MATH 208 C — FINAL EXAM — Autumn 2022

NAME: ____

To make it possible for Gradescope to recognize you, please write your name:

(1) **clearly** in **CAPITAL LETTERS**,
(2) **exactly on the line above**,
(3) use the **name you are registered under** for this class.

(1) Please put away all phones and earphones in your bag.
(2) There are 6 problems.
(3) Show all of your work and justify your answers.
(4) Write clearly.

(1) Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 - 2x_2 + 3x_3 - 4x_4 = 0 \right\}$.

(a) Is S a subspace? Give reasons.

(b) Find a basis for S.

(c) Express the equation $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ as $\det(M) = 0$ for a matrix M.

(d) Explain your logic in (c).

(2) Let D be the diamond in $\mathbb{R}^2$ with corners $(-1,0),(0,1),(1,0),(0,-1)$ and let Q be the quadrilateral with corners $(-2,0),(0,1),(2,0),(0,-1)$. Draw D and Q to help answer the following.

(a) Compute the linear transformation T that takes D to Q. Write it fully with domain, codomain and map.

(b) Is T invertible? If yes, find its inverse (written fully). If not, say why not.

(3) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that projects onto the (x_1, x_2)-plane.

(a) Write down the matrix of T, i.e., A such that $T(x) = Ax$.

(b) Calculate all eigenvalues and eigenspaces of A. Explain clearly and box your answers. (No long computations needed. Think about what projection does.)

(c) Diagonalize A as PDP^{-1} if possible. Identify P and then leave P^{-1} symbolic.

(4) Let A be the following matrix.

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & 5 & -6 & 2 \\ 1 & 3 & 4 & -3 & 3 \end{bmatrix}$$

(a) If $T(x) = Ax$, then what is the domain and codomain of T?

(b) Is T onto? If yes, say why. If not, find a vector b that is not in the range of T.

(c) Compute the range of T. Say what you did.

(d) Is T one to one? Explain.

(5) The following is the diagonalization of a 3×3 matrix A. Use it to answer the following questions, with reasons.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$$

(a) Is A invertible?

(b) What is the rank of A?

(c) What are the coordinates of $(1, 1, 2)$ in the basis of eigenvectors associated to this diagonalization?

(d) What is the characteristic polynomial of A^2 (up to sign)?

(6) Let $A = \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix}$.

(a) Find A^{-1}.

(b) If $Ax = b$ can you write a formula for x in terms of some, or all, of A, A^{-1}, b?

(c) Find a quadratic of the form $y = 1 + a_1 x + a_2 x^2$ that passes through $(-1, 2)$ and $(3, -3)$.

(d) Do you expect more than one such quadratic? Why?

SCRATCH PAPER