Definition. A positive measure μ on a Jordan curve Γ is a doubling measure if there is a constant C such that

$$\mu(I) \leq C\mu(J)$$

whenever I and J are adjacent subarcs of Γ with $\operatorname{diam}(I) \leq \operatorname{diam}(J)$.

Theorem. Suppose Γ is a Jordan curve in the plane and suppose Ω_1 and Ω_2 are the two components of the complement $\mathbb{C}^*\backslash\Gamma$. Let $z_j \in \Omega_j$ and let $\omega_j(E) = \omega(z_j, E, \Omega_j)$. Then Γ is a quasicircle if and only if both ω_1 and ω_2 are doubling measures on Γ.

Proof ($\Leftarrow$). Assume ω_1 and ω_2 are doubling measures both with a constant c and assume $z_2 = \infty \in \Omega_2$. Let $\phi_1 : \mathbb{D} \to \Omega_1$ and $\phi_2 : \mathbb{C}^*\backslash\bar{\mathbb{D}} \to \Omega_2$ be conformal maps such that $\phi_2(\infty) = \infty$ and $\phi_1(0) = z_1$. Define the welding map $h : \partial\mathbb{D} \to \partial\mathbb{D}$ by

$$h = \phi_2^{-1} \circ \phi_1.$$

Let I and J be adjacent subarcs of $\partial\mathbb{D}$ such that $|I| \leq \epsilon|J|$ where $0 < \epsilon < 1/c$ is fixed. We claim that $\operatorname{diam}(\phi_1(I)) \leq \operatorname{diam}(\phi_1(J))$. Suppose not. Then by the doubling condition for ω_1, we have that $\omega_1(\phi_1(J)) \leq c\omega_1(\phi_1(I))$. Hence, $|J| \leq c|I| \leq c\epsilon|J|$ which is a contradiction. This proves the claim. Now, using the doubling condition for ω_2 on $\phi_1(I)$ and $\phi_1(J)$, we have that $\omega_2(\phi_1(I)) \leq c\omega_2(\phi_1(J))$, which by composing with ϕ_2^{-1} gives that

$$|h(I)| \leq c|h(J)|. \tag{1}$$

Finally let I and J be adjacent subarcs of $\partial\mathbb{D}$ with $|I| = |J|$. Find a collection of subarcs I_j, $1 \leq j \leq n$ of I such that J and I_1 are adjacent, I_j and I_{j+1} are adjacent for all $1 \leq j \leq n-1$ and $|I_j| = \epsilon|J|$ for $1 \leq j \leq n-1$ while $|I_n| \leq \epsilon|J|$. Note that since

$$|J| \geq \sum_{j=1}^{n-1} |I_j| = (n-1)\epsilon|J|,$$

we must have that $n \leq 1 + 1/\epsilon$.

By applying (1) to the adjacent arcs I_j and $J \cup \bigcup_{k=1}^{j-1} I_k$, and using induction, we find that

$$|h(I_j)| \leq j|h(J)|(\frac{1 - c^{j+1}}{1 - c} - 1).$$

Since n is uniformly bounded from above by a constant that depends only on ϵ, we conclude that there is a constant $M = M(\epsilon, c)$ such that $|h(I_j)| \leq M|h(J)|$. Finally, observe that

$$|h(I)| = \sum_{j=1}^{n} |h(I_j)| \leq nM|h(J)| \leq M(1 + \frac{1}{\epsilon})|h(J)|.$$

We may interchange I and J to see that h is a quasisymmetric function. Now let $H : \mathbb{C} \to \mathbb{C}$ denote the quasiconformal extension of h and define

$$\Phi(z) = \phi_1(z)\mathbb{1}_{\{|z|\leq 1\}} + \phi_2(H(z))\mathbb{1}_{\{|z|>1\}}.$$

Then Φ is a quasiconformal map such that $\Phi(\partial\mathbb{D}) = \Gamma$, and Γ is a quasicircle.