Final Exam Rubric

Wednesday, March 8, 2023 8:56 AM

Math 120 Section **Final Exam, Mar 11th,** **Winter 2023**

HONOR STATEMENT
I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam.

Name

Signature

Student ID #

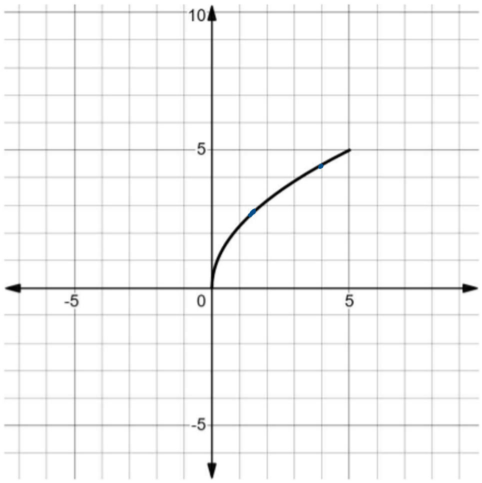
1.	2.	3.	4.	5.	6.	$\sum$
10	10	10	10	10	10	60

- You have 2 hours and 50 minutes for 6 problems. Check your copy of the exam for completeness.
- You are allowed to use a hand written sheet of paper (8x11 in), back and front.
- Calculator : TI 30 X.
- Justify all your answers and show your work for credit.
- All answers must be exact, no rounding.

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

Problem 1. Consider the function $g(x)$ restricted to the domain $0 \le x \le 5$ whose graph is shown in Illustration (a). We define the function $h(x) = 2 - g\left(\frac{1}{2}x + 3\right)$.



(a) Graph of $g(x)$

1

(a) Evaluate $h(h(2))$? Read off the values as best as you can.

$h(2) = 2 - g(1+3) = 2 - 4.5 = -2.5 \quad h(-2.5) = 2 - g(-1.25+3) = 2 - g(1.75) \approx 2 - 2.8 = -0.8$

(b) Below, list the horizontal and vertical shifts, horizontal and vertical dilations and reflections in the correct order that $g(x)$ must undergo to become $h(x) = 2 - g\left(\frac{1}{2}x + 3\right)$. Be precise: Give the direction (right, left, up, down), the number of units of the shift, and the scaling factor.

3

1. horiz. shift to the left by 3 units
2. horiz. dilation by factor 2
3. reflection about x-axis
4. vert shift by 2 units up

deduct $\frac{1}{2}$ points if order not correct / factor not correct

(c) With the help of part (b), find the domain and range of $h(x)$.

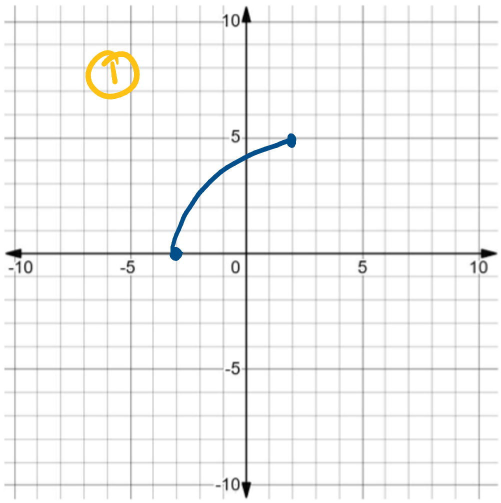
2

original domain: $0 \le x \le 5 \rightarrow \quad 0 \le \frac{1}{2}x + 3 \le 5 \rightarrow \boxed{-6 \le x \le 4}$ domain

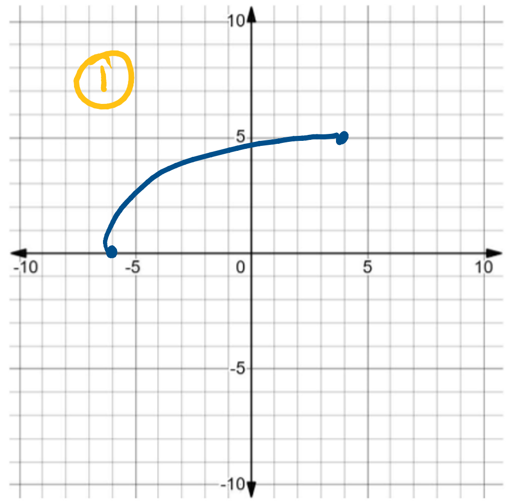
original range: $0 \le y \le 5 \rightarrow \quad 2-5 \le y \le 2-0 \rightarrow \quad -3 \le y \le 2$

Finding it with the help of the sketches is ok, too.

(d) Draw the graphs after each transformation for the functions you obtained in (b) in the same order.

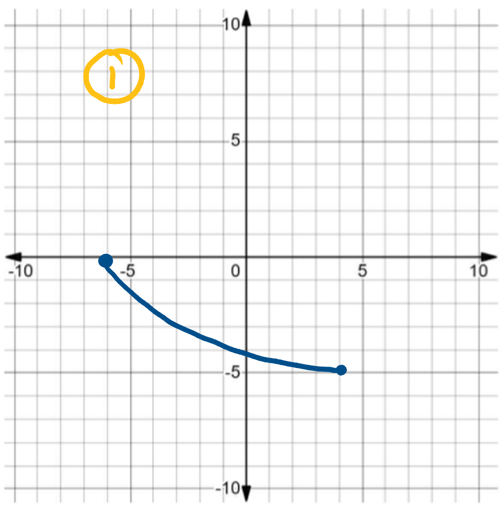


(a) Function after first transformation

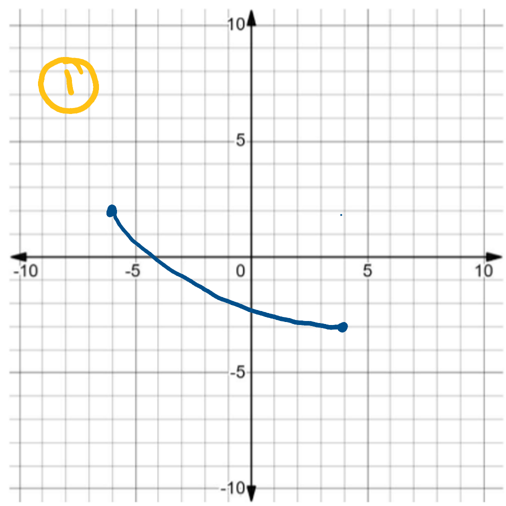


(b) Function after second transformation

should be according to what they have in b)



(c) Function after third transformation

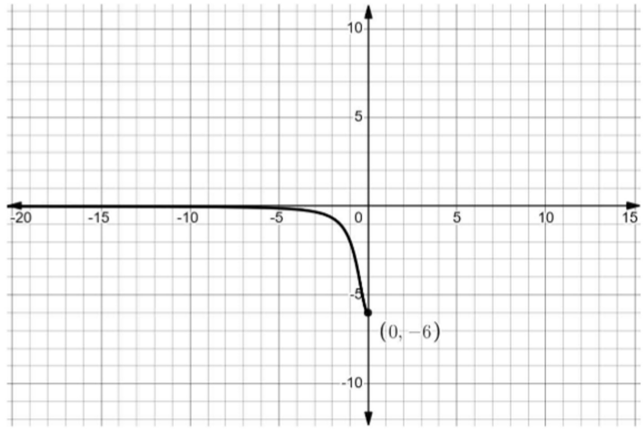


(d) Function after final transformation

$(0,0) \to (-3,0) \to (-6,0) \to (-6,0) \to (-6,2)$

$(5,5) \to (2,5) \to (4,5) \to (4,-5) \to (4,-3)$

Problem 2. Consider the function $f(x) = -\frac{6}{2x^2+1}$ whose graph is shown below.



(a) Graph of $f(x)$

(a) Explain briefly why $f(x)$ has an inverse function.

1

It is one-to-one because it passes the horiz. line test

(b) What is the domain of $f(x)$, what is the range of $f(x)$?

domain : $-\infty < x \leq 0$

range : $-6 \leq y < 0$

(c) Find the rule of the inverse function $f^{-1}(x)$.

$y = -\frac{6}{2x^2+1} \rightarrow x = -\frac{6}{2y^2+1} \rightarrow 2y^2+1 = -\frac{6}{x} \rightarrow$

4

$2y^2 = -\frac{6}{x} - 1 \rightarrow y^2 = -\frac{3}{x} - \frac{1}{2} \rightarrow y = \pm\sqrt{-\frac{3}{x}-\frac{1}{2}}$

as domain f : $-\infty < x \leq 0 \rightarrow$ range f^{-1} = $-\infty < y \leq 0$

$\rightarrow \boxed{y = -\sqrt{-\frac{3}{x}-\frac{1}{2}}}$

(d) What are the domain and range of $f^{-1}(x)$?

1

domain : $-6 \leq x < 0$

range : $-\infty < y \leq 0$

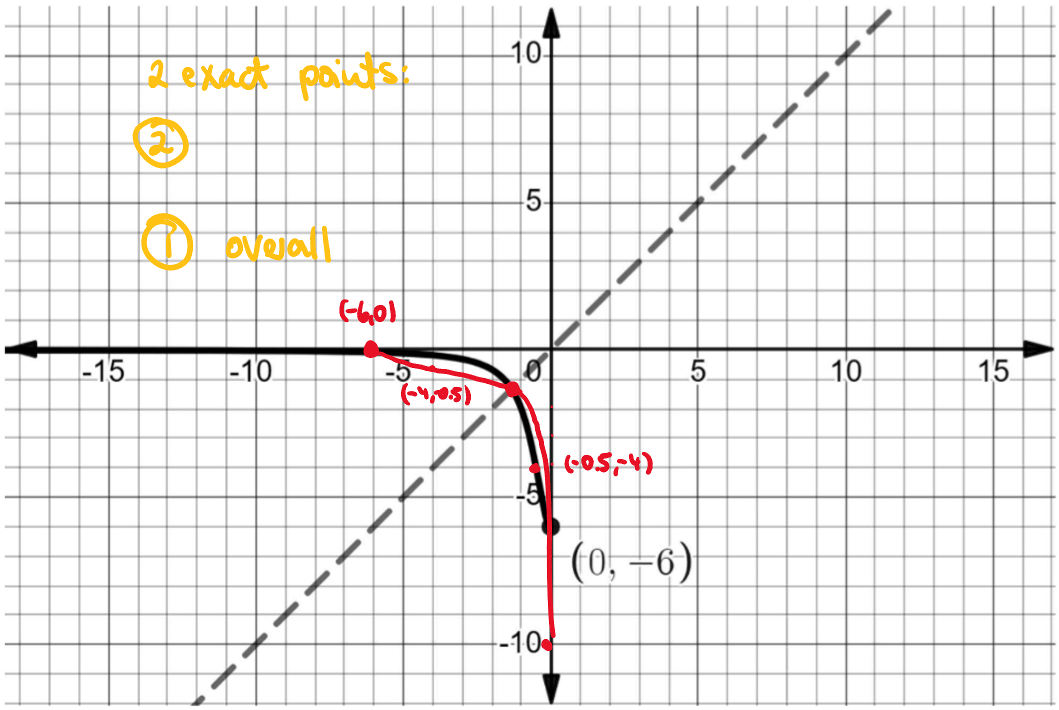
(e) Find $f^{-1}(f(-1))$ and $f(f^{-1}(-1))$. Precise values!

$\shortparallel$ $\shortparallel$

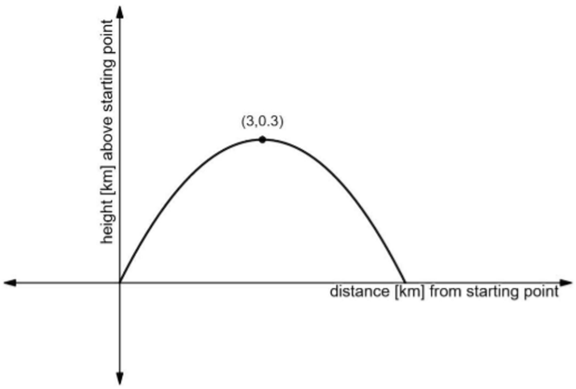
(e) Find $f^{-1}(f(-1))$ and $f(f^{-1}(-1))$. Precise values!

1 $\quad f^{-1}(f(-1)) = -1 \quad f(f^{-1}(-1)) = -1$ as f / f^{-1} undo each other

(f) In the coordinate system below, sketch the graph of the inverse function. The diagonal $y = x$ has been added (dashed line). At least two points of $f^{-1}(x)$ must be precise.



Problem 3. We are taking a balloon ride in the form of a parabola as illustrated below. The highest point is at a horizontal distance of 3km from the starting point and at a height of 0.3km. None of the illustrations below are at scale.



(a) Find the equation of the parabola. Do not round.

(0,0), (3,0.3) on the parabola, (3,0.3) vertex

3

$y = a(x-3)^2 + 0.3$ $\qquad 0 = a \cdot 9 + 0.3 \Rightarrow a = -\frac{\cancel{3}}{10 \cdot \cancel{9}3} = -\frac{1}{30}$

$y = -\frac{1}{30}(x-3)^2 + 0.3$

(b) For the balloon ride as shown in the illustration above, what is a reasonable domain of the function? Do not round!

1

we can exploit symmetry: The two zero's of the function are at $x=0$, $x=6$

OR

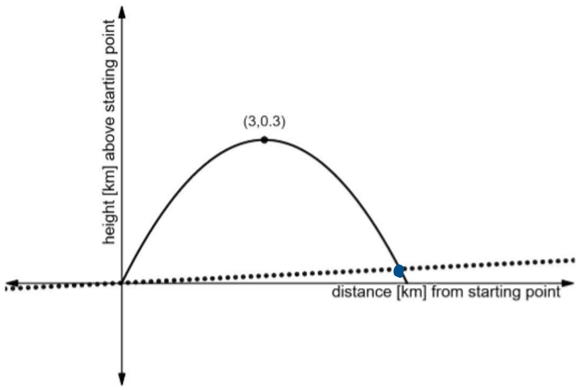
we can solve $-\frac{1}{30}(x-3)^2 + 0.3 = 0$

(c) We now assume that the ground is slanted so that every 1km it rises by 10 meters (see the dotted line in the illustration below). Find the equation of the line representing the ground? Use that 1000 meters = 1km.

2

$$\frac{10m}{1km} \cdot \frac{1km}{1000m} = \frac{1}{100}$$

$$y = \frac{1}{100}x$$



(d) Given the slanted ground, how far from the origin will the balloon land? Round to four(!) decimal places.

$$\frac{1}{100}x = -\frac{1}{30}(x-3)^2 + 0.3$$

$$\frac{1}{100}x = -\frac{1}{30}(x^2-6x+9) + 0.3 \quad | \cdot 300$$

3

$$3x = -10(x^2-6x+9) + 90$$

$$3x = -10x^2 + 60x$$

$$10x^2 - 57x = 0$$

$$x(10x-57) = 0 \Rightarrow x = 0 \text{ OR } x = 5.7 \Rightarrow y = \frac{5.7}{100} = 0.057$$

1

distance $\sqrt{5.7^2 + 0.057^2} = 5.7003$ km

The axis label could be interpreted to use as distance $x = 5.7$

↳ will also be accepted

Problem 4. A gardener is worried about her cherry trees which are infested with aphids. It seems that the population grows exponentially. In this problem, round each number that represents the aphids to the nearest integer in the final result, but work with 6 decimal places to get there.

(a) She finds that the aphid population triples every 5 days. Initially, (on day 0), there were 100 aphids. Find an exponential function that describes the population t days after the initial day.

3 $y = A \cdot b^t \quad A = 100 \quad b = \sqrt[5]{3} \rightarrow y = 100 \cdot \sqrt[5]{3}^t$

(b) On day 8 she buys ladybugs and releases them into the cherry trees. Ladybugs feast on aphids and the gardener observes that the number of aphids exponentially decreases by 15% every other day. Set up an exponential equation for the declining population of aphids. The independent variable t should be with respect to day 0, when the gardener first noticed the aphids.

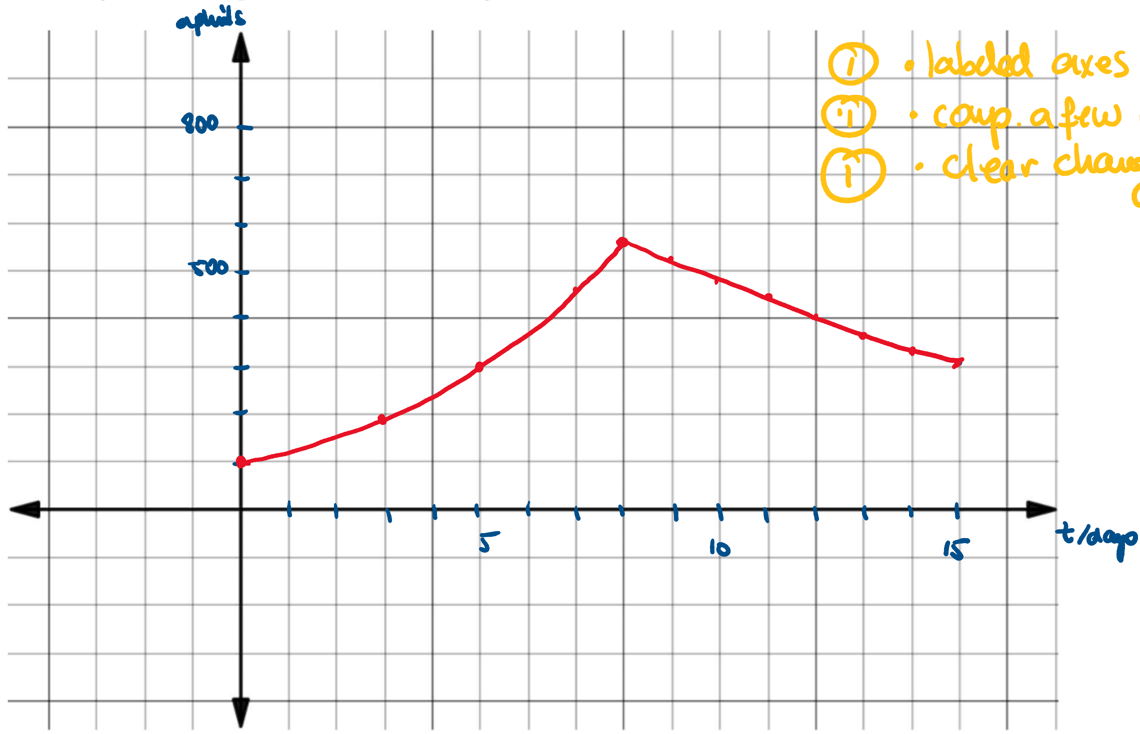
$y = A \cdot b^x \quad b = \sqrt[2]{0.85}$ 1 A = initial population = population @ $t=8$

on day 8: the # of aphids from a) is $y = 100 \cdot \sqrt[5]{3}^8 \approx 580.$ 1

$\underbrace{y = 580 \cdot \sqrt[2]{0.85}^{\,t-8}}_{\text{1 writing it out}}$ 1

(c) Sketch the multi-part function that describes the aphid population over at least 15 days. Use the provided coordinate system; label the axes.

(c) Sketch the multi-part function that describes the aphid population over at least 15 days. Use the provided coordinate system; label the axes.



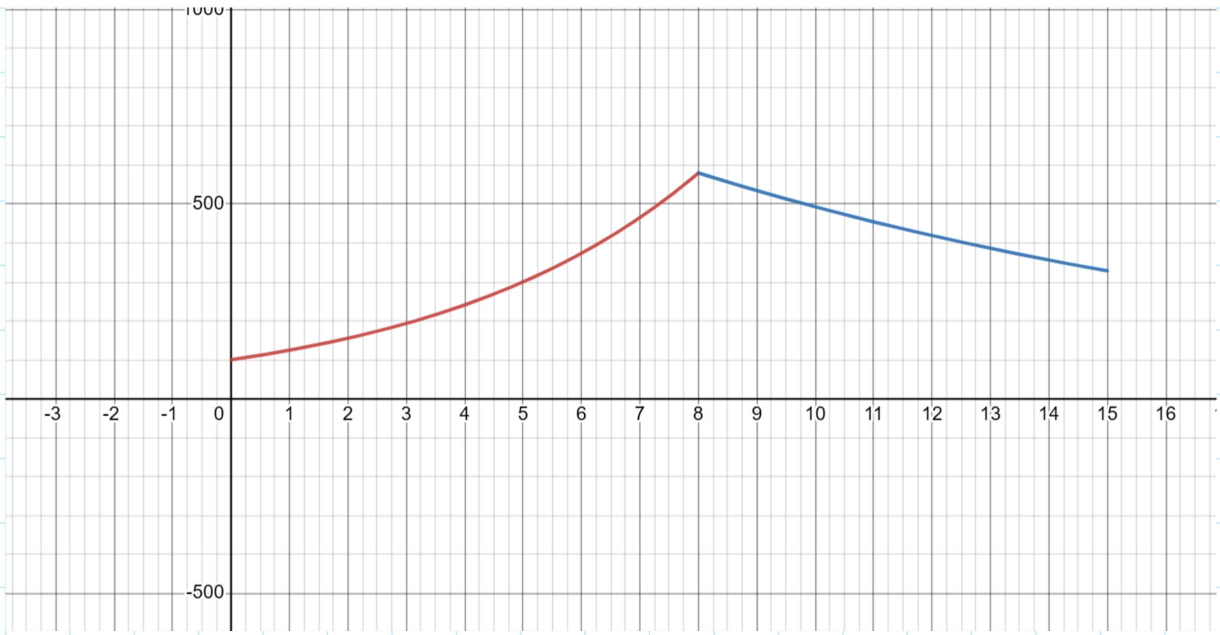
(1) • labeled axes

(1) • comp. a few points

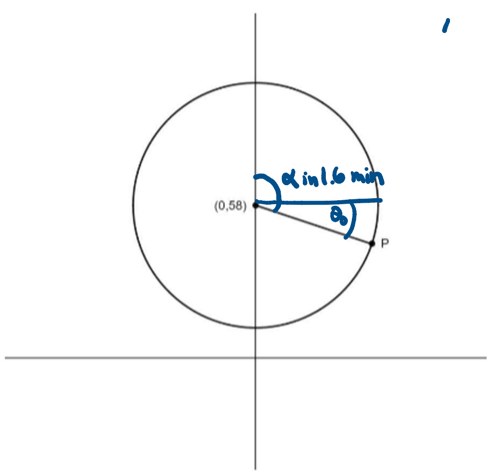
(1) • clear change at t=8

t=5: 300 t=11: 455 t=13: 386 t=15:

t=8: 580 t=12: 419 t=14: 356



Problem 5. Tom is riding a Ferris wheel. The center of the wheel is 58 feet above the ground. One complete revolution of the wheel takes 4 minutes. The wheel rotates counterclockwise. It takes Tom 1.6 minutes to reach the top of the wheel from the position P where he boards it. Tom travels 44π feet when he travels from P to the top of the wheel. Impose a coordinate system as shown in the illustration below.
In this problem, you round to two decimal places.



(a) Find the parametric equations that describe Tom's coordinates t minutes after he boards.

1 $\left.\begin{array}{l} x = x_c + r\cos(\omega t + \theta_0) \\ y = y_c + r\sin(\omega t + \theta_0) \end{array}\right\}$ $x_c = 0$ $y_c = 58$ 1

$\omega = \frac{2\pi}{4} = \frac{\pi}{2}\ \frac{\text{rad}}{\text{min}}$ 1.57 1

In 1.6 minutes: $\alpha = \omega \cdot 1.6 = \frac{\pi}{2}\cdot\frac{8}{5} = \frac{4\pi}{5}$ rad 2.51 $\qquad \frac{\pi}{2} = \theta_0 + \alpha \rightarrow \theta_0 = \frac{\pi}{2} - \frac{4\pi}{5} = -\frac{3\pi}{10}$ 1 −0.94

In 1.6 minutes: $\left.\begin{array}{l} 44\pi \text{ arclength covered} \\ \frac{4\pi}{5} \text{ angle covered} \end{array}\right\}$ $44\pi = \frac{4\pi}{5}\cdot r$ $\qquad r = 55$ 1

$\frac{4\pi}{5}$ angle covered)

$x = 55\cos\left(\frac{\pi}{2}t - \frac{3\pi}{10}\right)$

$y = 58 + 55\sin\left(\frac{\pi}{2}t - \frac{3\pi}{10}\right)$

(b) Find Tom's height above the ground 2.2 minutes after he boards the wheel.

$y = 58 + 55\sin\left(\frac{\pi}{2} \cdot 2.2 - \frac{3\pi}{10}\right) = 90.33$ ft

1

(c) After how many minutes will Tom be 85.5 feet above the ground for the first, second and third time?

$85.5 = 58 + 55\sin(\frac{\pi}{2}t - 0.3\pi)$

$27.5 = 55\sin(\frac{\pi}{2}t - 0.3\pi)$

$\frac{1}{2} = \sin(\frac{\pi}{2}t - 0.3\pi)$

$\frac{t}{2} = \frac{1}{6} + 0.3$

princ: $\frac{\pi}{2}t - 0.3\pi = \frac{\pi}{6} + 2k\pi \quad | \div \pi$

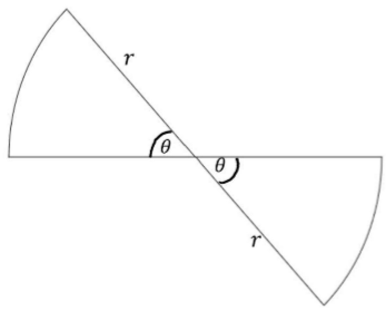
symm: $\frac{\pi}{2}t - 0.3\pi = \underbrace{\pi - \frac{\pi}{6}}_{\frac{5\pi}{6}} + 2k\pi \quad | \div \pi$

$$\boxed{\begin{array}{l} t = 0.6 + \frac{1}{3} + 4k \\ t = 0.6 + \frac{5}{3} + 4k \end{array}}$$

k	$0.6 + \frac{1}{3} + 4k$	$\frac{5}{3} + 4k$
0	0.93	2.27
1	4.93	6.27

1st after 0.93 min
2nd after 2.27 min
3rd after 4.93 min

Problem 6. Two identical circle sectors form the blade of a fan - see illustration below. Find the radius r and angle θ that gives the biggest area, given that the perimeter of the blade must be equal to 120cm. Round to two decimal places.



1) to be maximized:

2 $A = 2 \cdot \frac{1}{2}\theta r^2 = \theta r^2$

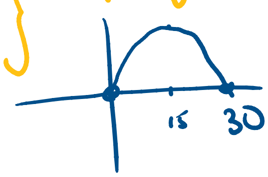
2) constraint: $120 = 4r + 2 \cdot r\theta$ 2

$60 = 2r + r\theta$

$\theta = \dfrac{60-2r}{r}$ 1

3) 2) in 1) 1 $A = \dfrac{60-2r}{r} \cdot r^2 = (60-2r)r$ } simplify 1

$A = -2r^2 + 60r$



vertex 1 at $-\dfrac{60}{-2 \cdot 2} = 15 = r \Rightarrow \theta = \dfrac{60 - 2 \cdot 15}{15} = \dfrac{60-30}{15} = 2$

maximal area when $r = 15$ cm, 1 $\theta = 2$ rad 1