$ which does not contain z_0. Thus $D_n = \Omega_n \setminus \Omega_{n+1}$ is the region between γ_n and γ_{n+1}. Write $\Omega_1 \setminus \sigma = \Omega^+ \cup \Omega^-$, where each set Ω^+ and Ω^- is connected, and set $D_n^+ = \Omega^+ \cap D_n$ and $D_n^- = \Omega^- \cap D_n$. See Figure 2.1.

The harmonic measure of a set $A \subset \partial D$ in the domain D, relative to z, will be denoted $\omega(z, A, D)$. We will say that the system of curves $\{\gamma_n\} \cup \sigma$ divide D into *hyperbolic blocks* tending to the prime end ζ if for some $\varepsilon > 0$ and all $n \geq 1$, the following conditions hold:

(i) $\varepsilon \leq \omega(z_0, \partial\Omega^+ \cap \partial D, D) \leq 1/2$ and $\varepsilon \leq \omega(z_0, \partial\Omega^- \cap \partial D, D) \leq 1/2$,
(ii) for all $n \geq 1$ and for all $z \in \partial D_n^+ \cup \{z_{n-1}\}$, we have $\omega(z, \partial D_n^+ \cap \partial D, D) \geq \varepsilon$,
(iii) for all $n \geq 1$ and for all $z \in \partial D_n^- \cup \{z_{n-1}\}$, we have $\omega(z, \partial D_n^- \cap \partial D, D) \geq \varepsilon$.

For every simply connected (and even finitely connected) domain and any prime end ζ, there exists a family of hyperbolic blocks. Here is one way to construct $\{\gamma_n\}_{n\geq 1}$ and σ. Suppose that φ is a conformal map of the upper half plane $\mathbb{H}$ onto D, such that $\varphi(0) = \zeta$ and $\varphi(i) = z_0$. Then we can take $\gamma_n = \varphi(\mathbb{H} \cap \{|z| = 2^{-n}\})$, $n \geq 1$, and $\sigma = \{\varphi(iy) : 0 < y \leq 1\}$. The conformal invariance of the harmonic measure makes it is easy to verify that $\{\gamma_n\} \cup \sigma$ divide D into hyperbolic blocks tending to ζ. We will later show by example how to construct hyperbolic blocks geometrically. The term "hyperbolic" in the name of the family $\{\gamma_n\} \cup \sigma$ is derived

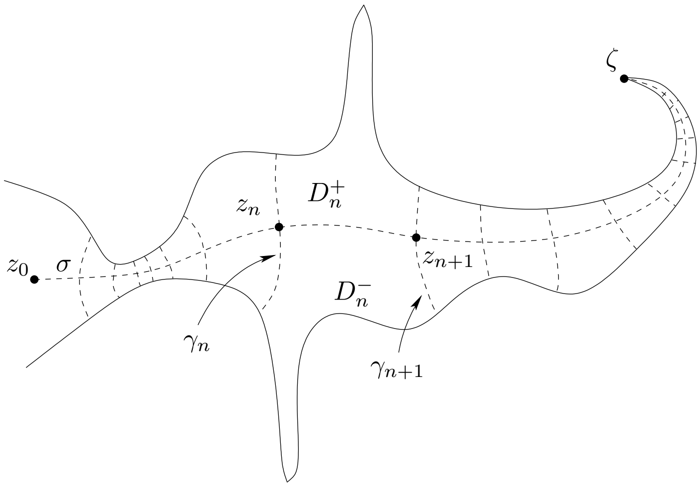

Fig. 2.1 Hyperbolic blocks

from the "hyperbolic distance" (see [P]). We will show in the proof of Theorem 2.2 that the hyperbolic distances between z_{n-1} and z_n for $n \geq 1$ are bounded below and above by constants.

Theorem 2.2. *A simply connected domain $D \subset \mathbb{C}$ having finite area is a non-trap domain if and only if there is a constant $\varepsilon > 0$ such that for each prime end $\zeta \in \partial D$ there is a system of curves $\{\gamma_n\} \cup \sigma$ dividing D into hyperbolic blocks with parameter ε and*

$$\sup_{\zeta} \sum_{n} |\Omega_n| \leq 1/\varepsilon, \tag{2.1}$$

where $|\Omega_n|$ denotes area or 2-dimensional Lebesgue measure of Ω_n.

Note that (2.1) is equivalent to

$$\sup_{\zeta} \sum_{n} n\, |D_n| \leq 1/\varepsilon, \tag{2.2}$$

and that we have not assumed in Theorem 2.2 that D is bounded.

Our proof of Theorem 2.2 yields some additional useful information. It shows that if one can find a system of hyperbolic blocks for some prime end ζ with $\sum_n |\Omega_n| = \infty$ then D is a trap domain. It follows that in such a case, there is no need to examine any other family of hyperbolic blocks.

2.2. *Maz'ja's domains*

We will define a class of multidimensional domains $D \subset \mathbb{R}^d$, $d \geq 2$, following Maz'ja [M]. We call a bounded open set $F \subset D$ "admissible" if the part of its boundary that lies in D, i.e., $\partial_i F = \partial F \cap D$, is a C^∞ manifold. Let $|F|$ denote the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and let $\mathcal{S}$ denote the $(d-1)$-dimensional surface area measure on $\partial_i F$.

Definition 2.2. *For $\alpha > 0$, we say that D belongs to class J_α if for some $\varepsilon > 0$, $c < \infty$ and all admissible sets $F \subset D$ with $|F| \le \varepsilon$, we have $|F|^\alpha \le c\mathcal{S}(\partial_i F)$.*

Clearly it follows from the definition that for $0 < \alpha < \beta$, $J_\alpha \subset J_\beta$.

Theorem 2.3. *Let $D \subset \mathbb{R}^d$ be a connected open set with finite volume.*

(i) Domains $D \in J_\alpha$ with $\alpha < 1$ satisfy the parabolic Harnack principle.
(ii) There exists a trap domain $D \in J_1$.

Part (ii) of Theorem 2.3 suggests that this result provides a sharp answer to Problems 1.1 and 1.3. It turns out that it is not a complete solution. We will show in Theorem 2.9 and Proposition 2.12 that there exist some natural classes of non-trap domains that are not contained in J_α for any $\alpha > 0$.

The intuitive meaning of the definition of a J_α domain is quite clear but proving that a given domain belongs to this class is far from trivial, because the definition involves a condition that is supposed to hold for a very large class of sets F. The methods used by Maz'ja to analyze concrete examples (see [M], Section 3.3.3, page 175) are based on explicit mappings and estimates of their Jacobians. This is sufficient to deal with regular horn-shaped domains but the method does not seem to be applicable to fractal domains. On the other hand, it is relatively easy to show that a domain does not belong to a class J_α because all one has to do is to find a sequence of admissible sets F_n with $|F_n|^\alpha/\mathcal{S}(\partial_i F_n) \to \infty$.

We recall another class of domains from [M], defined in terms of conductivity or relative capacity.

Definition 2.4. *For $\alpha > 0$, a domain $D \subset \mathbb{R}^d$ is said to belong to class $J_{2,\alpha}$ if for some $\varepsilon > 0$, $c > 0$ and for any bounded relatively closed set F in D and open subset G of D with $F \subset G$, $|G| \le \varepsilon$ and* $\mathrm{Cap}(F,G) > 0$, *we have* $|F|^\alpha \le c\,\mathrm{Cap}(F,G)^{1/2}$. *Here*

$$\mathrm{Cap}(F,G) = \inf\left\{ \int_D |\nabla f(x)|^2 dx : \ f \text{ is Lipschitz on } D, \right.$$
$$\left. f \ge 1 \text{ on } F \text{ and } f \le 0 \text{ on } D \setminus G \right\}.$$

It is clear that for $0 < \alpha < \beta$, $J_{2,\alpha} \subset J_{2,\beta}$. Domains in classes J_α and $J_{2,\alpha}$ can be characterized in terms of the Sobolev embedding. By Lemma 4.3.2 on page 199 and Theorem 4.3.3.1 on page 200 of [M] (taking $p = 1 = s, q^* = q = 1/\alpha$ there), a domain $D \subset \mathbb{R}^d$ is in J_α for some $\alpha < 1$ if and only if

$$\|u\|_{1/\alpha} \le c\,(\|\nabla u\|_1 + \|u\|_1) \quad \text{for } u \in W^{1,1}(D); \tag{2.3}$$

while by Theorem 4.3.3.1 on page 200 of [M] (taking $p = 2 = s, q^* = q = 1/\alpha$ there), D is a domain in $J_{2,\alpha}$ for some $\alpha < 1/2$ if and only if there is $c > 0$ such that

$$\|u\|_{1/\alpha} \le c\,(\|\nabla u\|_2 + \|u\|_2) \quad \text{for } u \in W^{1,2}(D). \tag{2.4}$$

Theorem 2.5. *Let $D \subset \mathbb{R}^d$ be a connected open set with finite volume.*

(i) Domains $D \in \mathcal{J}_{2,\alpha}$ with $\alpha < 1/2$ satisfy the parabolic Harnack principle.
(ii) There exists a trap domain $D \in \mathcal{J}_{2,1/2}$.

The definition of $J_{2,\alpha}$ is a bit more abstract than that of J_α. According to Proposition 4.3.4.2 on page 203 of [M], we have $J_{\alpha+\frac{1}{2}} \subset J_{2,\alpha}$. Hence Theorem 2.3(i) follows from Theorem 2.5(i), while Theorem 2.3(ii) implies Theorem 2.5(ii).

The rest of this subsection is devoted to the discussion of two classes of domains well known in analysis. Suppose $D \subset \mathbb{R}^d$. For $1 \le p \le \infty$, and integer $k \ge 1$, we denote by $W^{k,p}(D)$ the Sobolev space of functions having weak derivatives of all orders α, $|\alpha| \le k$, satisfying

$$\|f\|_{W^{k,p}(D)} = \sum_{0 \le |\alpha| \le k} \|\mathcal{D}^\alpha f\|_p < \infty.$$

An extension operator on $W^{k,p}(D)$ is a bounded linear operator $\Lambda : W^{k,p}(D) \to W^{k,p}(\mathbb{R}^d)$ such that $\Lambda f\big|_D = f$ for $f \in W^{k,p}(D)$. We say that D is a $W^{k,p}$*-extension domain* if there exists an extension operator for $W^{k,p}(D)$ (see, e.g., [J]).

Theorem 2.6. *Every $W^{1,1}$-extension domain is a $J_{\frac{d-1}{d}}$-domain and every $W^{1,2}$-extension domain is a $\mathcal{J}_{2,\alpha}$-domain with $\alpha = (d-2)/(2d)$.*

The definition of an extension domain is not easily verifiable. Jones [J] found an important class of extension domains with an intuitive geometric characterization—he called them (ε, δ)-domains. We say that D is an (ε, δ)*-domain* if $\delta, \varepsilon > 0$, and whenever $x, y \in D$ and $|x-y| < \delta$ then there exists a rectifiable arc $\gamma \subset D$ joining x and y with length$(\gamma) \le \varepsilon^{-1}|x-y|$ and moreover $\min\{|x-z|, |z-y|\} \le \varepsilon^{-1}\operatorname{dist}(z, \partial D)$ for all points $z \in \gamma$. Here $\operatorname{dist}(z, \partial D)$ is the Euclidean distance between a point z and the set ∂D.

Corollary 2.7. *The parabolic Harnack principle holds in every (ε, δ)-domain with finite volume.*

Corollary 2.7 follows from our Theorem 2.6 and Theorem 1 of Jones [J]. Note that *non-tangentially accessible domains* defined by Jerison and Kenig in [JK] are (ε, ∞)-domains (see (3.4) of [JK]).

A planar simply connected domain D is called a *John domain* if there exists $c < \infty$ such that for any curve $\Gamma \subset D$ with endpoints $x, y \in \partial D$, which cuts D into D_1 and D_2, we have $\operatorname{diam}(D_1) < c \operatorname{diam}(\Gamma)$ or $\operatorname{diam}(D_2) < c \operatorname{diam}(\Gamma)$ (see [P] p. 96). Since the area of a set is bounded by a constant times the square of its diameter, and the diameter of a rectifiable arc is bounded above by its length, it follows that every John domain is a $J_{1/2}$-domain. So by Theorem 2.3 we have the following.

Corollary 2.8. *Every John domain is a $J_{1/2}$-domain and so the parabolic Harnack principle holds in every John domain with finite volume.*

2.3. Twisted starlike domains

This subsection is devoted to some multidimensional domains which are not trap domains but do not necessarily belong to the family J_α for any $\alpha < 1$. There are two geometric reasons why a domain might not belong to J_α for any $\alpha < 1$. The first one is that it may contain many bottlenecks; we discuss such domains in Proposition 2.12. The second reason might be that the domain contains very thin and long channels—this is the class of domains we are going to discuss in this subsection.

We will temporarily drop the assumption that the vector of reflection for the RBM is normal to stress that our probabilistic method of proof does not depend on the assumption that the vector of reflection is normal. First suppose that $D \subset \mathbb{R}^d$, $d \geq 2$, has a C^2 boundary, $\mathbf{n}(x)$ is the unit inward normal vector at $x \in \partial D$, and $\mathbf{v}(x)$, $x \in \partial D$, is the unit reflection vector field satisfying for some $c_1 > 0$ and all $x \in \partial D$,

$$(\mathbf{v}(x), \mathbf{n}(x)) > c_1. \tag{2.5}$$

If B_t is a d-dimensional Brownian motion then the reflected Brownian motion starting from $x_0 \in D$ can be defined as the unique strong solution to the following stochastic differential equation,

$$X_t = x_0 + B_t + \int_0^t \mathbf{v}(X_s) dL_s, \tag{2.6}$$

where L_t is the local time of X_t on the boundary of D (see [LS]).

Note that reflected Brownian motion with the normal reflection on the boundary in an arbitrary open set $D \subset \mathbb{R}^d$, $d \geq 2$, will be defined in Section 3.1 below using the Dirichlet form approach.

The idea of a "twisted starlike" domain is best explained by an example such as

$$D = (-1, 1) \times (0, 2) \setminus \left(\bigcup_{k \geq 1} \{(2^{-k}, y) : 0 < y < 1\} \cup \{(0, y) : 0 < y < 1\} \right). \tag{2.7}$$

It is easy to see that this domain is not starlike but it is also clear that one can deform this domain in a smooth way to make it a starlike domain (see Remark 2.10 below).

We will call D a *twisted starlike* domain if there exists a continuous one-to-one mapping $F : \overline{D} \to \mathbb{R}^d$ such that $F(D)$ is bounded and starlike with respect to $0 \in \mathbb{R}^d$, $B(0, r_0) \subset F(D)$ for some $r_0 > 0$, $|F|$ is of class C^2 in D and its partial derivatives of second order are bounded above, and $0 < c_1 < |\nabla |F(x)|| < c_2 < \infty$ for $x \in D \cap F^{-1}(B(0, r_0/2))$.

Theorem 2.9. *(i) Assume that D is a bounded twisted starlike domain with C^2 boundary, $\mathbf{v}(x)$ satisfies (2.5) and $(\mathbf{v}(x), \nabla |F(x)|) \leq 0$ for every $x \in \partial D$. Define the reflected Brownian motion X_t in D with the oblique direction of reflection $\mathbf{v}(x)$ using (2.6). Then D is not a trap domain in the sense that the supremum in (1.1) is finite.*

(ii) Assume that D is a bounded twisted starlike domain (with no assumptions on the smoothness of the boundary) and let X_t be the reflected Brownian motion in D with the normal direction of reflection, in the sense of Section 3.1. Then D is not a trap domain.

Note that the twisted starlike domain in (2.7) does not belong to Maz'ja's J_α class for any $\alpha < 1$.

Remark 2.10. A bounded domain D in the plane is called *monotone* if it locally lies above the graph of a (not necessarily continuous) function, after a rotation. In other words, D is monotone if for every $x \in \partial D$, there exists a ball B centered at x, a function $f : \mathbb{R} \to \mathbb{R}$ and an orthonormal coordinate system such that in that system $x = 0$ and $D \cap B = \{(z, y) \in B : y > f(z)\}$. We now sketch an argument showing that every such domain is a twisted starlike domain and therefore is a non-trap domain.

First, consider the domain D in (2.7) and let $A = D \cap ((-1/4, 3/4) \times (0, 3/2))$. If $f(z) = \exp(i\pi z + 2)$ in the complex notation then $f(A) \cup B(0, e^{3/4})$ is a starlike domain with respect to 0. Since $\partial D \setminus A$ is a polygonal line, it is easy to see that f can be extended to a function F satisfying the conditions in the definition of a twisted starlike domain.

Next consider a general bounded monotone domain. By compactness, its boundary can be covered by a finite number of open sets $\{U_k\}$ as in the definition of a monotone domain. If the corresponding coordinate systems in two overlapping U_k's have parallel axes, we combine the two sets into one, so that we can assume that for every pair of overlapping U_k's, the coordinate systems are at a non-zero angle. The part of the boundary of D inside the intersection of any two U_k's is the graph of a function in each of two non-aligned coordinate systems corresponding to these U_k's. This easily implies that the boundary of D is Lipschitz in the intersection of any two U_k's. For every k, we can define F on $U_k \setminus \bigcup_{j \neq k} U_j$ using the same idea as in the special case of (2.7). It is easy to see that the separate pieces of F can be patched together using C^2 functions because the boundary of D is Lipschitz outside $\bigcup_k (U_k \setminus \bigcup_{j \neq k} U_j)$.

A similar argument seems to work in higher dimensions, at least for some classes of domains, but we will not try to provide the details of the proof here.

2.4. Examples

The geometric characterization of simply connected planar domains can be made even more explicit when we limit ourselves to "horn" domains. Suppose $f \colon [1, \infty) \to (0, \infty)$ is a Lipschitz function and let the corresponding *horn domain* D_f be defined by

$$D_f = \left\{ (x, y) \in \mathbb{R}^2 : x > 1, |y| < f(x) \right\}.$$

Proposition 2.11. *A horn domain $D_f \subset \mathbb{R}^2$ is a trap domain if and only if*

$$\int_1^\infty \left(\int_1^x \frac{1}{f(y)} dy \right) f(x) dx = \infty.$$

The classical von Koch snowflake may be defined as follows. Start with an equilateral triangle T_1. Consider one of its sides I and the equilateral triangle one of whose sides is the middle one third of I and whose interior does not intersect T_1. There are three such triangles; let T_2 be the closure of the union of these three triangles and T_1 (see Fig. 2.2).

We proceed inductively. Suppose I is one of the line segments in ∂T_j and consider the equilateral triangle one of whose sides is the middle one third of I and whose interior does not intersect T_j. Let T_{j+1} be the closure of the union of all such triangles and T_j. The snowflake D_{vK} is the interior of the closure of the union of all triangles constructed in all inductive steps.

We will illustrate our results by a variant of the von Koch snowflake which can be obtained from the snowflake D_{vK} as follows. Fix a function $f : (0, \infty) \to (0, \infty)$ with $f(a) \le a$ for all a. Consider any two triangles in the above construction whose boundaries have a common part I with length $a > 0$. Let I' be I with the middle $f(a)$-portion removed, i.e., if I has endpoints x and y then I' is the union of two closed line segments, the first with endpoints x and $x + \frac{a-f(a)}{2} \frac{y-x}{a}$, and the second

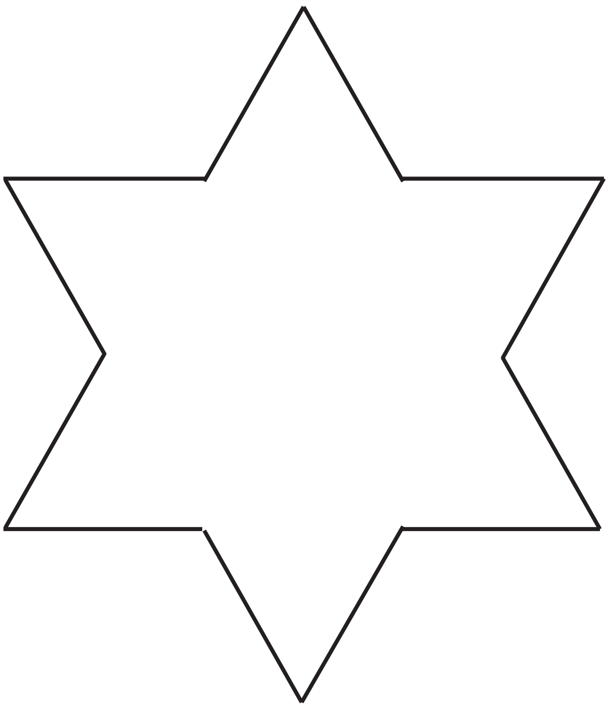

Fig. 2.2 Second step of snowflake construction, i.e., T_2

with endpoints $y - \frac{a-f(a)}{2}\frac{y-x}{a}$ and y. Let D_f be D_{vK} minus all sets of the form I' (see Fig. 2.3). The point of the construction is that the passage from a smaller triangle to a bigger triangle is blocked in D_f by a wall with a small opening. One may guess that if $f(a)/a$ tends to 0 rapidly as $a \to 0$ then D_f is a trap domain.

Proposition 2.12. *(i) Each of Corollaries 2.7 and 2.8 separately implies that the von Koch snowflake D_{vK} is not a trap domain. In other words, D_f is not a trap domain for $f(a) = a$.*

(ii) Suppose that $f(a) = a^\beta$ where $\beta < 2$. Then $D_f \in J_{\beta/2}$ and so Theorem 2.3(i) implies that D_f is not a trap domain.

(iii) Suppose that $f(a) = \exp(-a^{-\gamma})$. If $\gamma < 2$ then D_f is not a trap domain, but it is a trap domain if $\gamma > 2$.

Parts (i) and (ii) of Proposition 2.12 are much weaker than part (iii)—we stated them only to illustrate the strength of various results. Parts (ii) and (iii) of Proposition 2.12 show that Theorem 2.3(ii) must be interpreted with a great caution. Note that a domain D_f, with $f(a) = \exp(-a^{-\gamma})$ for some $0 < \gamma < 2$, is not in class J_α for any $\alpha > 0$. So one must not presume that a domain is a trap domain just because it does not belong to class J_α for any $0 < \alpha < \infty$.

Another example is a spiral domain, definition in the next proposition.

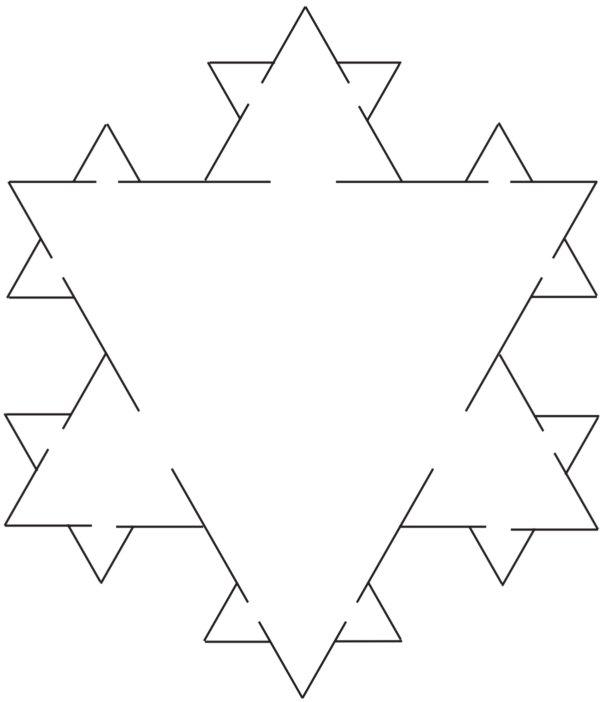

Fig. 2.3 Modified snowflake D_f

Proposition 2.13. *Let*

$$S_p = \mathbb{D} \setminus \{re^{i\theta} : r = \theta^{-p} \text{ and } \theta \geq 1\},$$

where $\mathbb{D}$ is the open unit disc. Then S_p is a trap domain if and only if $p \leq 1$.

3. RBM, Green's function and trap domains

As was pointed out previously, Problem 1.1 is meaningful only for domains with non-smooth boundary. There are many definitions of reflecting Brownian motion on smooth domains. The most elementary and the most powerful definitions, such as the (deterministic) Skorokhod problem method and the martingale problem method, apply only when D has a C^2-smooth boundary. Hence, we cannot use any of the relatively easy definitions of reflecting Brownian motion. For this reason, Subsection 3.1 will be devoted to the definition of RBM and its Green function in non-smooth domains. Subsection 3.2 will give the proofs to the results presented in Sections 1 and 2.

3.1. Reflecting Brownian motion and Green's function

Constructing a reflecting Brownian motion on a non-smooth domain D is a delicate problem. Let

$$W^{1,2}(D) := \{f \in L^2(D, dx) : \nabla f \in L^2(D, dx)\}$$

be the Sobolev space on D of order $(1, 2)$. Fukushima [Fu] used the Martin-Kuramochi compactification D^* of D to construct a continuous diffusion process X^* on D^* with transition semigroup denoted P_t, such that

$$\{f \in L^2(D, dx) : \sup_{t>0} \frac{1}{t} \int_D f(x)(f(x) - P_t f(x))dx < \infty\} = W^{1,2}(D)$$

and for $f \in W^{1,2}(D)$,

$$\mathcal{E}(f, f) := \lim_{t \to 0} \frac{1}{t} \int_D f(x)(f(x) - P_t f(x))dx = \frac{1}{2} \int_D |\nabla f(x)|^2 dx.$$

The pair $(\mathcal{E}, W^{1,2}(D))$ is called the Dirichlet space of X^* in $L^2(D^*, m)$, where m is Lebesgue measure on D extended to D^* by setting $m(D^* \setminus D) = 0$. See [FOT] for definitions and properties of Dirichlet spaces, including the notions of quasi-everywhere, quasi-continuous, etc. The process X^* could be called reflecting Brownian motion in D but it lives on an abstract space D^* that contains D as a dense open set. Chen [C] proposed referring to the quasi-continuous projection X of X^* from D^* into the Euclidean closure $\overline{D}$ as reflecting Brownian motion in D. The projection process X is a continuous process on $\overline{D}$, but in general X is not a strong Markov process on $\overline{D}$ (for example this is the case when D is the unit disk with a slit removed). However when D is a Lipschitz domain, it can be shown that

X is the usual reflecting Brownian motion in D as constructed in [BH]. See the introductions of [C] and [CFW] for the history of constructing reflecting Brownian motion on non-smooth domains.

Let $\{P_t, t \geq 0\}$ and $(\mathcal{L}, D(\mathcal{L}))$ denote the semigroup and the L^2-infinitesimal generator of X^*, respectively. We call $-2\mathcal{L}$ the Neumann Laplacian on D. Clearly its 1-resolvent R_1 is given by $R_1 f = \int_0^\infty e^{-t} P_{2t} f \, dt$. The following result might be known to experts. We present it here for the reader's convenience.

Lemma 3.1. *Suppose that D is a domain in $\mathbb{R}^d$ with finite volume. Then the following are equivalent.*

(i) The Neumann Laplacian in D has discrete spectrum.
(ii) The 1-resolvent R_1 of the Neumann Laplacian in D is a compact operator in $L^2(D, dx)$.
(iii) The embedding $W^{1,2}(D) \to L^2(D, dx)$ is compact.

Proof. The equivalence of (i) and (iii) follows immediately from Theorem 4.8.2 and Theorem 4.10.1.3 of Maz'ja [M]. If R_1 is compact, then R_1 has discrete spectrum, and so does the Neumann Laplacian in D; that is, (ii) implies (i). Now suppose (iii) holds. Since R_1 is a bounded operator from $L^2(D, dx)$ into $W^{1,2}(D)$, it follows that R_1 is a compact operator from $L^2(D, dx)$ into itself. Hence, (iii) implies (ii) and this completes the proof of the lemma. $\square$

As is pointed out in [BBC], the reflecting Brownian motion defined above is conformally invariant in planar domains in the following sense. Suppose that D and U are two planar domains, φ is a one-to-one conformal map from D onto U and X^* is the reflecting Brownian motion on D constructed above. Then $\varphi(X^*)$ is a time-changed RBM on U. Suppose that B is a closed ball in a planar domain D. Let Y^* be the subprocess of X^* killed upon hitting B, which is called the RBM on D^* killed upon hitting B. If we use process X instead of X^* in the above procedure, then the process Y obtained will be called RBM on $\overline{D}$ killed upon hitting B. Let $G_{D \setminus B}(x, y)$ be the Green function of Y^*, that is, for every Borel function $f \geq 0$ on $D \setminus B$,

$$\begin{aligned} \int_{D\setminus B} G_{D\setminus B}(x, y) f(y) dy &= \mathbb{E}_x \int_0^{T_B} f(X_s) ds \\ &= \mathbb{E}_x \int_0^{T_B} f(X_s^*) ds \qquad \text{for every } x \in D \setminus B. \end{aligned}$$

It follows from [Fu] and [FOT] that (i) $G_{D\setminus B}(x, y)$ is symmetric and continuous in $(D \setminus B) \times (D \setminus B) \setminus d$, where d denotes the diagonal; (ii) For every $y \in D \setminus B$, $x \mapsto G_{D\setminus B}(x, y)$ is harmonic in $D \setminus (B \cup \{y\})$. It follows from the conformal invariance of RBM that

$$G_{D\setminus B}(x, y) = G_{\varphi(D)\setminus\varphi(B)}(\varphi(x), \varphi(y)) \quad \text{for } x, y \in D \setminus B. \tag{3.1}$$

3.2. Trap domains

In this subsection, we give proofs for the results stated in Sections 1 and 2.

Proof of Proposition 1.4. (i) Obviously, (1.4) implies (1.3). For $s, t \geq t_0$ we have $p_{t+s}(x, y) = \int p_t(x, z)p_s(z, y)dz$. Now we can apply Lemma 6.1 of [BTW] (see Lemma 1 of [BK] for a more accessible version) to see that (1.3) implies convergence of $p_t(x, y)$ to the stationary density at an exponential rate, as in (1.4).

(ii) If we assume (1.3) holds then for some $c_4 > 0$ and all $x \in D$, $\mathbb{P}^x(T_B \leq t_1) \geq \int_B p_{t_1}(x, y)dy \geq c_4$. By the Markov property of X^* and the fact that $\mathbb{P}^x(X_t^* \in D^* \setminus D) = 0$ for every $x \in D$ and $t > 0$, we conclude that $\mathbb{P}^x(T_B \geq kt_1) \leq (1-c_4)^k$ for every $x \in D$ and $k \geq 1$. This implies that

$$\sup_{x\in D} \mathbb{E}^x T_B \leq \sup_{x\in D} \sum_{k=0}^{\infty} t_1 \mathbb{P}^x(T_B \geq kt_1) < \infty$$

and so D is not a trap domain.

(iii) The proof of part (iii) of this proposition will be given after the proof of Proposition 2.12. □

We will now present two elementary lemmas showing that our main problem is well posed.

Lemma 3.2. *If $D \subset \mathbb{R}^d$, $d \geq 1$, has finite volume and B is a closed ball in D, then $\mathbb{E}^x T_B < \infty$ for every $x \in D$.*

Proof. Recall the definition of RBM X^* on D^*, and the RBM Y^* on $D^* \setminus B$ killed upon hitting B given in Section 3.1. Since Y^* is transient, by Lemma 1.6.4 and Theorem 1.5.1 of [FOT], there is a function $g \in L^1(D \setminus B, dx)$ such that $g > 0$ and $Gg < \infty$ a.e. on $D \setminus B$. One can modify g as follows. Define $A_1 = \{x \in D \setminus B : Gg(x) \leq 2\}$ and for $k \geq 2$, $A_k = \{x \in D \setminus B : k < Gg(x) \leq k+1\}$. Note that $G(g1_{A_k})(x) \leq \sup_{y\in A_k} G(g1_{A_k})(y)$ for $x \in D \setminus B$. Let $f(x) = \sum_{k=1}^{\infty} 2^{-k}(k+1)^{-1}g(x)1_{A_k}(x)$. Then $f \leq g$, $f > 0$ and $Gf \leq 1$ a.e. on $D \setminus B$. Since D has finite volume and G is symmetric, we have

$$\int_{D\setminus B} f(x)G1(x)dx = \int_{D\setminus B} Gf(x)dx \leq |D| < \infty.$$

This implies that $\mathbb{E}^x T_B = G1(x) < \infty$ for a.e. $x \in D \setminus B$. Now for an arbitrary but fixed $x_0 \in D \setminus B$, let $r > 0$ so that $B(x_0, 2r) \subset D \setminus B$. By the strong Markov property of Y^*, we have

$$G1(x) = \mathbb{E}^x \tau_{B(x_0,r)} + \mathbb{E}^x \left[G1(Y^*_{\tau_{B(x_0,r)}}) \right] \quad \text{for } x \in B(x_0, r).$$

Clearly $\mathbb{E}^x \tau_{B(x_0,r)} < \infty$ for $x \in B(x_0, r)$ as $\{X_t^*, 0 \leq t < \tau_{B(x_0,r)}\}$ is the killed Brownian motion in $B(x_0, r)$. Function $u(x) := \mathbb{E}^x \left[G1(Y^*_{\tau_{B(x_0,r)}}) \right]$ is finite a.e. on $B(x_0, r)$ and harmonic in $B(x_0, r)$ so it is finite everywhere on $B(x_0, r)$. This implies that $G1(x) < \infty$ for every $x \in B(x_0, r)$ and hence for every $x \in D \setminus B$. □

Lemma 3.3. *If $D \subset \mathbb{R}^d$ is a connected open set with finite volume and B_1 and B_2 are closed non-degenerate balls in D then $\sup_{x\in D} \mathbb{E}^x T_{B_1} < \infty$ if and only if $\sup_{x\in D} \mathbb{E}^x T_{B_2} < \infty$.*

Proof. This is standard so we only sketch the proof. Suppose that $\sup_{x\in D} \mathbb{E}^x T_{B_1} < \infty$. Then $\sup_{x\in D} \mathbb{P}^x(T_{B_1} > t) \le \sup_{x\in D} \mathbb{E}^x T_{B_1}/t$, and so $\inf_{x\in D} \mathbb{P}^x(T_{B_1} \le t_0) \ge c_1$ for some $t_0 < \infty$ and $c_1 > 0$. Let $p_t(x, y)$ and $p_t^0(x, y)$ denote the transition density function for RBM X^* on D^* and the killed Brownian motion in D, respectively. Clearly $p_t(x, y) \ge p_t^0(x, y)$ on $(0, \infty) \times D \times D$ and so

$$\inf_{x\in B_1, y\in B_2} p_1(x, y) \ge \inf_{x\in B_1, y\in B_2} p_1^0(x, y) > c_2 > 0.$$

By the Markov property of X_t^* and the fact that $\mathbb{P}^x(X_t^* \in D^* \setminus D) = 0$ for every $x \in D$ and $t > 0$, we have $\sup_{x\in D} \mathbb{P}^x(T_{B_2} \le t_0 + 1) \ge c_1 c_2$ and by induction, $\sup_{x\in D} \mathbb{P}^x(T_{B_2} > k(t_0+1)) \le (1-c_1c_2)^k$. This implies that $\sup_{x\in D} \mathbb{E}^x T_{B_2} < \infty$. □

Note that there is nothing special about assuming B_j are balls. We could, for example, use compact sets with non-empty interior.

Proof of Proposition 1.2. Clearly (iii) implies (ii). Let B be a ball with $\overline{B} \subset D$ and for any set A, let $|A|$ denote the Lebesgue measure of A. If (ii) holds, then there is $t_0 \ge 0$ such that $\sup_{x\in D} \|\mathbb{P}^x(X_{t_0} \in \cdot) - \Pi_D\| \le (1/2)|B|/|D|$ and so $\inf_{x\in D} \mathbb{P}^x(X_{t_0} \in B) \ge (1/2)|B|/|D|$. This implies that $\inf_{x\in D} \mathbb{P}^x(T_B \le t_0) \ge (1/2)|B|/|D|$. By the same argument as that in the proof of Proposition 1.4(ii), we have $\sup_{x\in D} \mathbb{E}^x T_B < \infty$. Hence (ii) implies (i).

Now we will show that (i) implies (iii). Suppose that D is a non-trap domain. Let $x_0 \in D$ and $r > 0$ be such that $B(x_0, 3r) \subset D$. Since $\sup_{x\in D} \mathbb{E}^x T_{B(x_0,r)} = c_1 < \infty$, $\sup_{x\in D} \mathbb{P}^x(T_{B(x_0,r)} > t) \le c_1/t$ and so $\inf_{x\in D} \mathbb{P}^x(T_{B(x_0,r)} \le n_1) > 1/2$ for some integer $n_1 > 0$. Let W_t denote the Brownian motion in $\mathbb{R}^d$ and let $p_0 > 0$ be defined by $\mathbb{P}^x(W_t \in B(x, r) \text{ for } t \in [0, 1]) = p_0$. Recall from the beginning of this section that X^* denotes the reflected Brownian motion in D, in an appropriate sense, and it behaves like a Brownian motion inside D before hitting the boundary. By the strong Markov property of X^*,

$$\begin{aligned}\inf_{x\in D} \mathbb{P}^x(T_{B(x_0,r)} \le n_1 \text{ and } X_t^* \in B(x_0, 2r) \text{ for } t \in [T_{B(x_0,r)},\ T_{B(x_0,r)} + 1])\\ > (1/2)p_0 > 0.\end{aligned}$$

If we let $S = \inf\{n \ge 0 : n \in \mathbb{Z} \text{ and } X_n \in B(x_0, 2r)\}$, then the above implies that $\inf_{x\in D} \mathbb{P}^x(S \le n_1 + 1) > c_2 := p_0/2 > 0$. Using the Markov property of X^* at integer times $k(n_1 + 1)$ and the fact that $\mathbb{P}^x(X_n^* \in D^* \setminus D) = 0$ for every $x \in D$ and $n \in \mathbb{Z}$, we deduce that $\sup_{x\in D} \mathbb{P}^x(S \ge k(n_1 + 1)) \le (1 - c_2)^k$ and so

$$\sup_{x\in D} \mathbb{E}^x S \le \sup_{x\in D} \sum_{k=0}^{\infty} k(n_1 + 1)\mathbb{P}^x(S \ge k(n_1 + 1)) < \infty.$$

Clearly, Π_D is the invariant measure for X_t^*. Now applying Theorem 16.0.2 of [MT] to the Markov chain $\{X_n^*, n = 0, 1, 2, \cdots\}$, we have

$$\sup_{x \in D} \|\mathbb{P}^x(X_n \in \cdot) - \Pi_D\|_{TV} \le c_3 e^{-c_4 n}, \qquad \text{for every } n \ge 1,$$

where c_3, c_4 are two positive and finite constants. Using the semigroup property of X_t^* and the fact that Π_D is its invariant measure, we have

$$\sup_{x \in D} \|\mathbb{P}^x(X_t \in \cdot) - \Pi_D\|_{TV} \le \sup_{x \in D} \|\mathbb{P}^x(X_n \in \cdot) - \Pi_D\|_{TV},$$

for all real t and integer n such that $t \ge n$. This establishes (iii) and therefore completes the proof of Proposition 1.2. □

Proof of Theorem 2.2. We will use the Riemann mapping theorem. It will be convenient first to map the unit disc $\mathbb{D}$ onto D, and then switch to a different mapping, from the upper half-plane $\mathbb{H}$ to D. We write $z = x + iy$ and use $dxdy$ for 2-dimensional, or area, measure.

Let B be a closed ball contained in D and let f be a conformal map of the unit disc $\mathbb{D}$ onto D with $f(\{z : |z| < r_0\}) \subset B$. Let U be the "double" of $\mathbb{D} \setminus f^{-1}(B)$:

$$U = \left(\mathbb{D} \setminus f^{-1}(B)\right) \cup \partial\mathbb{D} \cup \left\{\frac{1}{\overline{z}} : z \in \mathbb{D} \setminus f^{-1}(B)\right\},$$

and let $g_U(z, a)$ be the classical Dirichlet Green's function for U with $g_U(z, a) = 0$ for $z \in \partial U$, $a \in U$ and $g_U(z, a) + \log|z - a|$ harmonic for $z \in U$. Then for $z,\ a \in \mathbb{D} \setminus f^{-1}(B)$ the function

$$G(z) = g_U(z, a) + g_U(z, 1/\overline{a})$$

satisfies $G(z) = 0$ for $z \in \partial f^{-1}(B)$, $\quad G(z) + \log|z - a|$ is harmonic for $z \in \mathbb{D} \setminus f^{-1}(B)$ and $\frac{\partial G}{\partial r} = 0$ on $\partial\mathbb{D}$, since $G(z) = G(1/\overline{z})$. By the reflection and maximum principles, and (3.1),

$$G(z) = G_{\mathbb{D}\setminus f^{-1}(B)}(z, a) = G_{D \setminus B}(f(z), f(a)). \tag{3.2}$$

By the maximum principle, for $z,\ a \in U$,

$$G(z) \le \log \frac{c_1}{|z - a||1 - \overline{a}z|}$$

since the difference of these two functions is harmonic, $G = 0$ on ∂U and $|z - a||1 - \overline{a}z| \le (1 + 1/r_0)^2 \equiv c_1$. Thus for $z,\ a \in \mathbb{D} \setminus f^{-1}(B)$,

$$G_{\mathbb{D}\setminus f^{-1}(B)}(z, a) \le \log c_1 + \log \frac{1}{|z - a|^2},$$

and by (3.2) D is non-trap if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \log \frac{1}{|z - a|} |f'(z)|^2 dxdy < \infty, \tag{3.3}$$

since $|D| = \int |f'(z)|^2 dxdy < \infty$. Note also that $\int_{\mathbb{D}} \log(1/|z-a|)dxdy < C < \infty$ and thus (3.3) holds if and only if

$$\sup_{1-\delta<|a|<1} \int \log \frac{1}{|z-a|} |f'(z)|^2 dxdy < \infty.$$

If δ is sufficiently small, then for $1-\delta < |a| < 1$ and for $z \in \partial U$,

$$|z-a||1-\overline{a}z| \geq c_2 > 0$$

so by the maximum principle again

$$\log \frac{c_2}{|z-a||1-\overline{a}z|} \leq G(z).$$

Thus for $z,\ a \in \mathbb{D} \setminus f^{-1}(B)$,

$$\log \frac{c_2}{2} + \log \frac{1}{|z-a|} \leq G_{\mathbb{D}\setminus f^{-1}(B)}(z,a)$$

and by (3.2) D is non-trap if and only if

$$\sup_{3/4\leq|a|\leq 1} \int_{\{|z-a|<1/2\}} \log \frac{1}{|z-a|} |f'(z)|^2 dxdy < \infty. \tag{3.4}$$

We will show that (3.4) holds if and only if

$$\sup_{|a|=1} \int_{\{|z-a|<3/4\}} \log \frac{1}{|z-a|} |f'(z)|^2 dxdy < \infty. \tag{3.5}$$

Consider $a \in \mathbb{D}$ with $3/4 < |a| < 1$ and let $B_a = \{z : |z-a| < (1-|a|)/2\}$ and $a' = a/|a|$. By Corollary 1.6 on page 10 of [P], for some constant $c_3 < \infty$ not depending on a,

$$\sup_{z\in B_a} |f'(z)| \leq c_3 \inf_{z\in B_a} |f'(z)|.$$

A straightforward calculation shows that

$$\int_{B_a} \log \frac{1}{|z-a'|} dxdy \geq c_4 \int_{B_a} \log \frac{1}{|z-a|} dxdy,$$

so

$$\int_{B_a} \log \frac{1}{|z-a'|} |f'(z)|^2 dxdy \geq c_5 \int_{B_a} \log \frac{1}{|z-a|} |f'(z)|^2 dxdy. \tag{3.6}$$

On B_a^c, we have

$$\log \frac{1}{|z-a'|} \geq c_6 \log \frac{1}{|z-a|}$$

so

$$\int_{B_a^c} \log \frac{1}{|z-a'|} |f'(z)|^2 dxdy \geq c_6 \int_{B_a^c} \log \frac{1}{|z-a|} |f'(z)|^2 dxdy. \tag{3.7}$$

Note that $\{|z-a| < 1/2\} \subset \{|z-a'| < 3/4\}$. Combining (3.6) and (3.7), we obtain

$$\int_{\{|z-a'|<3/4\}} \log \frac{1}{|z-a'|} |f'(z)|^2 dxdy \geq c_7 \int_{\{|z-a|<1/2\}} \log \frac{1}{|z-a|} |f'(z)|^2 dxdy,$$

and this proves that (3.4) and (3.5) are equivalent.

We transfer (3.5) to the upper half plane $\mathbb{H}$ by applying the conformal maps to $\mathbb{D}$ given by $\psi_a(z) = a(i-z)/(i+z)$, with $|a| = 1$. Thus (3.5) is equivalent to

$$\sup_{\varphi} \int_{\mathbb{H}\cap\{|z|<1\}} \log \frac{1}{|z|} |\varphi'(z)|^2 dxdy < \infty, \tag{3.8}$$

where the supremum is taken over all conformal maps φ of $\mathbb{H}$ onto D such that $\varphi(i) = z_0$, a fixed base point in D. We will split the rest of the argument into several lemmas. Recall the parameter ε from the definition of hyperbolic blocks.

Lemma 3.4. *If $\{\gamma_n\} \cup \sigma$ divides D into hyperbolic blocks tending to $\zeta = \varphi(0)$, where φ is a conformal map of $\mathbb{H}$ onto D mentioned above, then $\varphi^{-1}(\sigma \cap \Omega_1)$ lies in a non-tangential cone $\Gamma_\varepsilon = \{z \in \mathbb{H} : \pi\varepsilon < \arg z < \pi(1-\varepsilon)\}$.*

Proof. Recall condition (i) in the definition of hyperbolic blocks. It implies, by conformal invariance, that $I_n^+ = \varphi^{-1}(\partial D_n^+ \cap \partial D) \subset (0, \infty)$ for all $n \geq 1$ or all of these intervals belong to $(-\infty, 0)$. We will assume without loss of generality that $I_n^+ = \varphi^{-1}(\partial D_n^+ \cap \partial D) \subset (0, \infty)$ and $I_n^- = \varphi^{-1}(\partial D_n^- \cap \partial D) \subset (-\infty, 0)$. If $z \in \sigma \cap D_n$ and if $\operatorname{Re} \varphi^{-1}(z) \leq 0$, then

$$\begin{aligned} \varepsilon &\leq \omega(z, \partial D_n^+ \cap \partial D, D) = \omega(\varphi^{-1}(z), I_n^+, \mathbb{H}) \\ &\leq \omega(\varphi^{-1}(z), [0, +\infty), \mathbb{H}) = 1 - \frac{1}{\pi} \arg \varphi^{-1}(z), \end{aligned}$$

since the harmonic measure of an interval evaluated at z is equal to the angle subtended at z by the interval divided by π. Similarly if $z \in \sigma \cap D_n$ and $\operatorname{Re} \varphi^{-1}(z) > 0$, then

$$\begin{aligned} \varepsilon &\leq \omega(z, \partial D_n^- \cap \partial D, D) = \omega(\varphi^{-1}(z), I_n^-, \mathbb{H}) \\ &\leq \omega(\varphi^{-1}(z), (-\infty, 0], \mathbb{H}) = \frac{1}{\pi} \arg \varphi^{-1}(z). \end{aligned}$$

Thus

$$\pi\varepsilon \leq \arg \varphi^{-1}(z) \leq \pi(1-\varepsilon)$$

and the lemma follows. □

Recall that z_n is the intersection point of γ_n and σ in the definition of hyperbolic blocks for D.

Lemma 3.5. *There is a $\delta > 0$ depending on ε but not on n so that if $z \in D_n \cup \{z_{n-1}\}$, $n \geq 1$, then*

$$\delta \leq \left| \frac{\varphi^{-1}(z)}{\varphi^{-1}(z_n)} \right| \leq \frac{1}{\delta}. \tag{3.9}$$

Proof. Write $I_n^+ = [a_{n+1}^+, a_n^+]$ and $I_n^- = [a_n^-, a_{n+1}^-]$. As in the proof of Lemma 3.4, by condition (ii) in the definition of hyperbolic blocks,

$$\begin{aligned}\varepsilon &\le \omega(\varphi^{-1}(z_n), I_{n-1}^+, \mathbb{H}) \le \omega(\varphi^{-1}(z_n), [a_n^+, \infty), \mathbb{H})\\ &= 1 - \frac{1}{\pi}\arg(\varphi^{-1}(z_n) - a_n^+).\end{aligned}$$

This implies that

$$a_n^+ \le C|\varphi^{-1}(z_n)|,$$

which is perhaps easiest to see by scaling $\mathbb{H}$ by the factor $1/a_n^+$. Similarly by condition (ii),

$$\begin{aligned}\varepsilon &\le \omega(\varphi^{-1}(z_n), I_n^+, \mathbb{H}) \le \omega(\varphi^{-1}(z_n), [0, a_n^+], \mathbb{H})\\ &= \frac{1}{\pi}\arg\left(\frac{\varphi^{-1}(z_n) - a_n^+}{\varphi^{-1}(z_n)}\right).\end{aligned}$$

By Lemma 3.4, $\varphi^{-1}(z_n)$ lies in the non-tangential cone Γ_ε, so this implies $a_n^+ \ge c|\varphi^{-1}(z_n)|$. We conclude that a_n^+ is comparable to $|\varphi^{-1}(z_n)|$ for all n and similarly $|a_n^-|$ is comparable to $|\varphi^{-1}(z_n)|$.

Now suppose that $z \in \gamma_{n+1} \subset \partial D_n$. Then by conditions (ii) and (iii) either

$$\varepsilon \le \omega(\varphi^{-1}(z), I_n^+, \mathbb{H}) \le 1 - \frac{1}{\pi}\arg(\varphi^{-1}(z) - a_{n+1}^+), \tag{3.10}$$

or

$$\varepsilon \le \omega(\varphi^{-1}(z), I_n^-, \mathbb{H}) \le \frac{1}{\pi}\arg(\varphi^{-1}(z) - a_{n+1}^-). \tag{3.11}$$

Conditions (3.10) and (3.11) define two half-lines in $\mathbb{H}$. Let $\mathcal{T}$ be the open triangle with sides on these half-lines and the real axis. We have shown that $\varphi^{-1}(z) \notin \mathcal{T}$ for $z \in \gamma_{n+1}$ and hence for all $z \in D_n \cup \{z_{n-1}\}$, since $\varphi^{-1}(\gamma_{n+1})$ is a crosscut of $\mathbb{H}$. Two of the vertices of $\mathcal{T}$ are a_{n+1}^- and a_{n+1}^+ and its height is comparable to $|a_{n+1}^+ - a_{n+1}^-|$. Since a_{n+1}^+ and $|a_{n+1}^-|$ are comparable to $|\varphi^{-1}(z_{n+1})|$, this implies that $|\varphi^{-1}(z)| \ge \delta|\varphi^{-1}(z_{n+1})|$ for some $\delta > 0$ and all $z \in D_n \cup \{z_{n-1}\}$. Similarly, for $z \in \gamma_n \subset \partial D_n$, by conditions (ii) and (iii),

$$\varepsilon \le \omega(\varphi^{-1}(z), I_n^+, \mathbb{H}) \le \frac{1}{\pi}\arg\left(\frac{\varphi^{-1}(z) - a_n^+}{\varphi^{-1}(z)}\right),$$

or

$$\varepsilon \le \omega(\varphi^{-1}(z), I_n^-, \mathbb{H}) \le \frac{1}{\pi}\arg\left(\frac{\varphi^{-1}(z)}{\varphi^{-1}(z) - a_n^-}\right).$$

Since a_n^+ and $|a_n^-|$ are comparable to $|\varphi^{-1}(z_n)|$,

$$|\varphi^{-1}(z)| \le \frac{1}{\delta}|\varphi^{-1}(z_n)|,$$

which must then hold for all $z \in D_n$, since $\varphi^{-1}(\gamma_n)$ is a crosscut of $\mathbb{H}$. Likewise for $z = z_{n-1}$

$$|\varphi^{-1}(z_{n-1})| \leq \frac{1}{\delta}|\varphi^{-1}(z_n)|,$$

and so for all $z \in D_n \cup \{z_{n-1}\}$

$$\delta^2|\varphi^{-1}(z_n)| \leq \delta|\varphi^{-1}(z_{n+1})| \leq |\varphi^{-1}(z)| \leq \frac{1}{\delta}|\varphi^{-1}(z_n)|.$$

□

Lemma 3.6. *There are constants* $0 < c_1 < c_2 < \infty$ *depending on* ε *but not on* n *such that*

$$c_1 n < \log \frac{1}{|\varphi^{-1}(z_n)|} < c_2 n. \tag{3.12}$$

Proof. By (3.9)

$$\left|\frac{\varphi^{-1}(z_{n-1})}{\varphi^{-1}(z_n)}\right| \leq \frac{1}{\delta},$$

and hence

$$\log\left|\frac{\varphi^{-1}(z_0)}{\varphi^{-1}(z_n)}\right| \leq n \log \frac{1}{\delta}.$$

For the reverse inequality, recall that $\varphi^{-1}(z_n)$ lies in a cone at 0, so that $\operatorname{Im} \varphi^{-1}(z_n)$ is comparable to $|\varphi^{-1}(z_n)|$ and also is comparable to a_n^+. Since

$$\varepsilon \leq \omega(\varphi^{-1}(z_n), [a_{n+1}^+, a_n^+], \mathbb{H}) = \frac{1}{\pi} \arg \left(\frac{\varphi^{-1}(z) - a_n^+}{\varphi^{-1}(z) - a_{n+1}^+} \right),$$

there is a $\lambda < 1$, depending only on ε such that

$$a_{n+1}^+ \leq \lambda a_n^+.$$

Thus

$$|\varphi^{-1}(z_n)| \leq C_1 \operatorname{Im} \varphi^{-1}(z_n) \leq C_2 a_n^+ \leq C_3 \lambda^n,$$

and so

$$\log\left|\frac{C_3}{\varphi^{-1}(z_n)}\right| \geq n \log \frac{1}{\lambda}.$$

□

Proof of Theorem 2.2 (continued). By Lemmas 3.5 and 3.6, the symmetric difference A of the sets $\mathbb{H} \cap \{|z| < 1\}$ and $\varphi^{-1}(\Omega_1)$ lies in $\mathbb{H} \cap \{c_1 < |z| < c_2\}$, where $0 < c_1 < c_2 < \infty$ depend only on ε but not on φ (i.e., ζ). Hence,

$$\left| \int_A \log \frac{1}{|z|} |\varphi'(z)|^2 dxdy \right| \leq c_3 \int_A |\varphi'(z)|^2 dxdy \leq c_3 |D| < \infty,$$

and, therefore, (3.8) is equivalent to

$$\sup_{\varphi} \int_{\varphi^{-1}(\Omega_1)} \log \frac{1}{|z|} |\varphi'(z)|^2 dxdy < \infty. \tag{3.13}$$

We apply Lemmas 3.5 and 3.6 again to conclude that the ratio

$$\frac{\int_{\varphi^{-1}(\Omega_1)} \log \frac{1}{|z|} |\varphi'(z)|^2 dxdy}{\sum_{n=1}^{\infty} n \int_{\varphi^{-1}(D_n)} |\varphi'(z)|^2 dxdy}$$

is bounded below and above by constants depending only on ε. This implies that (3.13) is equivalent to

$$\sup_{\varphi} \sum_{n=1}^{\infty} n \int_{\varphi^{-1}(D_n)} |\varphi'(z)|^2 dxdy = \sup_{\varphi} \sum_{n=1}^{\infty} n\, |D_n| = \sup_{\varphi} \sum_{n=1}^{\infty} |\Omega_n| < \infty.$$

□

Remark 3.7. It is quite easy to extend Theorem 2.2 to finitely connected planar domains D. We will limit ourselves to a very sketchy outline of the argument. Using the remark immediately following Lemma 3.3, we can choose a compact subset K of D such that each component of $D \setminus K$ is doubly connected, and apply Theorem 2.2 to each component. Green's function can also be constructed on $D \setminus B$ by first using the Riemann mapping theorem, once for each boundary component to map to a region Ω bounded by analytic curves. Then the Riemann surface "double", call it R, is formed by attaching two copies of $\Omega \setminus B$ along $\partial\Omega$. If $a \in \Omega \setminus B$ and if a^* is the corresponding point on the second copy, then Green's function equals $g_R(z, a) + g_R(z, a^*)$ as before, where g_R is the classical Dirichlet Green's function for the Riemann surface R. One could leave the statement of the result and its proof as is, using the analytic language but it is possible to give a probabilistic interpretation of the argument. First, one can construct a Brownian motion on R using the fact that R is an analytic manifold, i.e., for every point z in R, including the part where the two leaves of R meet, one can find an analytic mapping of a neighborhood U of z onto a disc. The inverse mapping of the usual Brownian motion on the disc, appropriately time-changed, is a Brownian motion on U and its projection on Ω is the reflected Brownian motion on a subset of Ω. The standard piecing-together method then shows that the reflected Brownian motion on Ω is the projection of the Brownian motion on R.

Proof of Theorem 2.3. (i) As we mentioned previously, this part follows from Theorem 2.5(i), whose proof will be given immediately after the proof of part (ii) of this theorem.

(ii) One counterexample is the region

$$D = \{(x, y) : x > 1 \text{ and } |y| < e^{-x}\}.$$

It is easy to verify that D is a trap domain using Proposition 2.11. The proof that $D \in J_1$ is exactly like the proof in the example below. We include another counterexample, though for two reasons: it is a bounded region, and the proof has perhaps greater intuitive appeal for probabilists.

Our counterexample is a snake-like domain (see Fig 3.2).

Let

$$\begin{aligned}
A_k &= \{(x, y) \in \mathbb{R}^2 : 2^{-2k-1} < x < 2^{-2k}, 0 \le y \le 1\}, \qquad k \ge 0,\\
U_k &= \{z = (x, y) \in \mathbb{R}^2 : |z - (3 \cdot 2^{-2k-3}, 1)| > 2^{-2k-3},\\
&\qquad |z - (9 \cdot 2^{-2k-4}, 1)| < 7 \cdot 2^{-2k-4}, y \ge 1\}, \qquad k \ge 0, k \text{ even},\\
U_k &= \{z = (x, y) \in \mathbb{R}^2 : |z - (3 \cdot 2^{-2k-3}, 0)| > 2^{-2k-3},\\
&\qquad |z - (9 \cdot 2^{-2k-4}, 0)| < 7 \cdot 2^{-2k-4}, y \le 0\}, \qquad k \ge 0, k \text{ odd},\\
D &= \bigcup_{k\ge 0} A_k \cup \bigcup_{k \ge 0} U_k,\\
B &= B((3/4, 1/2), 1/8),\\
z_k &= (3 \cdot 2^{-2k-2}, 1/2), \qquad k \ge 1,\\
C_k &= \{(x, y) \in \mathbb{R}^2 : 2^{-2k-1} \le x \le 2^{-2k}, y = 1/2\}, \qquad k \ge 1,\\
F_k &= \{(x, y) \in \mathbb{R}^2 : 2^{-2k-1} \le x \le 2^{-2k}, y = 1/4 \text{ or } y = 3/4\}, \qquad k \ge 1,\\
S_k &= \inf\{t \ge 0 : X_t \in C_k\}, \qquad k \ge 1,\\
T_k &= \inf\{t \ge S_k : X_t \in F_k\}, \qquad k \ge 1.
\end{aligned}$$

Since the part of D between the two line segments comprising F_k is a rectangle whose the long side has length $1/2$, it is easy to see that the distribution of $T_k - S_k$ is

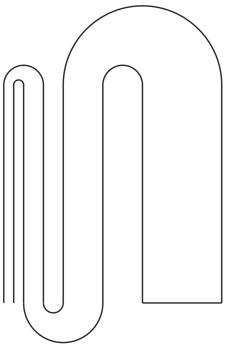

Fig. 3.2 Snake-like J_1 domain

the same as the distribution $\mathcal{Q}$ of the hitting time of $\{-1/4, 1/4\}$ by the one-dimensional Brownian motion starting from 0. By the strong Markov property of RBM X on $\overline{D}$, $\{T_k - S_k\}_{k\geq 1}$ are i.i.d. with distribution $\mathcal{Q}$. If the process X_t starts from z_k then it must go through the channels containing C_j and F_j for all $j < k$, before hitting B. Then $T_B \geq \sum_{j=1}^{k-1} T_j - S_j$ and this easily implies that $\sup_k \mathbb{E}^{x_k} T_B = \infty$. One can also reach this conclusion using Theorem 2.2.

It remains to show that $D \in J_1$. Let the two continuous curves comprising $\partial D \setminus \{(x, y) : 1/2 \leq x \leq 1, y = 0\}$ be called γ_1 and γ_2 and let σ be the set of points in D equidistant from γ_1 and γ_2. For $x \in \sigma$, let $\rho(x)$ be the distance from x to $(3/4, 0)$ along σ. For $x \in D \setminus \sigma$, find the point y on σ which is closest to x and set $\rho(x) = \rho(y)$.

Consider any admissible set $F \subset D$ with $|F| < 1/8$. Since $\alpha = 1$, it is enough to assume F is connected. Suppose ∂F does not touch one of the curves γ_1 and γ_2. Let $a = \inf_{x\in F} \rho(x)$ and $b = \sup_{x\in F} \rho(x)$. Then the length of $\partial F \cap D$ is bounded below by $c_1(b - a)$ (this may be infinite) and $|F| < c_2(b - a)$. Next suppose ∂F touches both γ_1 and γ_2. Let K be the connected part of $\partial F \cap D$ for which we have $\inf_{x\in K} \rho(x) = a$. If $x = (x_1, x_2) \in K$ with $\rho(x) \leq a + 1$ then the length of K is bounded below by $c_3 x_1$ and $|F| \leq c_4 x_1$. We conclude that $|F|$ is bounded by a constant times the length of $\partial F \cap D$. It follows that $D \in J_1$.

□

Proof of Theorem 2.5. (i) Suppose that D is a domain in $J_{2,\alpha}$ for some $\alpha < 1/2$ and has finite volume. By (2.4) there is a constant $c > 0$ such that

$$\|u\|_{1/\alpha} \leq c\,(\|\nabla u\|_2 + \|u\|_2) \qquad \text{for } u \in W^{1,2}(D). \tag{3.14}$$

By Varopoulos' theorem (see Theorem 2.4.2 in [D]), there is a constant $c_1 > 0$ so that

$$e^{-t} p_t(x, y) \leq c_1 t^{-\mu t} \qquad \text{for every } t > 0 \text{ and } x, y \in D^*,$$

where $\mu = (1 - 2\alpha)^{-1}$ and $p_t(x, y)$ is the the transition density function of the RBM X^* on D^* (see Section 3.1). In particular, $p_t(x, y)$ is a bounded function on $D^* \times D^*$ for every $t > 0$. Since D has finite volume and $m(D^* \setminus D) = 0$,

$$\int_D \int_D p_t(x, y)^2 dxdy = \int_D p_{2t}(x, x)dx < \infty,$$

that is, the semigroup P_t of X^* is a Hilbert-Schmidt operator. So P_t is a self-adjoint compact operator in $L^2(D, dx)$ (see Problem 5.1.4 of [Fr]) and hence P_t, and therefore the Neumann Laplacian in D, has a discrete spectrum (see Problems 6.7.4 and 6.7.5 in [Fr]). Now it follows from the argument on p. 6 of [BB] or Theorem 2.4 in [BH] that there are constants $c_2, c_3 > 0$ such that

$$\sup_{x,y\in D} \left| p_t(x, y) - \frac{1}{\mathrm{Vol}(D)} \right| \leq c_2\, e^{-c_3 t} \qquad \text{for } t \geq 1.$$

Therefore, by Proposition 1.4, the parabolic Harnack principle holds on D.

(ii) As we observed previously, this part follows from Theorem 2.3(ii). □

Proof of Theorem 2.6. The following proof might be known to experts. However for reader's convenience, we spell out the details.

It is well known (see Theorem 5.4 in [A]) that the Sobolev space $W^{1,p}(\mathbb{R}^d)$ can be continuously embedded into space $L^q(\mathbb{R}^d)$ for any $p \le q \le dp/(d-p)$ when $p < d$ and for any $p \le q < \infty$ when $p = d$; that is, there is a constant $c > 0$ such that

$$\|u\|_q \le c\,\left(\|\nabla u\|_p + \|u\|_p\right) := c\,\|u\|_{1,p} \quad \text{for } u \in W^{1,p}(\mathbb{R}^d). \tag{3.15}$$

(i) If D is a $W^{1,1}$-extension domain with finite volume, there is a continuous linear map $T : W^{1,1}(D) \to W^{1,1}(\mathbb{R}^d)$ such that $Tu = u$ a.e. on D for $u \in W^{1,1}(D)$. It follows then from (3.15) with $p = 1$ and $q = d/(d-1)$ that for $u \in W^{1,1}(D)$,

$$\|u\|_q \le \|Tu\|_q \le c_1\,\|Tu\|_{1,1} \le c_2\,\|u\|_{1,1}.$$

Now by (2.3), we conclude D is a domain in $J_{\frac{d-1}{d}}$.

(ii) If D is a $W^{1,2}$-extension domain with finite volume, there is a continuous linear map $T : W^{1,2}(D) \to W^{1,2}(\mathbb{R}^d)$ such that $Tu = u$ a.e. on D for $u \in W^{1,2}(D)$. When $d \ge 3$, by (3.15) with $p = 2$ and $q = 2d/(d-2)$, we have

$$\|u\|_q \le \|Tu\|_q \le c_3\,\|Tu\|_{1,2} \le c_4\,\|u\|_{1,2} \quad \text{for } u \in W^{1,2}(D).$$

By (2.4), D is a $J_{2,\alpha}$-domain with $\alpha = \frac{d-2}{2d}$ and so it is a non-trap domain. When $d = 2$, the same argument shows that

$$\|u\|_q \le c_5\,\|u\|_{1,2} \quad \text{for } u \in W^{1,2}(D)$$

holds for every $q < \infty$. By (2.4) D is in class $J_{2,\alpha}$ for any $\alpha > 0$. □

Proof of Theorem 2.9. (i) Let X_t be the reflected Brownian motion in D and set $B = F^{-1}(B(0,r_0))$ for $r_0 > 0$ such that $B(0, 2r_0) \subset F(D)$. We will estimate $\mathbb{E}^x T_B$. The estimate is trivial for $x \in B$ so assume that $x \in \overline{D} \setminus B$ and let $U_t = |F(X_{t\wedge T_B})|$. Our assumptions on the mapping F, the domain D and the vector field $\mathbf{v}$ easily imply, via the Itô formula, that U_t satisfies

$$U_{t\wedge T_B} - U_0 = \int_0^{t\wedge T_B} a(X_s)dW_s + \int_0^{t\wedge T_B} b(X_s)ds + V_{t\wedge T_B},$$

where W_t is a Brownian motion, V_t is a non-increasing process—a singular drift corresponding to the reflection on the boundary, and $\sup_{x\in\overline{D}}(|a(x)|, |a(x)|^{-1}, |b(x)|) = c_1 < \infty$, where the constant c_1 depends only on the bounds for the derivatives of F in $D \setminus B$. Let $c(t) = \int_0^t a(X_s)^{-2}ds$ and let $Z_t = U_{c(t)}$ be the corresponding time change of U_t. Note that for some constants $c_2, c_3 \in (0,\infty)$ and all $t \le T_B$, $c_2 t \le c(t) \le c_3 t$. Let $T_0 = c^{-1}(T_B)$. We obtain

$$Z_{t\wedge T_0} - Z_0 = \widetilde{W}_{t\wedge T_0} + \int_0^{t\wedge T_0} \widetilde{b}(X_s)ds + \widetilde{V}_{t\wedge T_0},$$

where $\widetilde{W}_t$ is a Brownian motion, $|\widetilde{b}(x)|$ is bounded by a constant $c_4 < \infty$ and $\widetilde{V}_t$ is non-increasing. Let r_1 be the diameter of $F(D)$. For some $p_1 > 0$,

$$\mathbb{P}(\widetilde{W}_{t+1} - \widetilde{W}_t < -c_4 - r_1 - 1) \geq p_1,$$

so for any $t \geq 0$,

$$\mathbb{P}(Z_{(t+1)\wedge T_0} = r_0 \mid Z_t) \geq p_1.$$

Let $T' = \inf\{t : Z_t = r_0\}$. By the Markov property applied at times k, for all $x \in \overline{D} \setminus B$, $\mathbb{P}(T' > k \mid X_0 = x) \leq c_5(1 - p_1)^k$, and, therefore,

$$\mathbb{P}(T_B > c_3 k \mid X_0 = x) \leq c_5(1 - p_1)^k, \tag{3.16}$$

where $c_5 < \infty$ depends only on the bound c_1. Hence we have $\sup_x \mathbb{E}^x T_B \leq c_6 < \infty$.

(ii) Let D be a twisted starlike domain and let F be the corresponding function. Find $r_0 > 0$ such that $B(0, 2r_0) \subset F(D)$ and let $B = F^{-1}(B(0, r_0))$ and $B_1 = F^{-1}(B(0, 3r_0/2))$. It is easy to see that there exists a monotone sequence of starlike domains $\widetilde{D}_k \uparrow F(D)$ with C^2 boundaries such that $\widetilde{D}_k \supset B(0, 2r_0)$ for every $k \geq 1$. Let $D_k = F^{-1}(\widetilde{D}_k)$ and note that if we take the vector field of reflection $\mathbf{v}_k(x)$ on ∂D_k to be the normal vector field $\mathbf{n}(x)$ then the assumptions of part (i) of the theorem are satisfied for D_k and $\mathbf{v}_k$. Fix any $x \in \overline{D} \setminus B_1$. When k is large enough, $x \in D_k$. Let X_t^k be the reflected Brownian motion in D_k defined as in (2.6), with $X_0^k = x$. Since $D^k \uparrow D$, by Theorem 2 in [BC], the processes X_t^k converge weakly to X_t with $X_0 = x$, the reflected Brownian motion in D starting from x. Recall that the estimates obtained in the first part of the proof depend only on the bounds for the derivatives of F and we can use the same mapping F for each D_k. Hence, by (3.16),

$$\mathbb{P}(T_B^{X^k} > c_3 k \mid X_0^k = x_k) \leq c_5(1 - p_1)^k,$$

where c_3 and c_5 do not depend on k or x. Here and in the sequel, whenever there is a danger of confusion, we use T_B^Z to denote the first hitting time of B by a process Z; that is, $T_B^Z := \inf\{t \geq 0 : Z_t \in B\}$. The last estimate implies that

$$\sup_{k \geq 1} \sup_{x \in D_k} \mathbb{E}^x \left[T_B^{X_k} \right] < \infty$$

and so $\sup_{x \in D} \mathbb{E}^x \left[T_{B_1}^X \right] < \infty$.

Proof of Proposition 2.11. Let $\sigma = \{(x, y) \in \mathbb{R}^2 : x \geq 2, y = 0\}$ and $z_0 = (x_0, 0) = (2, 0)$. We will define $z_n = (x_n, 0)$ and cuts γ_n inductively. If $\alpha < \infty$ denotes the Lipschitz constant of the function f defining the horn domain D_f, i.e., $|f(x) - f(y)| \leq \alpha|x - y|$ then we let $x_n = x_{n-1} + f(x_{n-1})/(2\alpha)$ and $\gamma_n = \{(x, y) \in \mathbb{R}^2 : x = x_n, |y| < f(x_n)\}$. It is easy to check (we leave it to the reader) that $\{\gamma_n\}_{n \geq 1} \cup \sigma$ divide D_f into hyperbolic blocks. Note that $\frac{1}{2} f(x_{n-1}) \leq$

$f(x) \le \frac{3}{2} f(x_{n-1})$ for $x \in [x_{n-1}, x_n]$ and so there are constants $0 < c_1 < c_2 < \infty$ depending only on α such that for all $n \ge 1$,

$$c_1 < \int_{x_{n-1}}^{x_n} \frac{1}{f(x)} dx < c_2.$$

Hence there are constants $0 < c_3 < c_4 < \infty$ depending only on α and $f(1)$, such that for large n and $y \in [x_n, x_{n+1}]$,

$$c_3 n < \int_1^y \frac{1}{f(x)} dx < c_4 n.$$

Since the area of D_n is $\int_{x_n}^{x_{n+1}} 2f(x)dx$,

$$c_3/4 < \frac{\int_1^{x_{m+1}} \int_1^y \frac{1}{f(x)} dx\, f(y)dy}{\sum_{n=1}^m n|D_n|} < c_4,$$

for large m. This proves that for the prime end representing the point at infinity, the condition $\sum_{n\ge1} n|D_n| < \infty$ is equivalent to $\int_1^\infty \int_1^y \frac{1}{f(x)} dx f(y)dy < \infty$. We omit a tedious but routine argument showing that if $\sum_{n\ge1} n|D_n| < \infty$ is satisfied for the prime end at infinity then the supremum of $\sum_{n\ge1} n|D_n|$ over all prime ends is finite. □

Proof of Proposition 2.12. (i) It is well known and not hard to verify that the snowflake domain is a John domain and an (ε, δ)-domain. One can use either Corollary 2.7 or 2.8 to conclude that the snowflake is not a trap domain.

(ii) We have mentioned in Section 2 that proving that a domain belongs to a class J_α is cumbersome when domain is not smooth. In view of part (iii), part (ii) of this proposition is meant only as an illustration of Theorem 2.3 so we will leave our claim at the heuristic level. Under the assumptions of part (ii), the opening between two adjacent triangles in the construction of the modified snowflake D_f is of size a^β, where a is the side length of the smaller triangle and $\beta < 2$. The area behind this opening is of order a^2 so if we take the admissible set F to be the set cut off by the line segment closing the opening, we obtain $|F|^{\beta/2} \le c_1\, a^\beta = c_1\, \mathcal{S}(\partial_i F)$. We see that $D_f \subset J_{\beta/2}$ and Theorem 2.3(i) implies that D_f is not a trap domain.

(iii) Consider a prime end ζ in D_f which is accessible only by going through an infinite sequence of triangles comprising the domain. Consider two adjacent triangles T_1 and T_2 in this sequence, with the side length of the smaller triangle equal to a. Let the size of the opening between the triangles be $\exp(-a^{-\gamma})$ and let y be its center. Let $\rho_m^1 = \{z \in T_1 : |y - z| = 2^{-m}\}$ and $\rho_m^2 = \{z \in T_2 : |y - z| = 2^{-m}\}$, and limit the range of m by $\exp(-a^{-\gamma}) \le 2^{-m} \le a/8$. Let σ be the polygonal line with vertices at the center of D_f and consecutive centers of openings between the triangles in the sequence leading to ζ. It is easy to see that the union of σ and all curves ρ_m^1 and ρ_m^2 corresponding to all pairs of adjacent triangles in the sequence, divides the domain into hyperbolic blocks. Relabel the family of all ρ_m^1's and ρ_m^2's as γ_n's and recall the definition of domains D_n from Section 2.1. We have to estimate $\sum_n n|D_n|$.

Consider D_n whose boundary contains ρ_m^1 or ρ_m^2, corresponding to a triangle with side length a. The area of this set D_n is at most $c_1 a^2$ and the number of such domains D_n corresponding to a single triangle is bounded by $c_2 a^{-\gamma}$. Hence, the portion of $\sum_n n|D_n|$ corresponding to the triangle with side length a is bounded by $c_1 a^2 c_2 a^{-\gamma} = c_3 a^{2-\gamma}$. The sequence of triangle diameters a_k along σ is geometric so $\sum_n n|D_n| \leq \sum_k c_3 a_k^{2-\gamma}$ is finite if $\gamma < 2$. A similar argument shows that $\sum_n n|D_n| = \infty$ for $\gamma > 2$. We omit a tedious but routine argument extending the estimates to prime ends which correspond to boundary points accessible via a finite sequence of triangles.

□

Our next proof involves the notion of the quasi-hyperbolic distance. This concept was used implicitly in Theorem 2.2 and its proof but this is the first time we will use it in an explicit way, because we want to quote a result of Smith and Stegenga ([SS]). The quasi-hyperbolic distance between points $x, y \in D$ is defined as

$$h(x, y) = \inf_{\Gamma} \int_{\Gamma} \frac{ds}{\operatorname{dist}(\Gamma(s), \partial D)},$$

where the infimum is taken over all rectifiable arcs $\Gamma(s) \subset D$, joining x and y. The quasi-hyperbolic distance is comparable to the standard hyperbolic distance. See [P] for this fact and other information about the quasi-hyperbolic distance in the two-dimensional setting.

Proof of Proposition 1.4 (iii). We will use one of the examples from [SS] so, for reader's convenience, we will describe the domain using the same notation as in [SS]. Let R_n denote the disc $B(x_n, c_n)$ with center $x_n \in \mathbb{R}^2$ and radius $c_n > 0$, for $n \geq 0$. We take $x_0 = 0$, $c_0 = 1$, and assume that $1 < |x_n| < 2$ for $n \geq 1$, and that the discs R_n are disjoint. For $n \geq 1$ let $x_n' = x_n/|x_n|$ and $b_n = |x_n - x_n'| - c_n$. Suppose that $a_n \in (0, c_n)$ and for $n \geq 1$ let $C_n = \bigcup_{0 \leq |x - x_n'| \leq b_n} B(x, a_n)$. Assume that $C_n \cup R_n$ are disjoint and let $D = \bigcup_{n=0}^{\infty} (R_n \cup C_n)$, where $C_0 = \emptyset$.

We will assume that $b_n/c_n \to 0$ and $a_n/c_n \to 0$ as $n \to \infty$ and that D has finite volume. Hence, the following condition needed to apply a result from [SS] holds: $a_n b_n / c_n^2 \to 0$.

Fix some $k \geq 1$ and let ζ_k be the prime end corresponding to the point in $\partial R_k \cap \partial D$ that lies on the line passing through x_0 and x_k. Let D_n be hyperbolic blocks corresponding to ζ_k as in Theorem 2.2. The largest of sets D_n inside R_k, say D_{n_0}, will have area comparable to the area of R_k, and it is easy to see that D_n's can be chosen so that $|D_{n+1}|/|D_n| < c < 1$ for $n \geq n_0$ and $|D_{n-1}|/|D_n| < c < 1$ for those $n \leq n_0$ with $D_n \subset R_k$. This implies that $\sum_{n=1}^{\infty} n|D_n|$ is comparable to $1 + n_0|D_{n_0}|$ and hence to $1 + n_0|R_k|$, since the sum of $n|D_n|$ over those D_n that are not in R_k is comparable to 1. Recall that z_n is the intersection point of γ_n and σ in the definition of hyperbolic blocks for D. It is clear from our proof of Theorem 2.2 that n_0 is comparable to the quasi-hyperbolic distance $h(x_0, z_{n_0})$ between x_0 and z_{n_0}, and it is easy to see that this distance is comparable to $h(x_0, x_k)$, so $\sum_n n|D_n|$ is comparable to $1 + h(x_0, x_k)|R_k|$. According to Theorem 2.2, D is not a trap domain if $\sup_k h(x_0, x_k)|R_k| < \infty$ (other prime ends can be analyzed in a similar way).

Theorem 15(ii) of [SS] says that the embedding $W^{1,2}(D) \to L^2(D, dx)$ is compact if and only if $\lim_{k\to\infty} h(x_0, x_k)|R_k| = 0$. Hence we conclude by Lemma 3.1 that the 1-resolvent R_1 of the Neumann Laplacian in D is compact if and only if

$$\lim_{k\to\infty} h(x_0, x_k)|R_k| = 0.$$

It follows that by a suitable choice of a_n, b_n and c_n, we can construct a non-trap domain where the 1-resolvent of the Neumann Laplacian is not compact. □

Remark 3.8. The quasi-hyperbolic distance can also be used to reinterpret Proposition 2.11 since for Lipschitz functions f there are constants c_1 and c_2 so that for $x \in \mathbb{R} \subset \mathbb{C}$ with $x > 2$,

$$c_1 \leq \frac{f(x)}{\operatorname{dist}(x, \partial D_f)} \leq c_2.$$

This implies $\int_2^y 1/f(x)dx$ is comparable to the hyperbolic distance from 2 to $y > 2$ in D_f. Note that the half-line $(1, \infty)$ is a hyperbolic geodesic in D_f by symmetry. Thus a horn domain D_f is non-trap if and only if

$$\int_2^\infty h(x) f(x)dx = \int_2^\infty |D_f \cap \{z : \operatorname{Re} z > x\}| dh(x) < \infty,$$

where $h(x)$ is the hyperbolic distance from 2 to x.

Proof of Proposition 2.13. Hyperbolic blocks for the origin in the spiral domain are formed by letting σ be a curve running down the "middle" of the channel, and using cross cuts that divide the channel into approximate squares. Consider the portion of the channel bounded by the curve $r = \theta^{-p}$ with $2\pi(n-1) \leq \theta \leq 2\pi(n+1)$. Then σ makes one "loop" around the origin within this portion of the channel. The width of this channel is comparable to $n^{-(p+1)}$ and the length of this portion of σ is comparable to n^{-p}, so that there are n approximate squares in this loop. If C_j is one of the cross cuts in this channel, then the component Ω_j of $S_p \setminus C_j$ which does not contain z_0 has area comparable to n^{-2p}. Thus the total contribution to (2.1) from this portion of the channel is n/n^{2p}, so that (2.1) is comparable to

$$\sum \frac{1}{n^{2p-1}}.$$

Thus S_p is not a trap domain if $p \leq 1$, and (2.1) is finite for $p > 1$. We leave the verification that (2.1) is uniformly bounded for all other boundary points if $p > 1$ to the reader. We also remark that when $p = 1$, the same reasoning as in the last paragraph of the proof of Theorem 2.3 shows that $S_p \in J_1$. □

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