1. **[5 points per part]** A bike is made up of two wheels with diameter 2 feet.

   The rear wheel is connected by an axle to a rear sprocket with diameter ¼ feet.

   The rear sprocket is connected by a chain to the front sprocket with diameter ½ feet.

   A biker pedals the front sprocket at a speed of 50 revolutions per minute.

   (a) Find the $\underbrace{\text{speed of the bike.}}_{\text{Linear speed of wheels}}$

	v (ft/min)	ω (rad/min)	r (ft)
R. Wheel	200π	$\boxed{200\pi}$	1
R. Sprocket	$\boxed{25\pi}$	$\boxed{200\pi}$	1/8
F. Sprocket	$\boxed{25\pi}$	100π	1/4

   Speed = $\boxed{200\pi}$ feet per minute

   (b) The biker is riding the bike counterclockwise around a circular track with radius 25 feet, starting at the northernmost point.

   Write parametric equations for the coordinates of the biker after t minutes.

   (Set the origin at the center of the circle, with north pointing upward.)

$r = 25$

$v = 200\pi$

$\omega = 8\pi$

$\theta_0 = \frac{\pi}{2}$

$(x_0, y_0) = (0,0)$

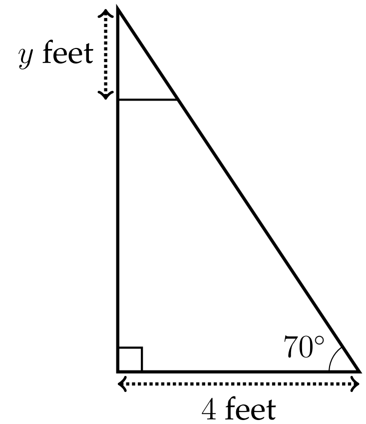
   Parametric equations:

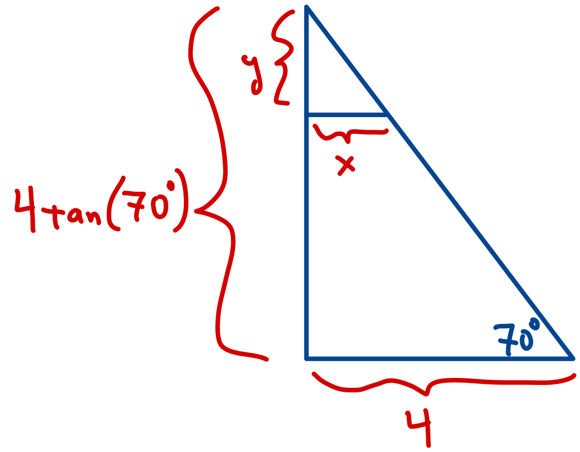
$$\boxed{\begin{aligned} x &= 25\cos\left(\frac{\pi}{2} + 8\pi t\right) \\ y &= 25\sin\left(\frac{\pi}{2} + 8\pi t\right) \end{aligned}}$$

2. **[10 points]** Steve's backyard contains a triangular hedge, as shown in the figure below.

He uses hedge trimmers to trim the top y feet of the hedge.

Write a function $A(y)$ for the area remaining after trimming.





$\frac{y}{x} = \tan(70^\circ)$

$x = \frac{y}{\tan(70^\circ)}$

Total area $= \frac{1}{2}(4)(4\tan(70^\circ)) = 8\tan(70^\circ)$

Trimmed area $= \frac{1}{2}xy = \frac{1}{2}\left(\frac{y}{\tan(70^\circ)}\right)y = \frac{y^2}{2\tan(70^\circ)}$

Remaining = total − trimmed

$A(y) = \boxed{8\tan(70^\circ) - \frac{y^2}{2\tan(70^\circ)}}$

3. The amount of water in a well is a linear-to-linear rational function of time.

In the year 2000, there was 7 feet of water in the well.

In the year 2002, there was 6 feet of water in the well.

In the year 2012, there was 5 feet of water in the well.

(a) **[7 points]** Write a function $f(t)$ for the amount of water in the well t years after 2000.

$$f(t) = \frac{at+b}{t+d}$$

$$f(0)=7 \rightarrow \frac{b}{d} = 7 \rightarrow b = 7d$$

$$f(2)=6 \rightarrow \frac{2a+b}{2+d} = 6 \rightarrow 2a+b = 12+6d \rightarrow 2a+7d = 12+6d$$

$$f(12)=5 \rightarrow \frac{12a+b}{12+d} = 5 \rightarrow 12a+b = 60+5d \rightarrow 12a+7d = 60+5d$$

$$-2(2a+d=12)$$
$$12a+2d=60$$

$$8a=36$$

$$\boxed{a=4.5}$$

$$d=12-2a$$

$$\boxed{d=3}$$

$$b=7d$$

$$\boxed{b=21}$$

$f(t) = \boxed{\dfrac{4.5t+21}{t+3}}$

(b) **[3 points]** In the long run, what will the water level approach?

Horizontal asymptote $= a$

Water level: $\boxed{4.5}$ feet

4. Greg and Paul are walking around the coordinate plane.

(a) **[3 points]** Greg starts at the point $(-2, 4)$, and walks towards the point $(4, 0)$ in a straight line at a constant speed, reaching it after 4 seconds.

Write parametric equations for Greg's location after t seconds.

$x_0 = -2 \quad y_0 = 4$

$x_1 = 4 \quad y_1 = 0$

$\Delta x = 6 \quad \Delta y = -4$

$\Delta t = 4$

Parametric equations: $\boxed{x = -2 + \frac{3}{2}t \quad y = 4 - t}$

(b) **[3 points]** Paul starts at $(5, 7)$ and runs towards $(-1, -1)$ at a constant speed of 2.5 units per second. Write parametric equations for Paul's position after t seconds.

$x_0 = 5 \quad y_0 = 7$

$x_1 = -1 \quad y_1 = -1$

$\Delta x = -6 \quad \Delta y = -8$

$\Delta t = 4$

$\text{dist} = \sqrt{(5+1)^2 + (7+1)^2} = 10$

$\Delta t = \frac{10}{2.5} = 4$

Parametric equations: $\boxed{x = 5 - \frac{3}{2}t \quad y = 7 - 2t}$

(c) **[4 points]** When are Greg and Paul closest together?

$$\text{dist} = \sqrt{\left((-2 + \tfrac{3}{2}t) - (5 - \tfrac{3}{2}t)\right)^2 + \left((4-t) - (7-2t)\right)^2}$$

$$= \sqrt{(-7+3t)^2 + (-3+t)^2} = \sqrt{49 - 42t + 9t^2 + 9 - 6t + t^2}$$

$$= \sqrt{\underbrace{10t^2 - 48t + 58}_{\text{quadratic, pointing up}}}$$

minimized when $t = h = \frac{-b}{2a} = \frac{48}{20}$

After $\boxed{2.4}$ seconds

5. For parts (a) and (b), put your answers in **standard exponential form**.

(a) **[3 points]** A band's popularity grows exponentially over time.

100 people will attend their concert today. The popularity grows by 7% every 5 days.

Write a function $a(t)$ for the attendance t days from now.

$a(t) = A_0 b^t$

$A_0 = 100 \qquad b^5 = 1.07$

$\downarrow$

$b = 1.07^{1/5}$

$\boxed{a(t) = 100 \cdot 1.07^{t/5}}$

(b) **[3 points]** The cost per ticket is also growing exponentially.

Right now, it's \$12 per ticket. The cost doubles every 30 days.

Write a function $c(t)$ for the cost t days from now.

$c(t) = A_0 b^t$

$A_0 = 12 \qquad b^{30} = 2$

$b = 2^{1/30}$

$\boxed{c(t) = 12 \cdot 2^{t/30}}$

(c) **[4 points]** When will the band make a total of \$10,000 per concert?

Round your answer to the nearest day. Assume every person at the concert buys one ticket.

Total money $= a(t)c(t) = 100(1.07^{1/5})^t \cdot 12 \cdot (2^{1/30})^t$

$= 1200\left(1.07^{1/5}\, 2^{1/30}\right)^t = 10000$

$\left(1.07^{1/5}\, 2^{1/30}\right)^t = \frac{100}{12} \rightarrow t = \log_{(1.07^{1/5}\, 2^{1/30})}\left(\frac{100}{12}\right) \approx 57.87$

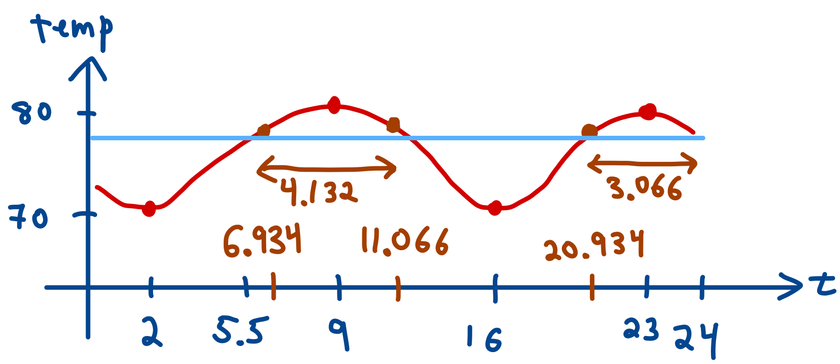
After $\boxed{57.87}$ days

6. **[10 points]** The temperature in Lake Wavia is a sinusoidal function of time.

2 hours from now, it will reach its minimum temperature of $70°$ F.

The temperature will then rise until it reaches a maximum of $80°$ F, 9 hours from now.

Over the next 24 hours (starting now), for how long will the temperature be above $78°$ F?



$A = 5$

$B = 14$

$C = 5.5$

$D = 75$

$$5\sin\left(\frac{2\pi}{14}(t-5.5)\right) + 75 = 78$$

$$5\sin\left(\frac{\pi}{7}(t-5.5)\right) = 3$$

$$\sin\left(\frac{\pi}{7}(t-5.5)\right) = \frac{3}{5}$$

principal soln

$$z = 5.5 + \frac{7}{\pi}\sin^{-1}\left(\frac{3}{5}\right) \approx 6.934$$

$$S = 2C + \frac{B}{2} - P = 11.066$$

Times when temp $= 78°$: 6.934, 11.066, 20.934, 25.066 (out of bounds)

+14

+14

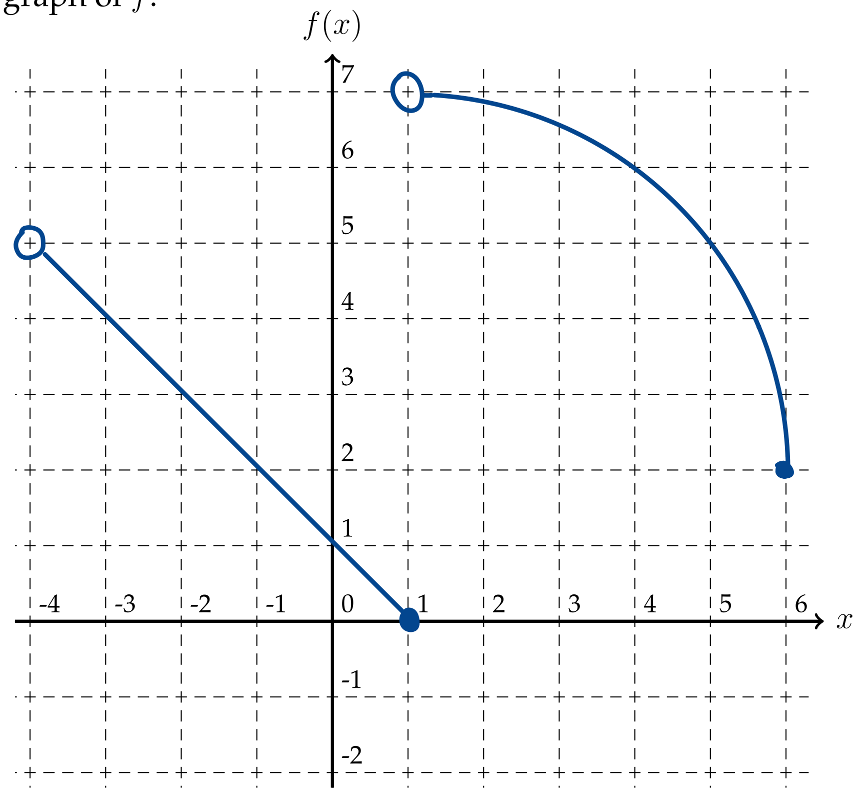
$$\text{Total} = 4.132 + 3.066$$

Total: $\boxed{7.199}$ hours

7. **[5 points per part]** For this problem, consider the following function:

$$f(x) = \begin{cases} 1 - x & \text{if } -4 < x \leq 1 \\ 2 + \sqrt{25 - (x-1)^2} & \text{if } 1 < x \leq 6 \end{cases}$$

(a) Sketch the graph of f:



(b) Find all values a such that $f(a) = a$.

if $-4 < a \leq 1$:

$1 - a = a$

$a = \frac{1}{2}$ ✓ Yes

if $1 < a \leq 6$:

$2 + \sqrt{25 - (a-1)^2} = a$

$\sqrt{25 - (a-1)^2} = a - 2$

$25 - (a-1)^2 = (a-2)^2$

$25 - a^2 + 2a - 1 = a^2 - 4a + 4$

$0 = 2a^2 - 6a - 20 = 2(a^2 - 3a - 10)$

$0 = 2(a-5)(a+2) \rightarrow a = 5$ ✓, or $\cancel{-2}$

$a = \boxed{\frac{1}{2} \text{ or } 5}$ (list all possibilities)