**Proof.** Actually we will give two examples. For the first example, let

$$\log(\varphi') = \varepsilon \sum z^{2^n}.$$

If $\varepsilon$ is small, then $\varphi$ maps $\mathbb{D}$ to a quasidisc; and by the remark following Theorem 1.3, $\varphi$ has an angular derivative at no $\zeta \in \partial\mathbb{D}$.

The second example is the von Koch snowflake, Example VI.4.3. The conformal map from $\mathbb{D}$ to the interior domain $\Omega$ has a non-zero angular derivative at no point of $\partial\Omega$. Indeed if there is an inner tangent at $\zeta$, then $\zeta$ belongs to $\Gamma_n$ for some n, as shown in Example VI.4.3. If $\zeta$ is a vertex of $\Gamma_n$, then $\Omega$ either contains a truncated cone with opening $4\pi/3$ or contains no truncated cone with opening greater than $\pi/3$ at $\zeta$. If $\zeta$ is not a vertex then $\Omega$ contains the union of a half disc and an equilateral triangle where the disc, centered at $\zeta$, can be arbitrarily small and the size of the equilateral triangle is comparable to the diameter of the disc. See Figure VII.5. By the easy half of Ostrowski's Theorem V.5.5, $\Omega$ does not have a nonzero angular derivative at $\zeta$. In fact the conformal map of the unit disc onto $\Omega$ is not conformal at any point of $\partial\mathbb{D}$. ■

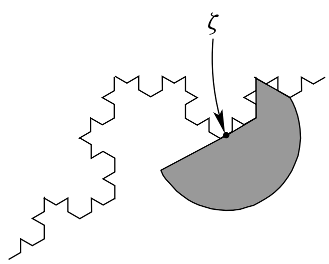


**Figure VII.5** Union of a half disc and triangle.

Duren, Shapiro, and Shields [1966] introduced (2.11) in order to construct a Jordan domain that is not a Smirnov domain. If $\Omega$ is a Jordan domain with rectifiable boundary $\Gamma$ the conformal map $\varphi : \mathbb{D} \to \Omega$ satisfies $\varphi' \in H^1$ and

$$\log|\varphi'(z)| = \int_{\partial\mathbb{D}} P_z(e^{i\theta}) \log|\varphi'(e^{i\theta})|d\theta - \int_{\partial\mathbb{D}} P_z(e^{i\theta}) d\mu_s(\theta),$$

where the singular measure $\mu_s$ is the weak-star limit

$$\mu_s = \lim_{r\to 1}\Big(\log|\varphi'(e^{i\theta})|d\theta - \log|\varphi'(re^{i\theta})|d\theta\Big).$$

Because $\varphi' \in H^1$,

$$\log|\varphi'(z)| \le \int P_z(e^{i\theta}) \log|\varphi'(e^{i\theta})|d\theta$$