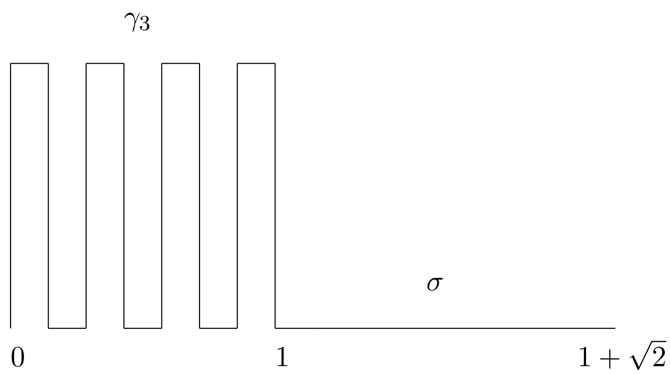
Let Γ be a Jordan curve and μ a positive measure on Γ. We say that μ satisfies condition (D) if there is a constant C such that $\mu(I) \leq C\mu(J)$ whenever I and J are adjacent arcs with $\text{diam}(I) = \text{diam}(J)$. We will say that μ satisfies condition (M) if there is a constant C such that $\text{diam}(I) \leq C\text{diam}(J)$ whenever I and J are adjacent arcs with $\mu(I) = \mu(J)$. The following examples show that (M) does not imply (D) in general. For set functions f and g, we will write $f(I) \sim g(J)$ when there exists a constant $c > 0$ independent of I and J such that $1/c \leq \frac{f(I)}{g(J)} \leq c$.

Example 1. Let $n \geq 0$ be an integer and for $k = 0, 1, \cdots, n$ set $z_{4k} = \frac{2k}{2n+1}$, $z_{4k+1} = \frac{2k}{2n+1} + i$, $z_{4k+2} = \frac{2k+1}{2n+1} + i$ and $z_{4k+3} = \frac{2k+1}{2n+1}$. Let γ_n be the polygonal line connecting $z_0 = 0$, $z_1, \cdots, z_{4n+3} = 1$ in this order with horizontal and vertical line segments. Set $\sigma = [1, 1+\sqrt{2}]$ and let μ be the arc length measure on $\gamma_n \cup \sigma$. The next picture shows the curve $\gamma_3 \cup \sigma$.



Let I,J be adjacent arcs on $\gamma_n \cup \sigma$ with $\mu(I) = \mu(J)$. Note that if $\mu(I) < 2$ then $\text{diam}(I) \sim \mu(I)$. On the other hand, if $\mu(I) \geq 2$, we have that $1 \leq \text{diam}(I) \leq \sqrt{2}$. This shows that

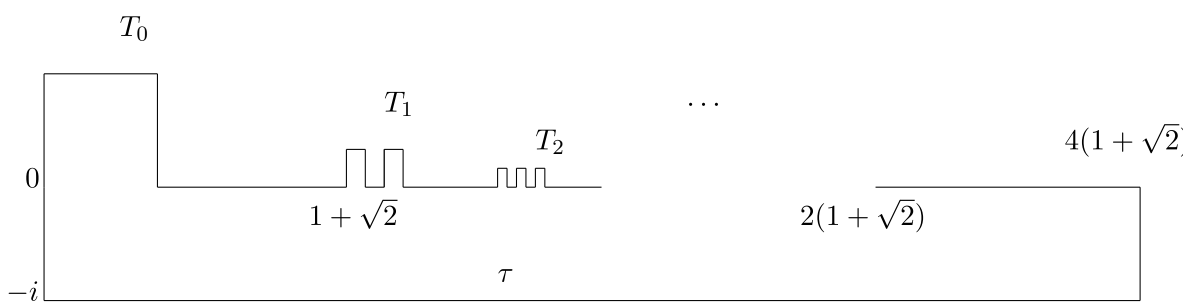
$$\frac{\text{diam}(I)}{\text{diam}(J)} \leq 2. \tag{1}$$

Moreover we have that $\operatorname{diam}(\gamma_n) = \operatorname{diam}(\sigma)$ while

$$\frac{\mu(\gamma_n)}{\mu(\sigma)} = \frac{2n+3}{\sqrt{2}}. \tag{2}$$

Observe that if we scale $\gamma_n \cup \sigma$ by a factor $c > 0$ and μ by a factor $d > 0$, then (1) and (2) remain true since $\frac{\operatorname{diam}(cI)}{\operatorname{diam}(cJ)} = \frac{\operatorname{diam}(I)}{\operatorname{diam}(J)}$ and $\frac{d\mu(cI)}{d\mu(cJ)} = \frac{\mu(I)}{\mu(J)}$.

Example 2. Let $T_n = 2^{-n}(\gamma_n \cup \sigma) + 2(1+\sqrt{2})(1-2^{-n})$ and $\Gamma = (\bigcup_{n\geq 0} T_n) \cup \tau$ where τ is the polygonal line connecting $2(1+\sqrt{2})$, $4(1+\sqrt{2})$ $4(1+\sqrt{2}) - i$, $-i$ and 0, in this order, with horizontal or vertical line segments. Observe that Γ is a closed Jordan curve.



The curve Γ.

For $E \subset \Gamma$, let E^* be the vertical projection of E onto $\mathbb{R}$ and define a measure μ on Γ by

$$\mu(I) = \sum_{n\geq 0} |T_n^*| \frac{|I \cap T_n|}{|T_n|} + |I \cap \tau|,$$

where $|\cdot|$ denotes arc length.
Now let I be an arc on Γ such that $T_n \subseteq I$ but $T_{n-1} \nsubseteq I$. Because

$$\mu(T_n) = |T_n^*| = \mu(\bigcup_{k>n} T_k) = 2^{-n}(1+\sqrt{2}),$$

we have that $\mu(I) \sim \max(|T_n^*|, |I \cup \tau|)$. Moreover, we have that

$$\operatorname{diam}(I) \sim \max(\operatorname{diam}(T_n), \operatorname{diam}(I \cap \tau),$$

because

$$\max(\operatorname{diam}(T_n), \operatorname{diam}(I \cap \tau) \leq \operatorname{diam}(I) \leq \sum_{j \geq n-1} \operatorname{diam}(T_j) + \operatorname{diam}(I \cap \tau).$$

We conclude that if I and J are two adjacent arcs on Γ with $\mu(I) = \mu(J)$ and such that $T_n \subseteq I$, $T_{n-1} \nsubseteq I$ for some n, then $\operatorname{diam}(I) \sim \operatorname{diam}(J)$.
If both I and J are included in some T_n, then $\operatorname{diam}(I) \sim \operatorname{diam}(J)$ by the proof of example 1. If I meets only T_{n-1} and T_n but does not contain either, then by considering the cases $\operatorname{diam}(J) \geq 2^{-n}$ and $\operatorname{diam}(J) < 2^{-n}$ it is not hard to see that we must have $\operatorname{diam}(I) \sim \operatorname{diam}(J)$. This shows that μ satisfies condition (M).
Finally note that μ does not satisfy condition (D) since the adjacent arcs $I_n = 2^{-n}\gamma_n + 2(1+\sqrt{2})(1-2^{-n})$ and $J_n = 2^{-n}\sigma + 2(1+\sqrt{2})(1-2^{-n})$ satisfy $\operatorname{diam}(I_n) = \operatorname{diam}(J_n)$ and $\frac{\mu(I_n)}{\mu(J_n)} = \frac{2n+3}{\sqrt{2}}$.