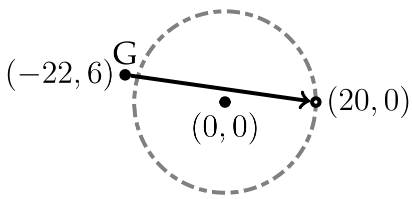
Math 120 Section A, Spring 2014
Final Exam: Solutions



1. Here's a picture. Gretel starts at $(-22, 6)$ and walks towards $(20, 0)$ along the line $y = \frac{-6}{42}(x - 20)$, which we can simplify to $y = \frac{-1}{7}(x - 20)$. The circle has equation $x^2 + y^2 = 20^2$, and we can find out where she enters the forest by substituting $\frac{-1}{7}(x - 20)$ in for y. Solve that to get $x = -19.2$, $y = 5.6$.

   How long does Gretel walk inside the forest? She walks in a straight line from $(-19.2, 5.6)$ to $(20, 0)$ at a speed of 3 miles per hour, which takes her $\left(\sqrt{(20 - (-19.2))^2 + (0 - 5.6)^2}\right)/3$, or about 13.199 hours.

2. (a) Asha walks from $(6, -5)$ to $(2, 3)$ in 8 seconds, so her parametric equations are straightforward:

$$x = 6 + \frac{-4}{8}t \qquad y = -5 + \frac{8}{8}t$$

   (b) Burt walks from $(-10, 7)$ to $(5, -1)$, but we don't know how long it takes him! Let's use his speed of 3.4 units per second to find out. The distance between those points is $\sqrt{(-10 - 5)^2 + (7 - (-1))^2} = 17$ units, so it takes him $17/3.4 = 5$ seconds to get there. So his parametric equations are:

$$x = -10 + \frac{15}{5}t \qquad y = 7 + \frac{-8}{5}t$$

   (c) To find out when Asha and Burt are closest, let's look at their distance using the distance formula:

$$d = \sqrt{\big((6 - t/2) - (-10 + 3t)\big)^2 + \big((-5 + t) - (7 - 1.6t)\big)^2}$$

   This eventually simplifies to:

$$d = \sqrt{19.01t^2 - 174.4t + 400}$$

   That's an upward-pointing quadratic inside a square root, so the distance is at its minimum at the vertex of the quadratic, which occurs at time $t = h = \frac{-b}{2a} = \frac{174.4}{38.02}$, or about 4.587 seconds.

3. (a) $f(g(x)) = |x - 2| - (x - 2)^2 + 2$. We want to write the multipart rule for this function. $|x - 2| = x - 2$ if $x - 2 \geq 0$, and $|x - 2| = -(x - 2)$ if $x - 2 < 0$. So we get:

$$f(g(x)) = \begin{cases} -(x-2) - (x-2)^2 + 2 & \text{if } x < 2 \\ x - 2 - (x-2)^2 + 2 & \text{if } x \geq 2 \end{cases}$$

   (b) Nah. $f(0) = 2$, but $f(1) = 2$ as well, so it's not one-to-one.

(c) Let's write $y = |x| - x^2 + 2$. Scaling vertically by a factor of 3 gives $y/3 = |x| - x^2 + 2$, and then shifting 2 units down gives $(y + 2)/3 = |x| - x^2 + 2$. So

$$h(x) = 3(|x| - x^2 + 2) - 2.$$

4. (a) Pretty straightforward: $A_0 = 4$ million and $b = 1.023$, so $\ell(t) = 4(1.023)^t$.

   (b) Andante's population doubles every 15 years, so $b^{15} = 2$ and therefore $b = \sqrt[15]{2}$.
   What's Andante's population in 2010? Well, $\ell(10) = 4(1.023)^{10} \approx 5.021$ million, and so Andante's population is about 2.511 million.
   That means $2.511 = A_0(\sqrt[15]{2})^{10}$, so $A_0 \approx 1.582$ million. So $a(t) = 1.582(\sqrt[15]{2})^t$.

   (c) We want to solve $a(t) = 3\ell(t)$.

$$1.582(\sqrt[15]{2})^t = 3 \cdot 4(1.023)^t$$

$$\left(\frac{\sqrt[15]{2}}{1.023}\right)^t = \frac{12}{1.582}$$

$$t = \frac{\ln\left(\frac{12}{1.582}\right)}{\ln\left(\frac{\sqrt[15]{2}}{1.023}\right)} \approx 86.33 \text{ years}$$

   So, around 2086.

5. (a) We want a function $f(x) = \dfrac{ax + b}{x + d}$ with $f(21) = 0$, $f(0) = 36$, and $f(1) = 40$. So:

$$\frac{21a + b}{21 + d} = 0$$

$$\frac{0a + b}{0 + d} = 36$$

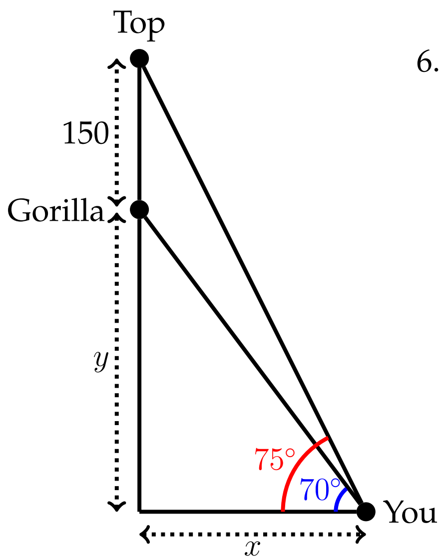
$$\frac{a + b}{1 + d} = 40$$

   Solving these three equations gives $a = 12$, $b = -252$, $d = -7$, so

$$f(x) = \frac{12x - 252}{x - 7}$$

   (b) The horizontal asymptote is at $y = a = 12$.

   The vertical asymptote is at $x = -d = 7$.



6. Here's a picture. Based on the two right triangles, we can get two equations:

$$\tan(75°) = \frac{150+y}{x} \qquad \tan(70°) = \frac{y}{x}$$

We want to know the height of the skyscraper, which is $y + 150$. Solving both equations for x and setting them equal gives

$$\frac{150+y}{\tan(75°)} = \frac{y}{\tan(70°)}$$

and solving that yields $y \approx 418.58$, so the building is 568.58 feet tall.

7. (a) In 11 seconds, Chloe runs 44 meters and Andy runs 33 meters, so they collectively travel 77 meters. And since they start on opposite sides of the track and run towards each other, that means half the circumference is 77 meters, and so the circumference is 154 meters and the radius is $\frac{154}{2\pi} \approx 24.50986$ meters.

   (b) Let's say that the center of the track is $(0, 0)$. We'll need to know θ_0 and ω for Chloe and Andy, and then plug those into the parametric equations for uniform circular motion (specifically the x-coordinates, since we want to know who's farther east).

   For Chloe, we have $r = \frac{77}{\pi}$, $\omega = \frac{4}{r} = \frac{4\pi}{77}$, $\theta_0 = \frac{\pi}{2}$, and $t = 600$ seconds. Plug that all in to get:

$$x = \frac{77}{\pi}\cos\left(\frac{\pi}{2} - \frac{4\pi}{77}(600)\right) \approx -12.399$$

   For Andy, we have $r = \frac{77}{\pi}$, $\omega = \frac{3}{r} = \frac{3\pi}{77}$, $\theta_0 = \frac{3\pi}{2}$, and $t = 600$ seconds. Plug that all in to get:

$$x = \frac{77}{\pi}\cos\left(\frac{3\pi}{2} + \frac{3\pi}{77}(600)\right) \approx -22.692$$

   So Chloe is farther east.

8. (a) A is the amplitude. $A = (\text{Max. temp} - \text{min. temp})/2 = 2.5$.

   B is the period. It takes 3 hours to go from maximum temperature to minimum temperature, so $B = 6$.

   C is the phase shift. We can take the time of maximum temperature ($t = 2$) and subtract $B/4$ to get $C = 0.5$.

   D is the average temperature, and that's just $D = 37.5$.

   Putting that all together:

$$f(t) = 2.5\sin\left(\frac{2\pi}{6}(t - 0.5)\right) + 37.5$$

(b) First, we want to solve $f(t) = 38$. So:

$$38 = 2.5 \sin\left(\frac{2\pi}{6}(t - 0.5)\right) + 37.5$$

Solve that to get

$$t = 0.5 + \frac{6}{2\pi} \sin^{-1}(0.2) \approx 0.6923$$

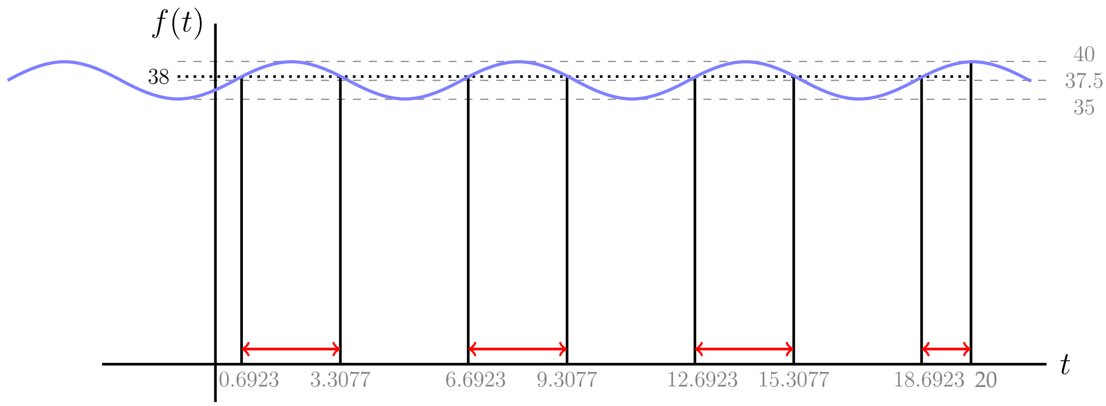
That's the principal solution. Then we can calculate the symmetry solution as

$$2(0.5) + 6/2 - 0.6923 = 3.3077.$$

Repeatedly adding the periods gives the following solutions on the interval $[0, 20]$:

- 0.6923
- 3.3077
- 6.6923
- 9.3077
- 12.6923
- 15.3077
- 18.6923

And here's what the curve looks like:



So the total time spent above $38°$ is

$$(3.3077 - 0.6923) + (9.3077 - 6.6923) + (15.3066 - 12.6923) + (20 - 18.6923) = 9.1539 \text{ hours}$$