Derivation of Chordal Loewner from Radial Loewner

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Abstract. We give an explicit analytic conversion between the chordal and radial versions of Loewner's differential equation, allowing the derivation of one from the other. This also allows us to translate some results known in one regime to results not previously known in the other.

1. Introduction.

Loewner's differential equation has been instrumental in solving important problems in classical function theory. More recently a variant, called the chordal Loewner equation, has arisen as a fundamental tool in studying random processes in the plane. The classical, or radial, version parameterizes a curve in the disk by minus the logarithm of the conformal radius, whereas the chordal version uses a different parameterization of curves in the upper half plane called the half plane capacity. Of course the disk and the half plane are conformally equivalent, but there is a fundamental difference between the radial and chordal equations. If we let the half plane parameter tend to ∞, then the associated curve tends to ∞, which is a boundary point of $\mathbb{H} = \{z : \mathrm{Im} z > 0\}$, whereas when the logarithm of the conformal radius tends to ∞, the associated curve in $\mathbb{D} = \{z : |z| < 1\}$ tends to the origin, an interior point of $\mathbb{D}$. Another aspect of this difficulty is that under the chordal flow, no point in the upper half plane can be fixed. For this reason some have suggested that the two regimes are quite different and could not be directly related. In [BCD], for example, a more general context is given which contains both the radial and chordal equations as special cases, but an explicit analytic relation between the two regimes does not seem to follow from their work.

Several results have been proved in one setting and the analogous result obtained in the other setting by imitating the main ideas of the proof. For example, a sufficient condition [MR] for the radial Loewner equation to have a trace which is a quasiarc is that the local Lip-$\frac{1}{2}$ semi-norm of its driving term is small. The easier chordal case also follows from similar ideas, as indicated in [MR]. Simple Examples in both the disk and the half plane show that norm equal to 4 is not sufficient. Lind [L] proved that any norm smaller than 4 suffices in the half plane setting. Prohkonov and Vasiliev [PV] indicated how to prove the same result in the disk setting using Lind's ideas. The

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local Lip-$\frac{1}{2}$ semi-norm has also been used in [LMR] to give a determistic version of the fundamental result of [RS] describing the nature of the SLE_κ curves for various ranges of the parameter κ.

Other results concerning the smoothness of Loewner curves are known in one setting but not the other. Earle and Epstein [EE] worked in the radial setting and proved that if the curve is C^n then the driving term is C^{n-1}, $n = 1, 2, \ldots$. They also proved that if the curve is analytic, then so is the driving term. Aleksandrov [Ale] proved in the radial case that if the driving term has bounded first derivative then the curve is C^1. Wong [CW] proved in the chordal case that if the driving term is $C^{\beta-\frac{1}{2}}$ as a function of half-plane capacity then the curve is C^β for $1 < \beta < 2$ and $2 < \beta < \frac{5}{2}$. The present short note is motivated by Wong's question of whether the same result holds in the radial case. His techniques are quite involved, and it does not appear to be simple to just translate the ideas to the disk.

In the second section we give an explicit change of variables in the domain and "time" variables to translate between the chordal and radial versions. We show how to derive one version of Loewner's equation from the other and show that the associated driving terms have the same smoothness in terms of C^β (Corollary 3) and in terms of the local Lip-$\frac{1}{2}$ (semi) norms (Corollary 4). In the third section attempt to resolve the apparent difficulty mentioned above about curves tending to boundary points and curves tending to interior points. We give a different derivation by viewing the point at ∞ in the chordal case as an interior point for the complement of the union of the curve and its reflection about the real line, and then derive the chordal equation from the slight extension of the classical radial case to the radial case for two symmetric curves. The discussion here will be limited to domains slit by Jordan arcs. The reader can easily extend the proofs, virtually unchanged, to more general hulls.

2. A change of variables.

Suppose γ is a Jordan arc in $\mathbb{H} \cup \{0\}$ beginning at $\gamma(0) \in \mathbb{R}$, where $\mathbb{H} = \{z : \mathrm{Im} z > 0\}$ is the upper half plane. Fix $z_0 \in \mathbb{H} \setminus \gamma$, let $\tau = (z - z_0)/(z - \overline{z_0})$, a conformal map of $\mathbb{H}$ onto $\mathbb{D} = \{\zeta : |\zeta| < 1\}$, and let

$$\alpha = \tau(\gamma). \tag{1}$$

Then $\alpha \subset \mathbb{D}$ with $\alpha(0) = z_0/\overline{z_0} \in \partial\mathbb{D}$ and $1 = \tau(\infty) \notin \alpha$. By the Riemann mapping theorem, there is a unique conformal map $F = F(s, \zeta)$ of $\mathbb{D}$ onto $\mathbb{D} \setminus \alpha[0, s]$ such that $F(0) = 0$ and $F'(0) > 0$. By Schwarz's lemma, we may choose the parameterization of α so that

$$F(\zeta) = F(s, \zeta) = e^{-s}\zeta + \mathrm{O}(\zeta^2).$$

The quantity e^{-s} is called the *conformal radius* of $F(s,\mathbb{D})$.

By Caratheodory's Theorem, F extends to be continuous on the closed unit disk with $F(I) = \alpha[0,s]$ and $F(\partial\mathbb{D}\setminus I) = \partial\mathbb{D}\setminus\alpha(0)$, for some (closed) circular arc $I = I(s) \subset \partial\mathbb{D}$. By the Schwarz Reflection Principle, F extends to be a conformal map of $\mathbb{C}^*\setminus I$ onto $\mathbb{C}^*\setminus(\alpha[0,s]\cup\alpha^*[0,s])$, where $\mathbb{C}^*$ is the extended plane and $\alpha^* = \{\zeta : 1/\overline{\zeta}\in\alpha\}$ is the reflection of α about the unit circle. Then by Loewner's equation (see e.g. Ahlfors, Conformal Invariants) the function F satisfies

$$\frac{\dot{F}}{F'} = \zeta\left(\frac{\zeta+\omega}{\zeta-\omega}\right), \qquad (R-LDE)$$

where $\dot{F} = \frac{dF}{ds}$, $F' = \frac{dF}{d\zeta}$ and $\omega = \omega(s)$ is a unimodular continuous function of s given by $F(\omega(s)) = \alpha(s)$. By our choice of τ, $1\notin\alpha$, so that $|F| = 1$ in a neighborhood of $A = F^{-1}(1)$. By the Schwarz Reflection Principle, F is analytic in a neighborhood of A and $F'(A)\neq 0$.

Let $\sigma_1 = A(z-i)/(z+i)$ be the conformal map of $\mathbb{H}$ onto $\mathbb{D}$ with $\sigma_1(\infty) = A$ and $\sigma_1(i) = 0$. Then

$$f_1 = \tau^{-1}\circ F\circ\sigma_1 = Dz + C + \mathrm{O}\left(\frac{1}{z}\right)$$

is a conformal map on $\mathbb{H}$, indeed on all of $\mathbb{C}\setminus\sigma_1^{-1}(I)$. By equating coefficients in the expansion about ∞ of the equation $\tau\circ f_1 = F\circ\sigma_1$, D and C can be determined explicitly in terms of A, $F'(A)$ and $F''(A)$:

$$D = \frac{z_0 - \overline{z_0}}{2iAF'(A)} \qquad (2)$$

and

$$C = \overline{z_0} + \frac{(z_0-\overline{z_0})}{2AF'(A)}\left(1 + \frac{AF''(A)}{F'(A)}\right). \qquad (3)$$

We claim that A, $F'(A)$ and $F''(A)$ are continuous functions of s. In fact they have one degree of smoothness more than ω. To see this, suppose that $\omega\in C^\beta[0,S]$ where $S<\infty$ and $\beta\geq 0$. Then for $s<S$, the function $g_s = F_s^{-1}\circ F_S$ satisfies the Loewner ordinary differential equation

$$\dot{g} = g\left(\frac{\omega+g}{\omega-g}\right). \qquad (4)$$

Fix z in a neighborhood of $\mathbb{D}\cup\{A\}$ and apply Picard iteration to solve this equation as a function of s. As a uniform limit of continuous functions, we have that $g_s\in C$. By induction, since $\omega\in C^\beta$, we have that $\dot{g}\in C^\beta$ and hence $g = g_s\in C^{1+\beta}$ as a function of s. In particular

$$A(s) = g_s(A_S)\in C^{1+\beta}. \qquad (5)$$

Differentiating (4) with repect to the domain variable z and then dividing by $g_s'(z)$ yields that $\log g_s' \in C^1$ and again by induction $\log g_s' \in C^{1+\beta}$, and hence $g_s' \in C^{1+\beta}$. By Cauchy's integral formula for w sufficiently close to $g_z(z_0)$

$$g_s^{-1}(w) = F_S^{-1} \circ F_s(w) = \int_{|\zeta - z_0| = \varepsilon} \frac{g_s'(\zeta)\zeta}{g_s(\zeta) - w} \frac{d\zeta}{2\pi i}.$$

We conclude that $g_s^{-1} \in C^{1+\beta}$ and hence $F_s(w) \in C^{1+\beta}$. Moreover by the Cauchy integral formula applied to a small circle Γ winding once around A we have that

$$F_s^{(k)}(A) = \frac{1}{k!} \int_\Gamma \frac{F_s(\zeta)}{(\zeta - A)^{k+1}} \frac{d\zeta}{2\pi i} \in C^{1+\beta}.$$

Thus by (2) and (3)

$$B = C + iD \in C^{1+\beta}. \tag{6}$$

We can normalize by setting

$$f(z) = f_1\left(\frac{z - C}{D}\right) = z - \frac{2t}{z} + \mathrm{O}\left(\frac{1}{z^2}\right). \tag{7}$$

Then f is a differentiable function of s, conformal in $z \in \mathbb{H}$ with $f(\mathbb{H}) = \mathbb{H} \setminus \tau^{-1}(\alpha[0,s]) \supset \mathbb{H} \setminus \gamma$ and $f(B) = z_0$. It is not hard to show that $t > 0$ and $dt/ds > 0$ by applying Schwarz's lemma to F, for instance. The quantity $2t$, or sometimes t, is called the *half-plane capacity.*

Set

$$\sigma(z) = A\left(\frac{z - B}{z - \overline{B}}\right) = \sigma_1\left(\frac{z - C}{D}\right), \tag{$X1$}$$

where $B = C + iD$. Then

$$\tau \circ f = F \circ \sigma,$$

so by the chain rule

$$\frac{\dot{f}}{f'} = \frac{\tau' \circ f \; \dot{f}}{\tau' \circ f \; f'} = \frac{\dot{F} \circ \sigma + F' \circ \sigma \; \dot{\sigma}}{F' \circ \sigma \; \sigma'} = \frac{\sigma}{\sigma'} \frac{\sigma + \omega}{\sigma - \omega} + \frac{\dot{\sigma}}{\sigma'}. \tag{8}$$

Note that

$$\frac{\dot{\sigma}}{\sigma} = \frac{\dot{A}}{A} - \frac{\dot{B}}{z - B} + \frac{\dot{\overline{B}}}{z - \overline{B}} \tag{9}$$

and

$$\frac{\sigma'}{\sigma} = \frac{B - \overline{B}}{(z - B)(z - \overline{B})}. \tag{10}$$

Set

$$\lambda(s) = \sigma^{-1}(\omega(s)) = C + iD\left(\frac{A + \omega}{A - \omega}\right). \tag{11}$$

Note that $\omega \in C^\beta$ implies $\lambda \in C^\beta$, for $\beta \geq 0$, as functions of s. By (8), (9), and (10)

$$\frac{\dot{f}}{f'} = \frac{(z-B)(z-\overline{B})}{B-\overline{B}} \frac{A(z-B)+\omega(z-\overline{B})}{A(z-B)-\omega(z-\overline{B})} + \frac{\dot{A}}{A}\frac{(z-B)(z-\overline{B})}{B-\overline{B}} - \frac{\dot{B}(z-\overline{B})}{B-\overline{B}} + \frac{\dot{\overline{B}}(z-B)}{B-\overline{B}}$$

$$= \frac{P}{z-\lambda},$$

where P is a polynomial of degree at most 3 in z, with coefficients depending on s.

By the normalization (7),

$$\frac{\dot{f}}{f'} = \frac{-2\frac{dt}{ds}}{z} + \mathrm{O}(\frac{1}{z^2}).$$

Thus the polynomial P must have degree 0 in z and is negative since $dt/ds > 0$. In other words, if we change variables from s to t where

$$\frac{dt}{ds} = -\frac{P(s)}{2}, \qquad\qquad t(0) = 0,$$

then

$$\frac{df}{dt} = \frac{2f'(z)}{\lambda - z},$$

where $\lambda(s(t))$ is a real-valued continuous function of t. This is the chordal version of Loewner's equation.

One observation that may shed some light is that each version of Loewner's equation implies the associated normalization.

We can now give the explicit "time change" from s to t. If we multiply equation (8) by $z-\lambda$ and let $z \to \lambda$, and use the identity $\omega = \sigma(\lambda)$ and (10), we obtain

$$P = \lim_{z\to\lambda}(z-\lambda)\frac{\dot{f}}{f'} = 2\left(\frac{\sigma(\lambda)}{\sigma'(\lambda)}\right)^2 = 2\left(\frac{(\lambda-B)(\lambda-\overline{B})}{B-\overline{B}}\right)^2,$$

and so

$$\frac{dt}{ds} = \frac{|\lambda - B|^4}{4\,(\mathrm{Im}B)^2}, \tag{12}$$

where $B(s)$ and $\lambda(s)$ are given by (2), (3), (6), and (11).

Summary: Given a Jordan curve $\gamma(0,T] \subset \mathbb{H}$, $T < \infty$ and $\gamma(0) \in \mathbb{R}$, suppose $z_0 \notin \gamma$, then set $\tau(z) = (z-z_0)/(z-\overline{z_0})$. Define α by (1) and let F be the conformal map of $\mathbb{D}$ onto $\mathbb{D} \setminus \alpha[0,s]$ where α is parameterized so that $F'(0) = e^{-s}$. Then $F(s,\zeta)$ satisfies (R-LDE). Set $A = F^{-1}(1)$

and define C, D, B, σ and λ by (2), (3), (6) and (X1). Define $t(s)$ by (12) with $t(0) = 0$, and set $f(t,z) = \tau^{-1} \circ F(s(t), \sigma(s(t), z))$. We have proved:

Theorem 1. *The function f satisfies (C-LDE) and $f(t, \lambda(t)) = \gamma(t)$.*

Conversely, given a Jordan curve $\alpha(0, S] \subset \mathbb{D}$ with $\alpha(0) \in \partial\mathbb{D}$ and $S < \infty$, as above set $\tau(z) = (z - z_0)/(z - \overline{z_0})$. Define $\alpha = \tau^{-1}(\gamma)$ and let f be the conformal map of $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma[0, t]$, where γ is parameterized so that for z near ∞ we have the expansion $f(t, z) = z - \frac{2t}{z} + O(\frac{1}{z^2})$. Then $f(t, z)$ satisfies (C-LDE). Set $B = B(t) = f^{-1}(z_0)$ and define $A(t)$ by

$$\frac{d}{dt} \log A(t) = -i\left(\frac{(\lambda - \mathrm{Re}B)4\mathrm{Im}B}{|\lambda - B|^4}\right),$$

with $A(0) = 1$. Define $s(t)$ by

$$\frac{ds}{dt} = \frac{4(\mathrm{Im}B)^2}{|\lambda - B|^4},$$

with $s(0) = 0$. Define σ by (X1) and set $\omega(s) = \sigma(s, \lambda(t(s))$. Then

$$\sigma^{-1}(s, \zeta) = \frac{B - \overline{BA}\zeta}{1 - \overline{A}\zeta},$$

where $B = B(t(s))$ iand $A = A(t(s))$. Set $F(s, \zeta) = \tau \circ f(t(s), \lambda(t(s), \sigma^{-1}(s, \zeta)$. Then

Theorem 2. *The function F satisfies (R-LDE) and $F(s, \omega(s)) = \alpha(s)$.*

The first Corollary gives the relation between the smoothness of the driving terms in the radial and chordal cases.

Corollary 3. *Suppose $\gamma(0, T)$ is a curve in $\mathbb{H}$, parameterized by half-plane capacity $2t$ with $\gamma(0) \in \mathbb{R}$. Let $\lambda(t)$ be the associated driving term for the chordal Loewner equation. If $z_0 \notin \gamma$, set $\tau(z) = \frac{z - z_0}{z - \overline{z_0}}$ and $\alpha = \tau(\gamma)$. Reparameterizing α so that the conformal radius of $\alpha[0, s]$ equals e^{-s}, and letting $\omega(s)$ be the driving term for the radial Loewner equation, then for $\beta \geq 0$, dt/ds given by (12) is bounded above and below, and*

$$\omega(s) \in C^{\beta} \textit{ if and only if } \lambda(t) \in C^{\beta}.$$

By (1) it is clear that $\gamma \in C^{\beta}$ if and only if $\alpha \in C^{\beta}$. So by Corollary 3, the results mentioned in the introduction about the relation between the smoothness of the driving term and the curve which have been proved in either the radial or chordal case hold in both the radial and chordal cases.

Proof. The relation between ω and λ is given by the identity

$$\omega = A\left(\frac{\lambda - B}{\lambda - \overline{B}}\right). \tag{13}$$

Since γ is a compact Jordan arc, there is a curve $\beta \subset U \equiv \mathbb{C}\backslash(\gamma \cup \overline{\gamma})$ connecting ∞ to z_0 with positive spherical distance to the boundary of U. By a normal families argument, or a direct estimate using Koebe's theorem, $B(s) = f_s^{-1}(z_0)$ lies in a compact subset of $\mathbb{H}$ and $\lambda(s(t)) = f^{-1}(\gamma(t))$ lies in a bounded interval in $\mathbb{R}$. We also have that $|\lambda - B| \geq \mathrm{Im}B$ since λ is real valued, and so the first statement of Corollary 3 follows from (12). The second statement follows from (13), (5), and (6).

We define the *local Lip*-1/2 *(semi-)norm* of a function f to be

$$||f||_{\ell_{\frac{1}{2}}} = \inf_{0<\varepsilon<1} \sup_{|u-v|<\varepsilon} \frac{|f(u) - f(v)|}{\sqrt{u-v}}.$$

The next Corollary says that the local Lip-$\frac{1}{2}$ semi-norm of the driving term is the same in the radial and chordal regimes. As mentioned earlier the local Lip-$\frac{1}{2}$ norm has been used in [MR], [L], [LMR], [LR] to give a determistic version of the fundamental result of [RS] describing the nature of the SLE_κ curves for various ranges of the parameter κ.

Corollary 4. *Under the assumptions of Corollary 3,*

$$||\omega||_{\ell_{\frac{1}{2}}} = ||\lambda||_{\ell_{\frac{1}{2}}},$$

where ω is viewed as a function of s and λ is viewed as a function of t.

Proof. We claim that if either ω or λ has bounded local Lip-$\frac{1}{2}$ semi-norm then

$$\frac{|\omega(s_2) - \omega(s_1)|}{|s_2 - s_1|^{\frac{1}{2}}} = \frac{|\lambda(s(t_2)) - \lambda(s(t_1))|}{|t_2 - t_1|^{\frac{1}{2}}} + \mathrm{O}(|s_2 - s_1|^{\frac{1}{2}}),$$

where $s_j = s(t_j)$, $j = 1, 2$ and $|s_2 - s_1| \leq 1$.

The functions $A(s)$ and $B(s)$ are differentiable functions of s so that we can write

$$A_2 = A_1 + \mathrm{O}(s_2 - s_1)$$

and

$$B_2 = B_1 + \mathrm{O}(s_2 - s_1)$$

where $A_j = A(s_j)$ and $B_j = B(s_j)$. Then by the triangle inequality and (13)

$$\omega(s_2) - \omega(s_1) = A_1 \frac{(B_1 - \overline{B_1})(\lambda(s_2) - \lambda(s_1))}{(\lambda(s_2) - \overline{B_1})(\lambda(s_1) - \overline{B_1}))} + \mathrm{O}(s_2 - s_1)$$

and hence

$$\frac{\omega(s_2) - \omega(s_1)}{\omega(s_1)} = \frac{(\lambda(s_2) - \lambda(s_1))(B_1 - \overline{B_1})}{(\lambda(s_1) - B_1)(\lambda(s_1) - \overline{B_1})} + \mathrm{O}((\lambda(s_2) - \lambda(s_1))^2) + \mathrm{O}(s_2 - s_1).$$

By (12)

$$|\omega(s_2) - \omega(s_1)| = |\lambda(s_2) - \lambda(s_1)| \left| \frac{s_2 - s_1}{t_2 - t_1} \right|^{\frac{1}{2}} + \mathrm{O}(s_2 - s_1),$$

where $s_j = s(t_j)$. Corollary 4 then follows. □

The process above can be reversed to obtain the classical version of Loewner's equation from the chordal version. Loewner's equation for the inverse of f or F can be derived from the equations above by calculus or the astute reader can deduce one from the other using the plot above. The converse to (radial) Loewner's equation states that for any continuous function ω there is a solution F to Loewner's equation which is a conformal map from $\mathbb{D}$ to $\mathbb{D} \setminus \alpha$ where $\alpha([0, s])$ is a compact subset of $\overline{\mathbb{D}}$, though not necessarily a curve. Using the process above, this statement easily transfers to the corresponding statement about λ for the chordal version of Loewner's equation.

From the point of view of SLE curves, there is a fundamental difference between the radial Loewner equation on the disk and the chordal Loewner equation on the half plane. If we let the the parameter $s \to \infty$, then since $F'(0) \to 0$, we must have that

$$\liminf_{s \to \infty} |\alpha(s)| = 0,$$

so that if α has a limit as $s \to \infty$, then $\alpha(s) \to 0$, an interior point of $\mathbb{D}$. Likewise, if we let the parameter $t \to \infty$ then

$$\limsup_{t \to \infty} |\gamma(s(t))| = \infty,$$

so that if γ has a limit in $\mathbb{C}^*$ as $t \to \infty$, then $\gamma(s(t)) \to \infty$, a boundary point of $\mathbb{H}$. It is not true in general that s and t tend to ∞ simulataneously. For example if γ is the positive imaginary axis and if $z_0 = 1 + i$ then α is the portion in $\mathbb{D}$ of a circle orthogonal to $\partial\mathbb{D}$. The map f is given by $f(t, z) = \sqrt{z^2 - 4t}$. The reader can easily find a formula for F which shows that the corresponding parameter s has a finite limit as $t \to \infty$. Indeed F will converge to the conformal map of the disk onto the region between α and $\partial\mathbb{D}$ containing 0. Similarly if $\gamma(t) = 2it$, $0 \le t < 1$ and $z_0 = 2i$, then t has a finite limit, but $s \to \infty$. Another example is given by a curve $\gamma(t)$ which is Jordan for $0 \le t < 2$ but closes up when $t = 2$, surrounding the point $z_0 = i$. Then both t and s are bounded but the limiting functions f and F are not related via linear fractional transformations because z_0 and ∞ are in different components of the complement of γ. The key fact that was used in the relation between radial and chordal Loewner, and which is not present in these examples, is that z_0 and ∞ are in the same component of the complement of γ as t varies over a closed interval $0 \le t \le T < \infty$.

3. Two Symmetric Curves.

There is another way to relate chordal Loewner's equation to a slightly more general radial Loewner for which the "time" parameterizations for both tend to ∞ simultaneously.

As before, suppose f is a conformal map of the upper half plane $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma$ where γ is a Jordan arc in $\mathbb{H} \cup \{0\}$ beginning at $\gamma(0) = 0$. By Caratheodory's Theorem, we may suppose that f extends to be continuous on the closed upper half plane with $f(\infty) = \infty$. By the Schwarz Reflection Principle, f extends to be a conformal map of $\mathbb{C}^* \setminus I$ onto $\mathbb{C}^* \setminus \{\gamma \cup \overline{\gamma} \cup J\}$ where $\mathbb{C}^*$ is the extended plane, $I \subset \mathbb{R}$ is an interval, $J = [-1, 1]$ and $\overline{\gamma} = \{\overline{z} : z \in \gamma\}$ is the reflection of γ about $\mathbb{R}$. By composing f with a linear map, we may normalize f so that near ∞

$$f(z) = z + \mathrm{O}(\frac{1}{z}).$$

Although f is defined on the upper half plane, the extension of f to $C^* \setminus I$ has been composed with a linear map depending upon s, and thus the "domain" of the extension is a function of s. The map $S(z) = \frac{1}{2}(z + \frac{1}{z})$ is a conformal map of the unit disk $\mathbb{D}$ onto $\mathbb{C}^* \setminus [-1, 1]$ with $S(0) = \infty$. Let $L(z) = \delta z + \beta$, $\delta > 0, \beta \in \mathbb{R}$, be the linear map of the interval $[-1, 1]$ onto I. Then

$$F(z) = S^{-1} \circ f \circ L \circ S(z)$$

is a conformal map of $\mathbb{D}$ onto $\mathbb{D} \setminus \{\alpha \cup \overline{\alpha}\}$, where α is a Jordan arc in $\mathbb{D} \cap \mathbb{H}$ beginning at $\alpha(0) = i$. The map F satisfies $F(0) = 0$ and $F'(0) > 0$ and maps the interval $[-1, 1]$ onto $[-1, 1]$. This process can be reversed, so that given a conformal map F of $\mathbb{D}$ onto $\mathbb{D} \setminus \{\alpha \cup \overline{\alpha}\}$, with $F(0) = 0$ and $F'(0) > 0$, where α is a Jordan arc in $\mathbb{D} \cap \mathbb{H}$ beginning at $\alpha(0) = i$, then the map

$$f(z) = S \circ F \circ S^{-1} \circ L^{-1}$$

restricted to $\mathbb{H}$ is a conformal map of $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma$ where γ is a Jordan arc in $\mathbb{H} \cup \{0\}$ beginning at $\gamma(0) = 0$.

We now consider f as a function of $t > 0$ and $z \in \mathbb{H}$, as a map from $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma[0, t]$. We choose the parameterization of γ so that

$$f(z) = z - \frac{2t}{z} + \mathrm{O}(\frac{1}{z^2}).$$

Similarly we consider the corresponding map F as a function of $s > 0$ and $z \in \mathbb{D}$, as a map of $\mathbb{D}$ onto $\mathbb{D} \setminus \{\alpha[0, s] \cup \overline{\alpha[0, s]}\}$. We choose the parameterization $\alpha(s)$ so that

$$F'(0) = e^{-s}.$$

The classical version of Loewner's equation on the disk (see e. g. Ahlfors, Conformal Invariance) deals with the case of maps F of $\mathbb{D}$ onto $\mathbb{D} \setminus \alpha$ where α is a single Jordan arc. A virtually identical approach gives the differential equation for two symmetric slits, and yields:

$$\dot{F} = -zF'(z)\frac{1}{2}\left[\frac{1 + \omega z}{1 - \omega z} + \frac{1 + \overline{\omega} z}{1 - \overline{\omega} z}\right], \tag{14}$$

where $\dot{} = \frac{d}{ds}$. The function $\omega = \omega(s)$ is a continuous function of s with $|\omega(s)| = 1$ and $F(\omega(s)) = \alpha(s)$ and $F(\overline{\omega(s)}) = \overline{\alpha(s)}$. Since $F(\mathbb{D} \cap \mathbb{H}) \subset \mathbb{D} \cap \mathbb{H}$ by construction, we also have that $\mathrm{Im}\omega > 0$. Setting

$$\lambda = \frac{\delta}{2}(\omega + \overline{\omega}) + \beta, \tag{15}$$

we conclude $f(\lambda) = \gamma(t)$, where $t = t(s)$. We will later derive the relationship between the parameterizations s and t.

We seek the differential equation satisfied by $f(z, t)$. Write

$$F(z, s) = e^{-s}[z + a_2 z^2 + \ldots] \tag{16}$$

where $a_2 = a_2(s)$. Using the identity

$$\frac{1}{2}(F + \frac{1}{F}) = f(\frac{\delta}{2}(z + \frac{1}{z}) + \beta) \tag{17}$$

we obtain

$$\frac{e^s}{2z} - \frac{e^s a_2}{2} + \mathrm{O}(z) = \frac{\delta}{2z} + \beta + \mathrm{O}(z),$$

for z near 0. Equating coefficients, we obtain

$$\delta = e^s \qquad \text{and} \qquad \beta = -\frac{e^s a_2}{2}. \tag{18}$$

Substituting (16) into (14) and equating coefficients we also obtain that

$$\dot{a_2} = -a_2 - (\omega + \overline{\omega})$$

and hence

$$\dot{\beta} = \Big(\frac{\omega + \overline{\omega}}{2}\Big)e^s. \tag{19}$$

Differentiating (17) with respect to s and z yields

$$\frac{1}{2}\left(1 - \frac{1}{F^2}\right)\dot{F} = \dot{f}\left(\frac{\delta}{2}(z + \frac{1}{z}) + \beta\right) + 1Gf'\left(\frac{\delta}{2}(z + \frac{1}{z}) + \beta\right)\left(\frac{\dot{\delta}}{2}(z + \frac{1}{z}) + \dot{\beta}\right), \text{and}$$

$$\frac{1}{2}\left(1 - \frac{1}{F^2}\right)F' = f'\left(\frac{\delta}{2}(z + \frac{1}{z}) + \beta\right)\left(\frac{\delta}{2}(1 - \frac{1}{z^2})\right).$$

Setting $\zeta = \frac{\delta}{2}(z + \frac{1}{z}) + \beta$, taking the ratio of the above two equations and using (14) we obtain

$$\frac{\dot{f}(\zeta)}{f'(\zeta)} = \frac{\delta(1 - z^2)^2}{2z(1 - \omega z)(1 - \overline{\omega} z)} - \frac{\dot{\delta}}{2}(z + \frac{1}{z}) - \dot{\beta}. \tag{20}$$

Note that

$$\frac{(1 - z^2)^2}{z^2} = (z + \frac{1}{z})^2 - 4 = \left((\zeta - \beta)\frac{2}{\delta}\right)^2 - 4, \tag{21}$$

and by (15)

$$\frac{(1 - \omega z)(1 - \overline{\omega} z)}{z} = z + \frac{1}{z} - (\omega + \overline{\omega}) = \frac{2}{\delta}(\zeta - \lambda). \tag{22}$$

By (15), (18), and (19), we also have

$$\frac{\dot{\delta}}{2}(z + \frac{1}{z}) + \dot{\beta} = \zeta + \lambda - 2\beta. \tag{23}$$

Substituting (21), (22), and (23) into (20) we obtain

$$\frac{\dot{f}(\zeta)}{f'(\zeta)} = \frac{(\lambda - \beta)^2 - e^{2s}}{\zeta - \lambda} = \frac{\left(\omega - \overline{\omega}\right)^2 e^{2s}/4}{\zeta - \lambda}.$$

Thus if

$$\frac{dt}{ds} = \left(\frac{\omega(s) - \overline{\omega(s)}}{2i}\right)^2 \frac{e^{2s}}{2} \qquad \text{and} \qquad t(0) = 0,$$

then

$$\frac{d}{dt}f(\zeta) = -\frac{2f'(\zeta)}{\zeta - \lambda}, \tag{24}$$

which is the chordal version of Loewner's equation.

Given any continuous unimodular function ω, with $\text{Im}\omega > 0$, Loewner's equation (14) has a solution $F = F(z, s)$ which is a conformal map of the disk into the disk, though the complement of the image is not always the union of two symmetric curves. This can be translated to a similar statement for the driving term λ. Given any continuous real valued function λ, we can find a solution to (24) which is conformal on $\mathbb{H}$.

The relation between λ and ω when we use symmetric curves is somewhat simpler than what was obtained by the first method. By (15), (18), and (19), and using $\beta(0) = 0$, we have that

$$\lambda = \frac{e^{2s}}{2}(\omega + \overline{\omega}) + \int_0^s e^r \text{Re } \omega(r) dr.$$

We mention one geometric connection between the parameters t and s. By (16) and (17), $e^s/2$ is the logarithmic capacity, or transfinite diameter, of the set $E = \gamma[0, t] \cup \overline{\gamma[0, t]} \cup [-1, 1]$. It is also one-fourth of the length of $f^{-1}(E)$.

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