Lens chains and the geodesic algorithm for conformal mapping

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Abstract. We describe a modification of the geodesic algorithm for the numerical computation of conformal maps. This modification while improving the accuracy also allows us to give a simpler proof than in Marshall and Rohde [MR] of convergence for C^1 curves.

0. Introduction.

A variant of the zipper algorithm for the numerical computation of conformal maps is described in Marshall and Rohde[MR]. Briefly, if $z_0, \dots, z_n$ are distinct points in the plane $\mathbb{C}$, then a closed curve γ_c is constructed passing through $z_0, \dots, z_n$ such that if γ_k denotes the portion of the curve from z_0 to z_k, then $\gamma_{k+1} \setminus \gamma_k$ is the hyperbolic geodesic in $\mathbb{C} \setminus \gamma_k$ from z_k to z_{k+1}, for $k = 1, \dots, n$, where $z_{n+1} \equiv z_0$. The initial arc γ_0 is a straight line segment. The conformal maps from the upper and lower half planes to the interior and exterior (respectively) of γ_c are then computed as a composition of finitely many explicit elementary maps. This variant is called the geodesic algorithm in [MR].

Given a Jordan curve γ and a sequence of points $\{z_k\}$ on γ, the conformal maps to the interior and exterior of γ are approximated by the conformal maps to the interior and exterior of the curve γ_c, given by the goedesic algorithm. How close these conformal maps are depends on how close the curves γ and γ_c are (see [MR]). In other words, we need to understand the behaviour of γ_c between the data points $\{z_k\}$. It is proved in [MR], for example, that if $\{D_k\}_0^n$ is a sequence of disjoint open disks with ∂D_{k-1} tangent to ∂D_k at z_k, then

$$\gamma_{k+1} \setminus \gamma_k \subset D_k,$$

for $k = 1, \dots, n$. This result was deduced rather easily using an old result of Jørgensen [J], which says that disks are convex in the hyperbolic geometry of a region. Given a sequence of points $\{z_k\}$, it is in fact rare that such a sequence of pairwise tangential disks can be found. The emphasis in [MR] for the application of this result was rather on finding points $z_1, \dots, z_n$ on or near a given curve so that such disks can be found. As a result, the algorithm computes a curve close to the given curve.

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For smooth curves a much more complicated argument in [MR] located the geodesics $\gamma_{k+1}\setminus\gamma_k$ in smaller regions. This allowed us to prove C^1 convergence of a sequence of computed curves $\gamma_c^{(n)}$ to the given C^1 curve, as the mesh size $\max_j |z_{j+1}^{(n)} - z_j^{(n)}|$ decreases to zero.

In this article we describe a modification of the geodesic algorithm that allows us to locate the hyperbolic geodesics $\gamma_{k+1}\setminus\gamma_k$ in smaller regions called lenses with a simple geometric description, and with a simpler proof relying only on Jørgensen's theorem as in the disk case. The modification also improves the numerical accuracy of the geodesic algorithm.

1. The geodesic algorithm.

For the convenience of the reader we give a short description of the geodesic algorithm. For more details see [MR].

The geodesic algorithm constructs a Jordan curve through a collection of (distinct) points $z_0, \dots, z_n$ in $\mathbb{C}$. We will describe the algorithm using the right half plane $\mathbb{H}^+ = \{z : \mathrm{Re} z > 0\}$ instead of the traditional upper half plane because because of the usual convention that $-\frac{\pi}{2} < \arg\sqrt{z} \le \frac{\pi}{2}$. Using the right half plane will avoid errors due to choosing the wrong branch of the square root, as several people have encountered when programming the algorithm. See the end of this section for more details.

If $\zeta = a + ib \in \mathbb{H}^+$ then

$$L_\zeta(z) = \frac{cz}{1 + idz}$$

with $c = a/(a^2 + b^2) > 0$ and $d = b/(a^2 + b^2) \in \mathbb{R}$ is a conformal map of the right half plane $\mathbb{H}^+$ onto $\mathbb{H}^+$ with $L_\zeta(0) = 0$ and $L_\zeta(\zeta) = 1$. The map

$$S(z) = \sqrt{z^2 - 1}$$

is a conformal map of $\mathbb{H}^+ \setminus [0, 1]$ onto $\mathbb{H}^+$. The composed function

$$f_\zeta(z) = \sqrt{L_\zeta(z)^2 - 1}$$

is then a conformal map of $\mathbb{H}^+ \setminus \sigma$ onto $\mathbb{H}^+$, where σ is the circular arc from 0 to ζ which is orthogonal to the imaginary axis $i\mathbb{R}$ at 0.

The complement in the extended plane of the line segment from z_0 to z_1 can be mapped onto $\mathbb{H}^+$ with the map

$$\varphi_1(z) = \sqrt{\frac{z - z_1}{z - z_0}}$$

and $\varphi_1(z_1) = 0$ and $\varphi_1(z_0) = \infty$. Set $\zeta_2 = \varphi_1(z_2)$ and $\varphi_2 = f_{\zeta_2}$. Repeating this process, define

$$\zeta_k = \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_1(z_k)$$

and

$$\varphi_k = f_{\zeta_k}.$$

for $k = 2, \ldots, n$. Map the inside and outside of half-disc to the upper and lower half planes by letting

$$\zeta_{n+1} = \varphi_n \circ \ldots \circ \varphi_1(z_0) \in \mathbb{R}$$

be the image of z_0 and setting

$$\varphi_{n+1} = \pm \left(\frac{z}{1 - z/\zeta_{n+1}} \right)^2$$

The $+$ sign is chosen in the definition of φ_{n+1} if the data points have negative winding number (clockwise) around an interior point of $\partial\Omega$, and otherwise the $-$ sign is chosen. Set

$$\varphi = \varphi_{n+1} \circ \varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1$$

and

$$\varphi^{-1} = \varphi_1^{-1} \circ \varphi_2^{-1} \circ \ldots \circ \varphi_{n+1}^{-1}.$$

Then φ^{-1} is a conformal map of $\mathbb{H}^+$ onto a region Ω_c such that $z_j \in \gamma_c = \partial\Omega_c$, $j = 0, \ldots, n$. If γ_j denotes the subarc of γ_c from z_0 to z_j, then the portion $\gamma_{j+1} \setminus \gamma_j$ of γ_c between z_j and z_{j+1} is the image of the arc of a circle in the right half plane by the analytic map $\varphi_1^{-1} \circ \ldots \circ \varphi_j^{-1}$ and thus a geodesic in the hyperbolic geometry of $\mathbb{C} \setminus \gamma_j$.

As an aside, we make a few comments. The curve γ_c is piecewise analytic. A curve is called C^1 if the arc length parameterization has a continuous first derivative. In other words, the direction of the unit tangent vector is continuous. It is easy to see that γ_c is also C^1 since the inverse of the basic map f_ζ doubles angles at 0 and halves angles at $\pm c$. Actually it is shown in [MR] that $\gamma_c \in C^{\frac{3}{2}}$. Note also that φ^{-1} is a conformal map of the lower half plane onto the region complementary to Ω_c.

Branching difficulties occur when, through round-off error or analytic continuation, the map $\sqrt{z^2 - 1}$ is applied to points with $\text{Re}z < 0$. This function should have positive imaginary part when $\text{Im}z > 0$. For example if $z = -\varepsilon + 2i$ with $\varepsilon > 0$, then $\text{Im}z^2 < 0$ so that $\text{Im}\sqrt{z^2 - 1} < 0$ if the usual branch cut along the negative reals is used (as is the case in most programming languages). This difficulty can be avoided by adding a simple test: Set

$$w = \sqrt{z^2 - 1}$$

If $(\text{Im}w)(\text{Im}z) < 0$, then replace w with $-w$.

2. Lenses

If D^+ and D^- are open disks with $b, c \in \partial D^+ \cap \partial D^-$, then $L = D^+ \cap D^-$ is called a *lens with vertices* b *and* c.

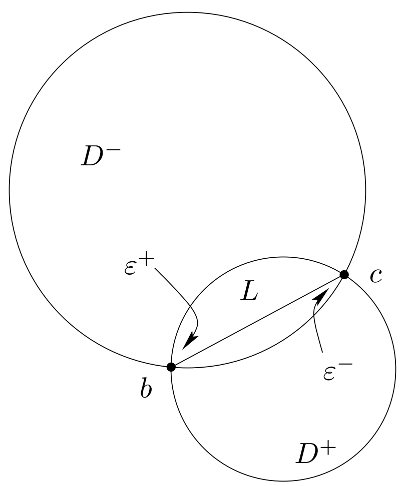

Figure 1. **A lens with vertices b and c and lens angle $\varepsilon = \varepsilon^+ + \varepsilon^-$.**

Let ε^+ denote the angle between the segment $[b, c]$ and the tangent to ∂D^+ at b with $0 < \varepsilon^+ \leq \frac{\pi}{2}$. Similarly ε^- denotes the angle between $[b, c]$ and ∂D^- at b with $0 < \varepsilon^- \leq \frac{\pi}{2}$. Note that ∂D^+ and ∂D^- form the same angles with $[b, c]$ at c. The angle $\varepsilon = \varepsilon^+ + \varepsilon^-$ is called the angle of the lens at b and c.

If $z_0, \ldots, z_n$ are points in $\mathbb{C}$ such that the polygonal curve through these points is Jordan, we define an ε*-tangential lens chain* for $z_0, \ldots, z_n$ to be a sequence of lenses $L_j = \partial D_j^+ \cap \partial D_j^-$ with vertices z_j and z_{j+1} such that the tangents to ∂D_j^+ and ∂D_j^- at z_j are also tangent to ∂D_{j-1}^- and ∂D_{j-1}^+. All of the lens angles of an ε-tangential lens chain are equal and ε will denote this common angle. The lens L_j contains the segment $[z_j, z_{j+1}]$. If the polygonal curve is closed, in other words $z_{n+1} = z_0$ then we do not require the last lens L_n to have the same tangents at z_0 as L_0.

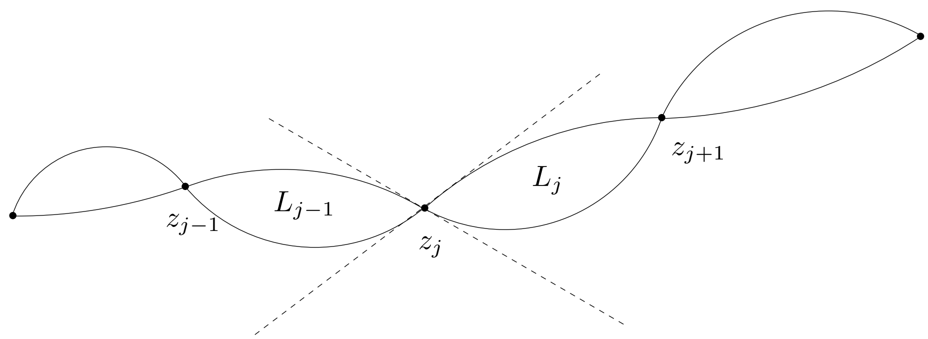

Figure 2. **An ε-tangential lens chain.**

A hyperbolic geodesic in the unit disk $\mathbb{D}$ is an arc of a circle orthogonal to the unit circle $\partial\mathbb{D}$. Hyperbolic geodesics in a simply connected domain Ω are images of hyperbolic geodesics in $\mathbb{D}$ by a conformal map of $\mathbb{D}$ onto Ω. The following is just an application of the proof of Theorem X.X in [MR] to lenses.

Theorem 1. *Suppose $\{L_j\}$ is an ε-tangential lens chain with vertices $z_0, \ldots, z_{n+1}$ with $z_{n+1} = z_0$, and suppose γ_c is a curve containing $\{z_j\}$ such that $\gamma_{k+1} \setminus \gamma_k$ is a hyperbolic geodesic in $\mathbb{C} \setminus \gamma_k$, where γ_k is the portion of γ from z_0 to z_k and γ_0 is the line segment $[z_0, z_1]$. Then*

$$\gamma \subset \bigcup_{j=0}^{n} L_j \cup \{z_j\}_{j=1}^{n},$$

provided the associated disks D_j+ and D_j^- do not intersect any of the previous lenses:

$$\left(D_j^+ \cup D_j^-\right) \cap \{L_k\}_{k=0}^{j-1} = \emptyset,$$

for $j = 1, \ldots, n$.

Proof. The hyperbolic geodesic $\gamma_{j+1} \setminus \gamma_j$ must intersect γ_j at z_j with angle π. This can be seen either by direct observation of the construction of γ_j in the geodesic algorithm, or by an appeal to Theorem V.5.5 in [GM] after applying a square root map at z_j. In other words, γ is a C^1 curve. By construction $\gamma_0 \subset L_0$. Suppose that $\gamma_j \subset \cup_{k=0}^{j-1} L_k$. Write $L_j = D_j^+ \cap D_j^-$. Since the tangent to ∂D_j^+ at z_j is also tangent to L_{j-1} and since $\gamma_j \setminus \gamma_{j-1} \subset L_{j-1}$, we conclude that $\gamma_{j+1} \setminus \gamma_j$ enters D_j^+ at z_j. By assumption $D_j^+ \cap \bigcup_{k=0}^{j-1} L_j = \emptyset$ and hence $D_j^+ \subset \mathbb{C} \setminus \gamma_j$. Jørgensen [J] proved that disks in a simply connected region are convex in the hyperbolic geometry of the region. Thus the hyperbolic geodesic $\gamma_{j+1} \setminus \gamma_j$ is contained in D_j^+. Similarly $\gamma_{j+1} \setminus \gamma_j$ is contained in D_j^-. Theorem 1 then follows by induction. $\square$

The proof of Theorem 1 shows why we chose the lenses to form a tangential chain. The inductive assumption is that γ_{j-1} is contained in the union of the first $j-1$ lenses. The proof requires first that the two disks $D_j^{\pm}$ do not intersect the previous lenses, so that the lens at stage j cannot be made any bigger than the tangential lens L_j. Secondly the proof requires that a geodesic beginning at z_j forming an angle of π at z_j with γ_j must enter the subsequent lens, so the subsequent lens cannot be any smaller than the tangential lens L_j.

Figure 3 illustrates the difficulty in creating successive lenses. The *bend angle at* z_j for the polygonal line through $z_0, \dots, z_n$ is given by

$$\delta_j = \arg\left(\frac{z_{j+1} - z_j}{z_j - z_{j-1}}\right).$$

For an ε-tangential lens chain, the lens angle $\varepsilon_j = \varepsilon_j^+ + \varepsilon_j^-$ satisfies

$$\varepsilon_j^- = \varepsilon_{j-1}^+ + \delta_j$$

and

$$\varepsilon_j^+ = \varepsilon_{j-1}^- - \delta_j = \varepsilon - \varepsilon_j^-.$$

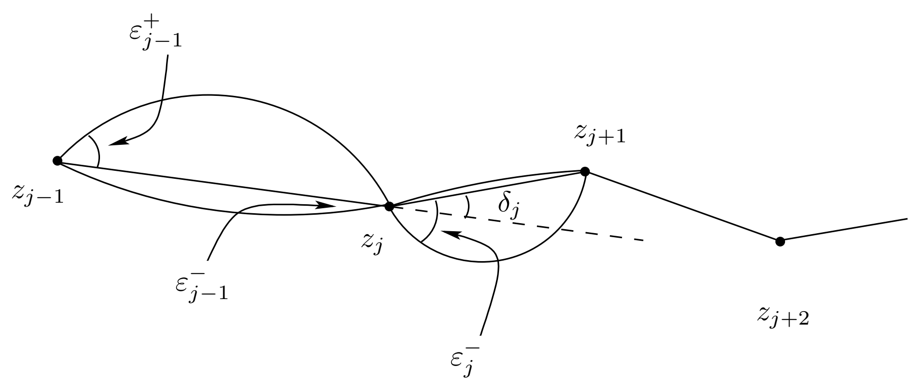

Figure 3. **A lens chain that cannot be extended to z_{j+2}.**

Applying this argument once more we obtain

$$\varepsilon_{j+1}^+ = \varepsilon_{j-1}^+ + \delta_j - \delta_{j+1}.$$

So if the bend angles are alternating in sign, the upper angles are increasing every two steps but bounded by ε, potentially leading to the impossibility of extending the lens chain. Indeed in Figure 3 we can't find a lens with vertices z_{j+1} and z_{j+2} having the same angles at z_{j+1} as L_j.

This difficulty was overcome in [MR] by proving that the hyperbolic geodesic $\gamma_{j+1} \setminus \gamma_j$ is actually contained in a smaller region than a lens. The important point is that the region has a smaller angle at the vertex z_{j+1} than at z_j, allowing for a bend, albeit small, in the polygon at the next data point z_{j+1}. Proving this result required a much more complicated argument than just using induction and Jørgensen's theorem.

We can also overcome this difficulty by altering the algorithm slightly so that additional points are occasionally added to the sequence $\{z_j\}$. For example, in the situation illustrated in Figure 3, where ε_{j-1}^+ is too large, we can add an additional point $z_j' = \frac{1}{2}(z_{j-1} + z_j)$ and replace the geodesic $\gamma_j \setminus \gamma_{j-1}$ from z_{j-1} to z_j by a geodesic γ_j' in $\mathbb{C} \setminus \gamma_{j-1}$ from z_{j-1} to z_j' followed by a geodesic γ_j'' in $\mathbb{C} \setminus (\gamma_{j-1} \cup \gamma_j')$ from z_j' to z_j. The lens L_{j-1} is replaced by two lenses where the "lower" angle ε_j^- at z_j is now smaller than ε_{j-1}^+ and hence ε_j^+ will be larger and ε_j^- will be smaller.

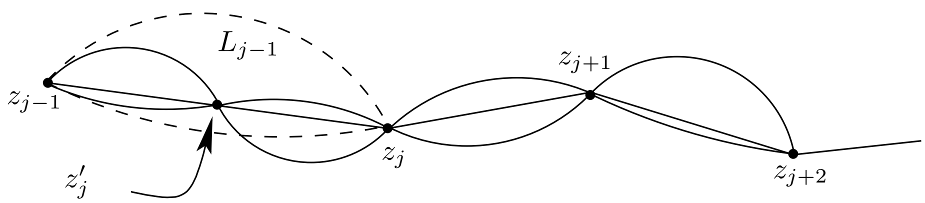

Figure 4. **Extending a lens chain with an extra step.**

We can do this systematically by keeping both angles $\varepsilon_j^\pm$ between $\varepsilon/2$ and $3\varepsilon/2$ as follows: Given $\varepsilon > 0$, suppose $\{z_j\}_0^{n-1}$ are the vertices of a closed Jordan polygon with bend angles δ_j satisfying

$$|\delta_j| < \frac{\varepsilon}{2}.$$

Construct a 2ε-tangential lens chain as follows: Let L_0 be the symmetric lens with vertices z_0 and z_1 and lens angle $2\varepsilon = \varepsilon_0^+ + \varepsilon_0^-$ where $\varepsilon_0^+ = \varepsilon_0^- = \varepsilon$. Suppose we have constructed a 2ε-tangential lens chain from z_0 to z_j, with lens angles $\varepsilon_k = \varepsilon_k^+ + \varepsilon_k^-$ satisfying

$$\frac{\varepsilon}{2} \le \varepsilon_k^+ \le \frac{3\varepsilon}{2} \tag{1}$$

for $k = 0, \ldots, j-1$. Note that (1) implies $\varepsilon_k^- = 2\varepsilon - \varepsilon_k^+$ also satisfies

$$\frac{\varepsilon}{2} \le \varepsilon_k^- \le \frac{3\varepsilon}{2}.$$

Moreover, assume that $\varepsilon_j^+ = \varepsilon_{j-1}^- - \delta_j = 2\varepsilon - \varepsilon_{j-1}^+ - \delta_j$ satisfies

$$\frac{\varepsilon}{2} \le \varepsilon_j^+ \le \frac{3\varepsilon}{2}. \tag{2}$$

If

$$\frac{\varepsilon}{2} \leq \varepsilon_j^- - \delta_{j+1} \leq \frac{3\varepsilon}{2}, \tag{3}$$

then let L_j be the lens with vertices z_j and z_{j+1} and lens angle $2\varepsilon = \varepsilon_j^+ + \varepsilon_j^-$ where $\varepsilon_j^+ = \varepsilon_{j-1}^- - \delta_j$ and $\varepsilon_j^- = 2\varepsilon - \varepsilon_j^+$. Note that $\varepsilon_{j+1}^+ = \varepsilon_j^- - \delta_{j+1}$, so that by (3), the inequalities in (2) hold with j replaced by $j+1$. On the other hand, if

$$\varepsilon_j^- - \delta_{j+1} < \frac{\varepsilon}{2} \quad \text{or} \quad \varepsilon_j^- - \delta_{j+1} > \frac{3\varepsilon}{2},$$

then let $z_j' = \frac{1}{2}(z_j + z_{j+1})$ be the midpoint of the segment $[z_j, z_{j+1}]$, and let L_j' be the lens with vertices z_j and z_j' and lens angle $2\varepsilon = \varepsilon_j^+ + \varepsilon_j^-$ and let L_j be the lens with vertices z_j' and z_{j+1} with lens angle $2\varepsilon = \varepsilon_j^- + \varepsilon_j^+$. Note that we have switched ε_j^- and ε_j^+ for the lens L_j since there is no bend at z_j'. The lens L_j will have the same tangents at z_j' as L_j' and will satisfy (3) since $|\delta_j| < \frac{\varepsilon}{2}$ and since we switched the magnitudes of the two angles at z_j'. See Figure 4.

By induction we then create a 2ε-tangential lens chain from z_0 to z_n. At the very last step there is no need to check the inequality (3) since the last lens does not need to have the same tangents as the initial lens at z_0. By the construction of the final map φ_{n+1} in the geodesic algorithm, the computed curve is C^1 at z_0.

Suppose $\{\widetilde{L}_j\}_0^n$ is a chain of lens (not tangential) with vertices $\{z_j\}_0^n$ such that $\widetilde{L}_j$ is symmetric about the line segment $[z_j, z_{j+1}]$ for each j, and such that each lens $\widetilde{L}_j$ has the same vertex angle δ. We call such a chain a $\delta-$*symmetric lens chain.* It is of course easier to construct symmetric lens chains than to construct tangential chains. Part of the next theorem is that a $\delta-$symmetric lens chain contains a $\delta/3$-tangential lens chain if

$$6 \max_j \left| \arg\left(\frac{z_{j+1} - z_j}{z_j - z_{j-1}} \right) \right| \leq \delta \tag{4}$$

Theorem 2. *Suppose $\delta > 0$ and set $\varepsilon = \delta/3$. If γ is a C^1 Jordan curve and if $\{z_j\}$ are points on γ together with midpoints whenever required in the modification described above and with mesh size*

$$\mu = \min_{0 \leq j \leq n-1} |z_{j+1} - z_j|$$

sufficiently small, then the geodesic algorithm constructs a C^1 curve

$$\gamma_c \subset \bigcup_0^n L_k \cup \{z_k\},$$

where L_k is the lens which is symmetric about the line segment $[z_k, z_{k+1}]$ with vertex angles equal to δ. The algorithm simultaneously computes conformal maps of the interior and exterior regions of γ_c onto the upper and lower half planes (respectively) along with their inverse maps. Moreover, if $\zeta \in \gamma_c$ and if $\alpha \in \gamma$ with $|\zeta - \alpha| < \mu$, then

$$|\eta_\zeta - \eta_\alpha| < \delta, \tag{5}$$

where η_ζ and η_α are unit tangent vectors to γ_c and γ at ζ and α respectively.

Proof. Let $\{L_j\}$ denote the 2ε-tangential lens chain constructed by the algorithm given in this section. If the mesh size μ is sufficiently small then (4) holds and thus

$$\delta_j = \arg\left(\frac{z_{j+1} - z_j}{z_j - z_{j-1}}\right)$$

satisfies $|\delta_j| < \varepsilon/2$. Since $\frac{\varepsilon}{2} \leq \varepsilon_j^+, \varepsilon_j^- \leq \frac{3\varepsilon}{2}$, the region $D_j^+ \cup D_j^-$ will be small if $|z_{j+1} - z_j|$ is small. The region $D_j^+ \cup D_j^-$ does not intersect L_{j-1} by construction. Since $\gamma \in C^1$, the region $D_j^+ \cup D_j^-$ will not meet any of the previous lenses if $|z_{j+1} - z_j|$ is sufficiently small. By Theorem 1, the computed curve γ_c lies in the union of the lenses and their vertices. Note that each of the lenses in the 2ε-tangential lens chain is contained in the corresponding symmetric lens $\widetilde{L}_j$ with vertex angle $\delta = 3\varepsilon$, even if a midpoint (and therefore two lenses of the tangential chain) is added.

To prove the statement about tangent vectors, note that for each point $\zeta \in \gamma_{j+1} \setminus \gamma_j$, we can construct a lens with vertices z_j and ζ which has the same tangents as L_{j-1} at z_j. Moreover this lens is contained in $D_j^+ \cup D_j^-$. The geodesic exits this new lens at ζ and hence the tangent to $\gamma_{j+1} \setminus \gamma_j$ at ζ differs from the direction of the line segment from z_j to ζ by at most $\frac{3\varepsilon}{2}$ and hence differs from the direction of the line segment from z_j to z_{j+1} by at most 3ε. Since $\gamma \in C^1$, we can then choose a sufficiently small mesh size μ to guarantee that (5) holds. $\square$

We remark that the computed curve γ_c can be parameterized so that if $p(t)$ is the polygonal curve through the data point $\{z_j\}$, then

$$\sup_t |p(t) - \gamma_c(t)| \leq \mu\varepsilon,$$

where μ is the mesh size, as in the statement of Theorem 2. The angle ε can be taken to be bounded by a constant times the modulus of continuity of the unit tangent vector to γ.

Since Jørgensen's theorem is a key component of the proof of the convergence of the geodesic algorithm, we include a short self-contained proof.

Theorem A.1 (Jørgensen). *Suppose Ω is a simply connected domain. If Δ is an open disc contained in Ω and if γ is a hyperbolic geodesic in Ω, then $\gamma \cap \Delta$ is connected.*

Proof. Without loss of generality, Ω is bounded by a Jordan curve and $\overline{\Delta} \subset \Omega$. Let f be a conformal map of Ω onto $\mathbb{H}$ such that $f(\gamma)$ is the positive imaginary axis which we denote by I. If J is the subinterval of the imaginary axis from 0 to ic, then the conformal map $\tau(z) = \sqrt{z^2 + c^2}$ of $\mathbb{H} \setminus J$ onto $\mathbb{H}$ maps $I \setminus J$ onto I. Replacing Ω with $f^{-1}(\mathbb{H} \setminus J)$, and replacing f with $\tau \circ f$, we may suppose that $f^{-1}(iy) \to z_1 \in \partial\Omega \cap \partial\Delta$ as $y \to 0$. Similarly we may suppose that $f^{-1}(iy) \to z_2 \in \partial\Omega \cap \partial\Delta$ as $y \to +\infty$. The points z_j divide $\partial\Delta$ into two arcs α_1 and α_2. Then $\sigma_j = f(\alpha_j)$, for $j = 1, 2$, are arcs in $\mathbb{H}$ connecting 0 to ∞. We may suppose that σ_2 lies to the left of σ_1.

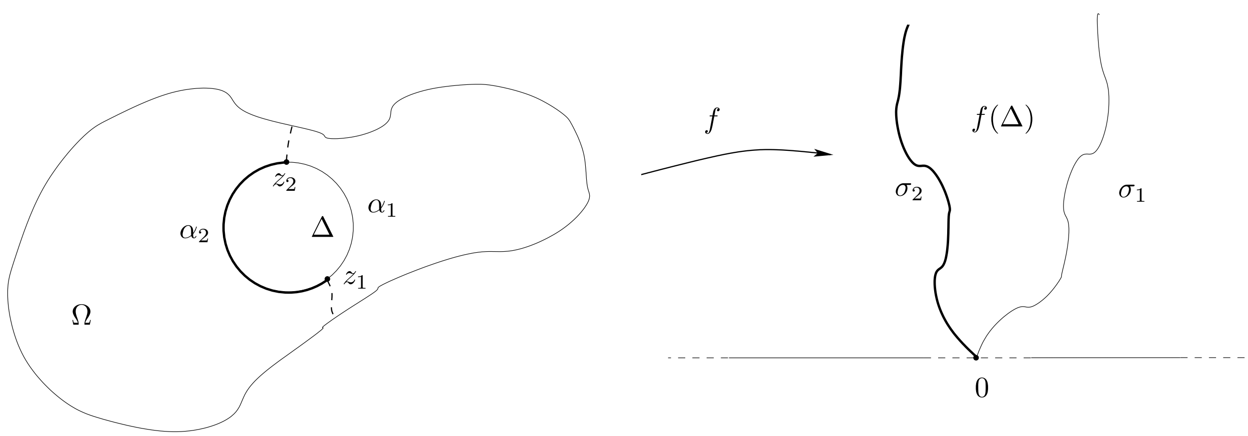

Figure 12. Proof of Jørgensen's theorem.

Let Ω_1 be the component of $\mathbb{H} \setminus \sigma_2$ containing $f(\Delta)$. Let ω_1 be the bounded harmonic function in $\mathbb{C} \setminus \alpha_2$ such that

$$\omega_1(z) \to 1 \text{ as } z \in f^{-1}(\Omega_1) \to \alpha_2^\circ$$

and

$$\omega_1(z) \to 0 \text{ as } z \in \Omega \setminus \overline{f^{-1}(\Omega_1)} \to \alpha_2^\circ.$$

The function ω_1 can be found explicitly using the conformal map of $\mathbb{C}^* \setminus \alpha_2$ onto $\mathbb{H}$. Then by comparison of boundary values and the maximum principle, $\arg z < \pi\omega_1(f^{-1}(z))$ for all $z \in \Omega_1$. Since $\omega_1 = \frac{1}{2}$ on α_1°, we conclude

$$\arg z < \frac{\pi}{2}$$

on $\sigma_1 \cap \mathbb{H}$. Similarly

$$\pi - \arg z < \frac{\pi}{2}$$

on $\sigma_2 \cap \mathbb{H}$. Thus $f(\Delta) \supset I$. $\square$

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