Win 25 Final Solutions

1. Evaluate the following integrals. Show your work.

   (a) (6 points) $\int_0^1 \dfrac{x}{x^2+3x+2}dx$

$$= \int_0^1 \frac{x}{(x+1)(x+2)}\,dx$$

$$= \int_0^1 \frac{-1}{x+1} + \frac{2}{x+2}\,dx$$

$$= -\ln|x+1|\Big|_0^1 + 2\ln|x+2|\Big|_0^1$$

$$= -\ln 2 + \ln 1 + 2\ln 3 - 2\ln 2$$

$$= 2\ln 3 - 3\ln 2$$

$$\left(\begin{array}{l} = \ln 9 - \ln 8 \\ = \ln(9/8) \end{array}\right)$$

Partial Fractions:

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\Leftrightarrow x = A(x+2) + B(x+1)$$

$$\Rightarrow \begin{cases} -1 = A(1) + (0) \Rightarrow \boxed{A = -1} \\ -2 = A(0) + B(-1) \Rightarrow \boxed{B = 2} \end{cases}$$

Answer: $\boxed{2\ln 3 - 3\ln 2}$

   (b) (6 points) $\int_0^2 \dfrac{x}{x^4+2x^2+2}dx$

Substitute $u = x^2$
$\frac{1}{2}du = x\,dx$

$$= \int_{u=0}^{u=4} \frac{1}{u^2+2u+2}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_0^4 \frac{1}{(u+1)^2+1}\,du$$

Substitute $v = u+1$
$dv = du$

$$= \frac{1}{2}\int_1^5 \frac{1}{v^2+1}\,dv$$

$$= \frac{1}{2}\arctan v\Big|_1^5 = \frac{1}{2}(\arctan 5 - \underset{=\pi/4}{\arctan 1})$$

$$= \boxed{\frac{1}{2}\left(\arctan 5 - \frac{\pi}{4}\right)}$$

Answer: $\boxed{\dfrac{1}{2}\arctan 5 - \dfrac{\pi}{8}}$

2. Evaluate the following integrals. Show your work.

(a) (6 points) $\int \dfrac{e^x}{\sqrt{4+e^{2x}}}\,dx$

① $u = e^x$, $du = e^x\,dx$

$$= \int \frac{1}{\sqrt{4+u^2}}\,du$$

② $u = 2\tan\theta$, $du = 2\sec^2\theta\,d\theta$
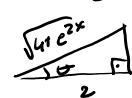
$$= \int \frac{1}{2\sec\theta}\,2\sec^2\theta\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$\tan\theta = \dfrac{u}{2} = \dfrac{e^x}{2} = \dfrac{opp}{adj}$

$e^x \Rightarrow \sec\theta = \dfrac{hyp}{adj} = \dfrac{\sqrt{4+e^{2x}}}{2}$

$$= \ln|\tan\theta + \sec\theta| + C$$

$$\boxed{= \ln\left|\frac{e^x}{2} + \frac{\sqrt{4+e^{2x}}}{2}\right| + C}$$

$$= \ln\left(e^x + \sqrt{4+e^{2x}}\right) - \ln 2 + C$$

Answer: $\boxed{\ln\left(e^x + \sqrt{4+e^{2x}}\right) + C}$

(b) (6 points) $\int \sec^3(x)\,dx$ (hint: try integration by parts)

$$I = \int \underbrace{\sec x}_{u}\,\underbrace{\sec^2 x\,dx}_{dv}$$

① By Parts:

$u = \sec x$ $dv = \sec^2 x\,dx$

$du = \tan x \sec x\,dx$ $v = \int \sec^2 x\,dx = \tan x$

$$= \tan x \sec x - \int \tan^2 x \sec x\,dx$$

$$= \tan x \sec x - \int (\sec^2 x - 1)\sec x\,dx$$

$$= \tan x \sec x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$I = \tan x \sec x - I + \ln|\tan x + \sec x|$$

$$I = \frac{1}{2}\left(\tan x \sec x + \ln|\tan x + \sec x|\right) + C$$

Answer: $\boxed{\dfrac{1}{2}\left(\tan x \sec x + \ln|\tan x + \sec x|\right) + C}$

3. (8 points) For each integral below, state which method applies best.

Your answer should be in one of the following forms:

- u-substitution, with $u =...$ (specify the substitution)
- integration by parts, with $u =...$, and $dv = ...$ (specify the parts)
- trigonometric substitution, with $x =...$ (specify the trig sub)
- partial fractions, with fractions: $\frac{A}{(...)} + ...$ (specify the fractions, do not calculate A, etc)

No need to justify or compute anything – and <u>do not evaluate the integrals!</u>

(a) $\int \sin(x)\cos^2(x)\,dx$ Method: <u>u-substitution</u>, with <u>u = cos θ</u>

(b) $\int x\sec^2(x)\,dx$ Method: <u>Integration by Parts</u>

$\underline{u = x}, \quad \underline{dv = \sec^2 x\, dx}$

(c) $\int \frac{2x+1}{x^4+x^3+x^2}\,dx$ Method: Partial Fractions

$\int \frac{2x+1}{x^2(x^2+x+1)}$ $\underline{\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}}$

(d) $\int \frac{x^2}{(x^2-4)^{3/2}}\,dx$ Method: <u>Trigonometric Substitution</u>

$\underline{x = 2\sec\theta}$

4. A drone starts from a height of 2 m above the ground at $t = 0$, and flies straight up, with velocity at t seconds, $0 \le t \le 10$, given by

$$v_1(t) = 0.3t^2 \, m/s.$$

At $t = 10$ seconds, its battery fails, so from that time on the acceleration acting on the drone is

$$a_2(t) = -9.8 \, m/s^2,$$

causing it to eventually fall down and crash on the ground.

(a) (5 points) What is the drone's height above ground at the moment when its battery fails?

Let $S_1(t)$ denote the position of the drone above the ground during the first 10 seconds.

Then $S_1(0) = 2$ m and

$$S_1(10) - S_1(0) = \int_0^{10} v_1(t)\,dt = \int_0^{10} 0.3t^2\,dt = 0.1\,t^3 \Big|_0^{10} = 0.1\,(1000) = 100$$

So $S_1(10) = 100 + 2 = 102$ meters

Answer: __102__ meters

(b) (5 points) How long does the drone take to crash on the ground, from the time its battery fails? Round your answer to the nearest tenth of a second.

Reset $t = 0$ when the battery fails.

Then: $a_2(t) = -9.8$

$v_2(t) = -9.8t + C$, with $v_2(0) = v_1(10) = 0.3(10^2) = 30$.

So $v_2(t) = -9.8t + 30$

The drone's position above ground t seconds after its battery fails is:

$h_2(t) = -4.9t^2 + 30t + D$ with $h_2(0) = 102$ (part (a))

so $h_2(t) = -4.9t^2 + 30t + 102$

It falls on the ground when $h_2(t) = 0$:

$-4.9t^2 + 30t + 102 = 0$

Quadratic Formula: $t = \dfrac{-30 \pm \sqrt{900 + 4(4.9)(102)}}{2(-4.9)} \approx \begin{cases} -2.43 \\ 8.56 \end{cases}$

Answer: __8.6__ seconds

5. (8 points) Find the area bounded by the curves:

$$y = \sin x \text{ and } y = \sin^2 x, \text{ for } 0 \le x \le \pi/2.$$
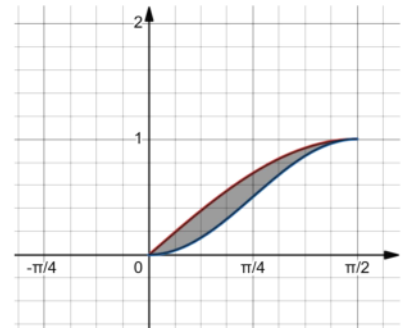
$$\text{Area} = \int_0^{\pi/2} \sin x - \sin^2 x \; dx$$

$$= \int_0^{\pi/2} \sin x \; dx - \int_0^{\pi/2} \frac{1 - \cos 2x}{2} \; dx$$

$$= \left(-\cos x - \frac{x}{2} + \frac{\sin 2x}{4} \right) \Big|_0^{\pi/2}$$

$$= -\cos \frac{\pi}{2} + \cos 0 - \frac{\pi/2}{2} + 0 + \frac{\sin \pi}{4} - \frac{\sin 0}{4}$$
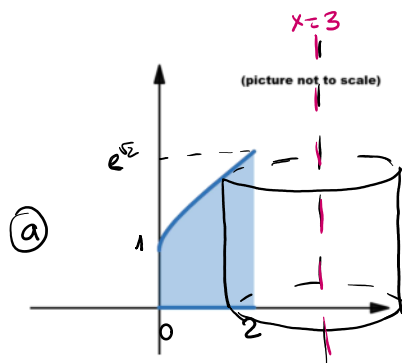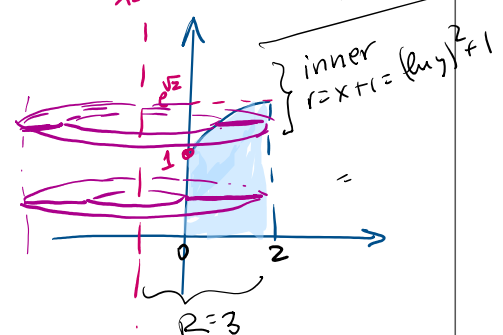
$$= 1 - \pi/4$$

Answer: Area = $\boxed{1 - \pi/4 = (4-\pi)/4}$

6. Let R be the region in the xy-plane bounded by the curves:

$$y = e^{\sqrt{x}}, x = 0, x = 2 \text{ and } y = 0.$$

$x = 3$

$y = e^{\sqrt{x}}$
$\ln y = \sqrt{x}$
$x = (\ln y)^2$

$x = -1$

inner
$r = x + 1 = (\ln y)^2 + 1$

=

(a)

(b)

$R = 3$

(a) (5 points) **Set up, but do not evaluate,** an integral (or a sum of integrals) equal to the volume of the solid of revolution obtained by revolving R about the vertical line $x = 3$ using the **method of cylindrical shells**.

$$\text{Volume} = \int_0^2 2\pi \, (\text{radius})(\text{height}) \, dx$$

$$= \int_0^2 2\pi \, (3-x) \, e^{\sqrt{x}} \, dx$$

Answer: $\boxed{\int_0^2 2\pi \, (3-x) \, e^{\sqrt{x}} \, dx}$

(b) (5 points) **Set up, but do not evaluate,** an integral (or a sum of integrals) equal to the volume of the solid of revolution obtained by revolving R about the vertical line $x = -1$ using the **method of disks and washers.**

The disks/washers would have to be horizontal so the integral has to be in terms of y and separated into $0 \le y \le 1$ and $1 \le y \le e^{\sqrt{2}}$.

or $\quad 9\pi - \pi + \int_1^{e^{\sqrt{2}}} \left[\pi (3)^2 - \pi \left((\ln y)^2 + 1 \right)^2 \right] dy$

or $\quad \underbrace{\int_0^{e^{\sqrt{2}}} \pi (3)^2 \, dy}_{(9\pi)e^{\sqrt{2}}} - \underbrace{\int_0^1 \pi (1)^2 \, dy}_{\pi} - \int_1^{e^{\sqrt{2}}} \pi \left((\ln y)^2 + 1 \right)^2 dy$

Answer: $\boxed{\int_0^1 \pi (3)^2 - \pi (1)^2 \, dy + \int_1^{e^{\sqrt{2}}} \pi (3)^2 - \pi \left[\ln^2 y + 1 \right]^2 dy}$

7. (10 points) Find the mass of an empty leaky bucket given the following information:

- The bucket held 10 liters of water at the ground level and was lifted to the height of 16 meters.
- While being lifted, water leaked out of the bucket at the rate of 1/8 liters per meter.
- The total work done was 1960 J.

Assume that the mass of the rope used to lift the bucket was negligible and can be ignored.
The mass of 1 liter of water is 1 kilogram, and the acceleration due to gravity is 9.8 meters/sec².

Let m be the mass of the bucket, in kgs. (constant)

The bucket holds 10 liters of water initially, and loses $1/8$ L/m

So, at y meters above ground, it has

$$10 - \tfrac{1}{8}y \quad \text{liters of water in it,}$$

which means the mass of water in the bucket is

$$10 - \tfrac{1}{8}y \quad \text{kgs. (at } y \text{ meters)}$$

Total mass: $m + (10 - \tfrac{1}{8}y)$ kgs

The work done is:

$$1960 \text{ J} = \int_0^{16} (\text{Force})\, dy = \int_0^{16} \left(m + 10 - \tfrac{1}{8}y\right)(9.8)\, dy$$

Divide by 9.8 and integrate:

$$\frac{1960}{9.8} = 200 = \left(my + 10y - \tfrac{1}{16}y^2\right)\Big|_0^{16}$$

$$= \left(16m + 160 - \tfrac{1}{16}16^2\right) - (0)$$

$$= 16m + 144$$

$$200 = 16m + 144$$

$$16m = 56$$

$$m = 3.5$$

ANSWER: mass of bucket $m =$ 3.5 kg

8. (10 points) Consider the region in the first quadrant bounded below by the *x*-axis, bounded on the left by the *y*-axis, bounded above by the graph of $y = 2 - x^2$ for $0 \le x \le 1$, and bounded above by the graph of $y = \sqrt{2-x}$ for $1 \le x \le 2$. See the picture.

Find the center of mass. Give the answer in the exact form.

Note: You can use symmetry but to get full credit, you have to provide a computation supporting the use of symmetry. Saying that the picture looks symmetric is not enough.
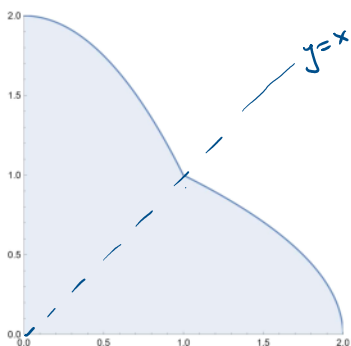
The two functions are inverses of each other

$$y = 2 - x^2 \iff x^2 = 2 - y \underset{(x > 0)}{\iff} x = \sqrt{2-y}$$

so their graphs are symmetric relative to the line $y = x$. Hence $\boxed{\bar{x} = \bar{y}}$

1) $\text{area} = \int_0^1 (2 - x^2)\, dx + \int_1^2 \sqrt{2-x}\, dx \qquad \begin{array}{l} u = \sqrt{2-x} \\ u^2 = 2 - x \\ 2u\, du = -dx. \end{array}$

$= \left(2x - \frac{1}{3}x^3\right)\Big|_0^1 + \int_1^0 u(-2u)\, du$

$= \left(2 - \frac{1}{3}\right) + \int_0^1 2u^2\, du = \frac{5}{3} + \frac{2}{3}u^3\Big|_0^1 = \frac{5}{3} + \frac{2}{3} = \frac{7}{3}$
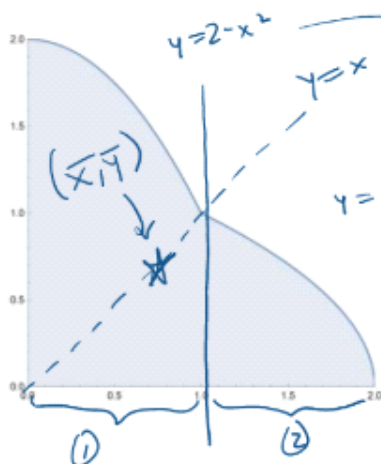
$\boxed{\text{area} = 7/3}$

2) $\bar{y} = \frac{1}{\text{area}}\left[\int_0^1 \frac{1}{2}(2-x^2)^2\, dx + \int_1^2 \frac{1}{2}(\sqrt{2-x})^2\, dx\right]$

$= \frac{3}{7}\frac{1}{2}\left[\int_0^1 4 - 4x^2 + x^4\, dx + \int_1^2 (2-x)\, dx\right]$

$= \frac{3}{7}\frac{1}{2}\left[\left(4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 + \left(2x - \frac{1}{2}x^2\right)\Big|_1^2\right]$

$= \frac{3}{7}\frac{1}{2}\left[\left(4 - \frac{4}{3} + \frac{1}{5}\right) + \left(4 - \frac{4}{2} - 2 + \frac{1}{2}\right)\right]$

$= \frac{3}{7}\frac{1}{2}\left(\frac{120 - 40 + 6 + 15}{(3)(5)(2)}\right) = \frac{101}{140}$

ANSWER: $(\bar{x}, \bar{y}) = \boxed{\left(\frac{101}{140}, \frac{101}{140}\right)}$

Alternative Approach:

8. (10 points) Consider the region in the first quadrant bounded below by the x-axis, bounded on the left by the y-axis, bounded above by the graph of $y = 2 - x^2$ for $0 \le x \le 1$, and bounded above by the graph of $y = \sqrt{2-x}$ for $1 \le x \le 2$. See the picture.

Find the center of mass. Give the answer in the exact form.

Note: You can use symmetry but to get full credit, you have to provide a computation supporting the use of symmetry. Saying that the picture looks symmetric is not enough.

Interchanging x and y, then solving:

$$x = 2 - y^2$$
$$y^2 = 2 - x$$
$$y = \sqrt{2-x}$$

$\therefore$ $y = 2 - x^2$ and $y = \sqrt{2-x}$ are reflections about $y = x$.

$$\therefore \bar{x} = \bar{y}.$$

① $A_1 = \int_0^1 (2 - x^2)\,dx = \left[2x - \frac{x^3}{3} \right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$

$\bar{x}_1 = \frac{1}{A_1} \int_0^1 (2 - x^2)\, x \,dx = \frac{3}{5} \int_0^1 (2x - x^3)\,dx = \frac{3}{5} \left[x^2 - \frac{x^4}{4} \right]_0^1$

$= \frac{3}{5} \left[1 - \frac{1}{4} \right] = \frac{9}{20}$

② $A_2 = \int_1^2 \sqrt{2-x}\, dx = \left[-\frac{2}{3}(2-x)^{\frac{3}{2}} \right]_1^2 = \frac{2}{3}$

$\bar{x}_2 = \frac{3}{2} \int_1^2 \sqrt{2-x}\, x\, dx = -\frac{3}{2} \int_1^0 \sqrt{u}\,(2-u)\,du = \frac{3}{2} \int_0^1 \left(2\sqrt{u} - u^{\frac{3}{2}} \right) du$

$u = 2 - x$
$du = -dx$
$-du = dx$

$= \frac{3}{2} \left[\frac{4}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right]_0^1$

$= \frac{3}{2} \left[\frac{4}{3} - \frac{2}{5} \right]$

$\therefore \bar{X} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2}{m_1 + m_2} = \frac{\frac{3}{4} + \frac{4}{3} - \frac{2}{5}}{A_1 + A_2}$

ANSWER: $(\bar{x}, \bar{y}) = \boxed{\left(\dfrac{101}{140}, \dfrac{101}{140} \right)}$

9. (10 points) Find the solution of the differential equation that satisfies the given initial condition:

$$\frac{dy}{dt} = 1 + t^2 + y^2 + t^2 y^2, \quad y(0) = 1.$$

For full credit, write your answer in explicit form, $y = f(t)$, and simplify it.

$$\frac{dy}{dt} = 1 + t^2 + y^2(1 + t^2) = (1 + t^2)(1 + y^2)$$

$$\int \frac{1}{1 + y^2} \, dy = \int (1 + t^2) \, dt$$

$$\arctan y = t + \frac{1}{3}t^3 + c$$

$$y(0) = 1: \quad \arctan 1 = \frac{\pi}{4} = 0 + \frac{1}{3}(0)^3 + c \implies c = \frac{\pi}{4}$$

$$\arctan y = t + \frac{1}{3}t^3 + \frac{\pi}{4}$$

$$y = \tan\left(t + \frac{1}{3}t^3 + \frac{\pi}{4}\right)$$

ANSWER: $y = \boxed{\tan\left(t + \frac{1}{3}t^3 + \frac{\pi}{4}\right)}$

10. A pan of lasagna has an internal temperature of $40°$ F at the time when it is placed in an oven whose temperature is kept constant at $380°$ F.

(a) (3 points) Newton's Law of Cooling states that the rate of cooling (or heating) of an object is proportional to the temperature difference between the object and its surroundings. Use this to write a differential equation and an initial condition for the internal temperature $y(t)$ of the lasagna at t hours after it was placed in the oven. Your equation should involve an unknown proportionality constant k.

Differential Equation: $\boxed{\dfrac{dy}{dt} = k(y - 380)}$ and $y(0) = \boxed{40}$

(b) (5 points) After a half hour in the oven, the internal temperature of the lasagna reaches $140°$ F. Compute the constant k and find the temperature y as a function of time t, in hours.

$$\int \frac{1}{y-380}\, dy = \int k\, dt \implies \ln|y-380| = kt + C$$

$$y - 380 = C_1 e^{kt} \qquad (C_1 = \pm e^C)$$

$$y = 380 + C_1 e^{kt}$$

① $\left(\begin{array}{l}\text{initially: } t=0,\ y(0)=40 \\ \implies 40 = 380 + C_1 \implies C_1 = -340\end{array}\right) \implies y = 380 - 340 e^{kt}$

② at $t = \frac{1}{2}$ $y = 140$ $\implies 140 = 380 - 340 e^{\frac{1}{2}k}$

$$\implies +240 = +340\, e^{\frac{1}{2}k}$$

$$\implies e^{\frac{1}{2}k} = \frac{240}{340} \implies \frac{1}{2}k = \ln\left(\frac{240}{340}\right) \implies k = 2\ln\left(\frac{240}{340}\right)$$

$$= 2\ln(12/17)$$

Answer: $y(t) = \boxed{380 - 340\, e^{2\ln(12/17)t}}$

(c) (2 points) The lasagna is finished baking when its internal temperature reaches $165°$ F. When will this happen?

$$380 - 340\, e^{2\ln(12/17)t} = 165$$

$$e^{2\ln(12/17)t} = \frac{215}{340}$$

$$t = \frac{\ln(215/340)}{2\ln(12/17)} \approx 0.6579\ldots$$

Answer: At $t = \boxed{0.66}$ hours
(round to the nearest two decimals)