Final Exam Key

Tuesday, November 19, 2024 1:53 PM

Math 120 **Final Exam 12/07/2024** **Fall 2024**
Instructor: Natalie Naehrig **Sections A,B**

HONOR STATEMENT
I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam.

Name	Signature

Student ID #

- Silence your phone and put it away.
- You have 170 minutes for 6 problems. Check your copy of the exam for completeness.
- You are allowed to use a hand written sheet of paper (8x11 in), back and front.
- Calculator : TI 30 XIIS.
- Justify all your answers and show your work for credit.
- Unless otherwise instructed, do not round.
- Each problem is worth 15 points.
- The last page can be used for scratch paper work and will not be graded unless otherwise indicated.
- If applicable, $\boxed{\text{Put a box around your final answers.}}$

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

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Problem 1. Consider the function $f(x) = \ln(x^2+1)+2$ with domain $-9 \le x \le 0$. The graph of f is shown on the next page. In this problem you may round to 1 decimal place when needed.

(a) Compute the range of the function.

$$\left.\begin{array}{l} f(-9) = \ln(82)+2 \approx 6.4 \\ f(0) = \ln(1)+2 = 2 \end{array}\right\} \text{range} : \boxed{[2, 6.4]}$$

(b) Justify why $f(x)$ has an inverse function. You may use the graph of f for your reasoning.

- it is one-to-one
- it passes horizontal line test

(c) What are domain and range of $f^{-1}(x)$?

domain: $[2, 6.4]$

range : $[-9, 0]$

(d) Find the inverse function $f^{-1}(x)$ of $f(x)$.

$x = \ln(y^2+1)+2$

$\Rightarrow x-2 = \ln(y^2+1)$

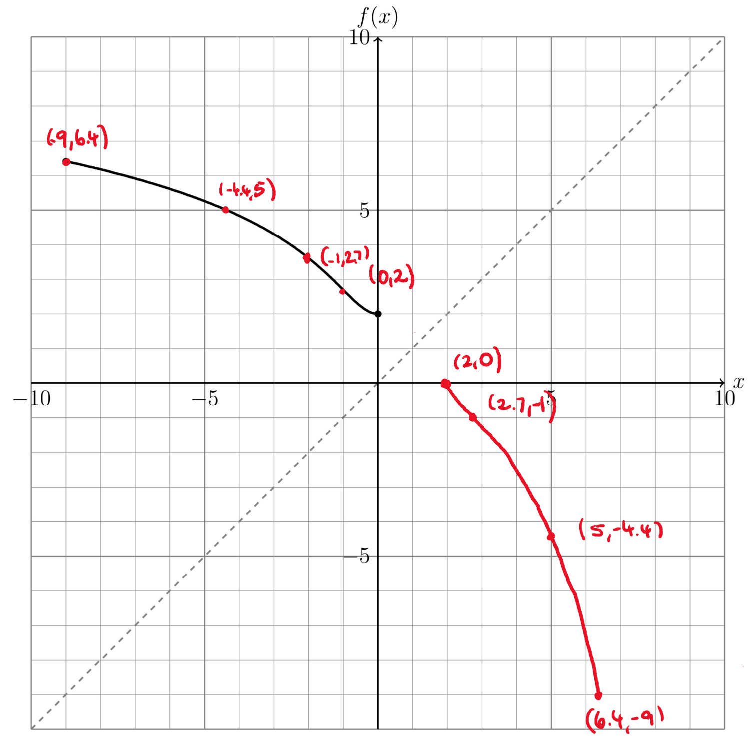
$\Rightarrow e^{x-2} = y^2+1$

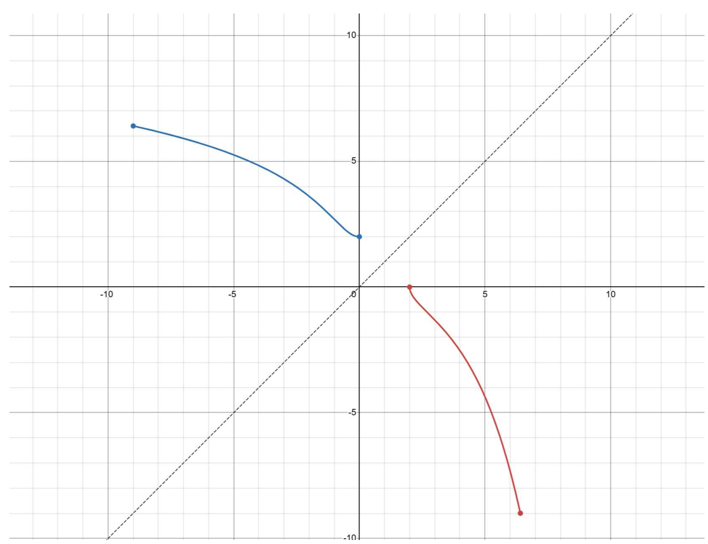
$\Rightarrow e^{x-2}-1 = y^2 \qquad y = \pm\sqrt{e^{x-2}-1}$

In order to have range $[-9,0]$, we must pick neg. root

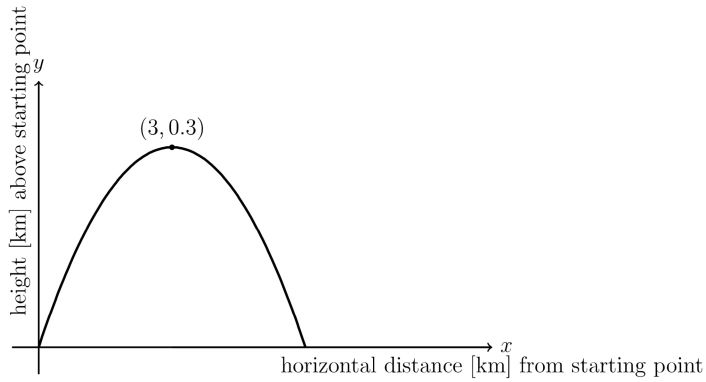
$$\boxed{f^{-1}(x) = -\sqrt{e^{x-2}-1}}$$

(e) Using the diagonal $y = x$, sketch the inverse function of $f(x)$ in the given coordinate system. At least 3 points of the graph should be used and labeled.





Problem 2. We are taking a balloon ride in the shape of a parabola as illustrated below. The highest point is at a horizontal distance of 3 km from the starting point and at a height of 0.3 km. None of the illustrations below are at scale.



(a) Find the equation of the parabola. Do not round.

$y = a(x-h)^2 + k$ vertex $(3, 0.3)$

$y = a(x-3)^2 + 0.3$ $(0,0)$ on parabola

$0 = a(-3)^2 + 0.3 \Rightarrow a = -\frac{3}{10 \cdot 9} = -\frac{1}{30}$

$$\boxed{y = -\frac{1}{30}(x-3)^2 + 0.3}$$

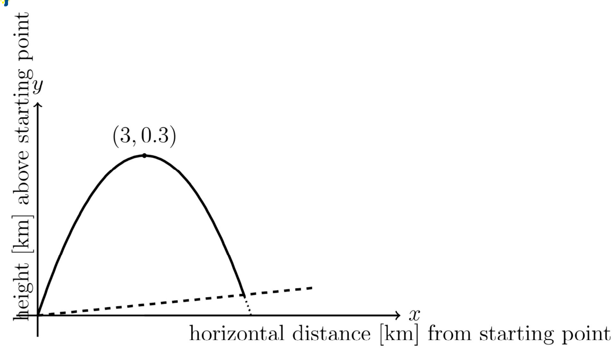
(b) For the balloon ride as shown in the illustration above, what is a reasonable domain of the function? Do not round.

exploit symmetry: zero's at 0 and 6:

$\boxed{[0,6]}$ is reasonable domain.

(c) We now assume that the ground is slanted so that for every 1 kilometer horizontally, it rises by 10 meters (see the dashed line in the illustration below). Find the equation of the line representing the ground. Use that 1000 meters = 1 kilometer.

$$\boxed{y = \frac{10}{1000}x \Rightarrow \qquad y = \frac{1}{100}x}$$



(d) Given the slanted ground, how far from the origin will the balloon land? Round to **four(!)** decimal places.

intersect: $\quad y = -\frac{1}{30}(x-3)^2 + \frac{3}{10} \qquad y = \frac{1}{100}x$

$\frac{1}{100}x = -\frac{1}{30}(x-3)^2 + \frac{3}{10} \qquad | \cdot 30$

$\frac{3}{10}x + (x-3)^2 - 9 = 0$

$x^2 - \frac{57}{10}x = 0$

$x(x - 5.7) = 0$

$x = 0 \quad \text{OR} \quad x = 5.7$

$\Rightarrow \quad (5.7, 0.057)$

$$\boxed{d = \sqrt{5.7^2 + 0.057^2} = 5.7003 \text{ km}}$$

Problem 3. A gardener is worried about her cherry trees which are infested with aphids. It seems that the population grows exponentially. In this problem, round each number that represents the aphids to the nearest integer in the final result, but work with 6 decimal places to get there.

(a) She finds that the aphid population triples every 5 days. Initially, (on day 0), there were 100 aphids. Find an exponential function $f_1(t)$ that describes the population t days after the initial day.

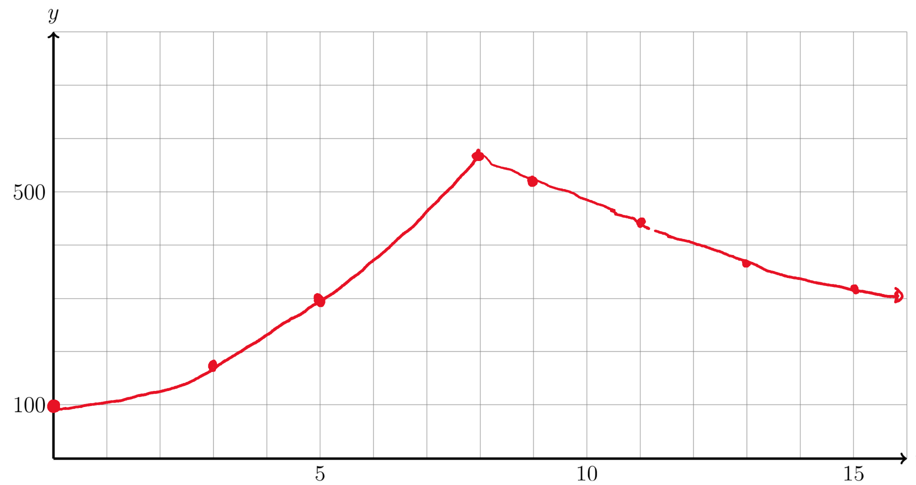
$$\boxed{f_1(t) = 100 \cdot \sqrt[5]{3}^{\,t} \approx 100 \cdot 1.245731^t}$$

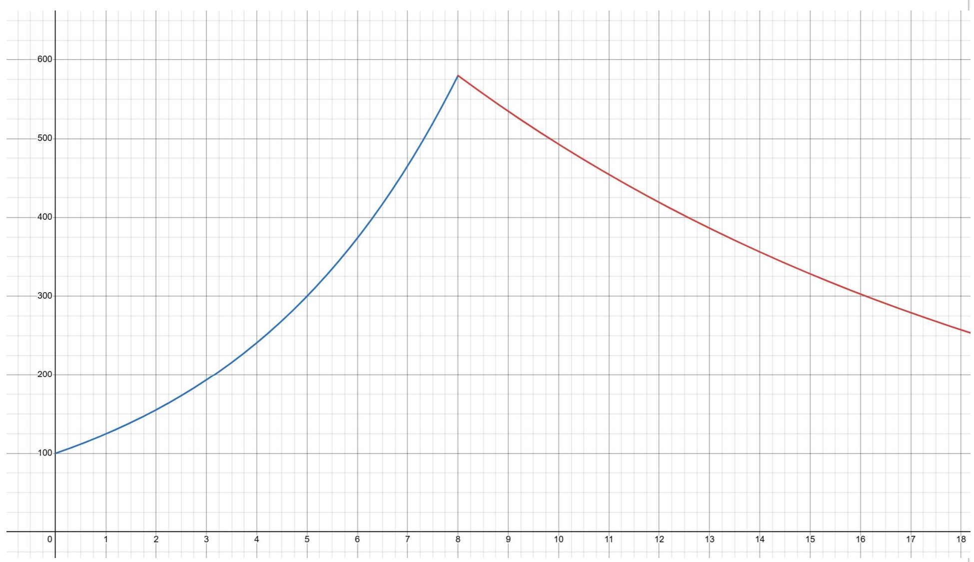
(b) On day 8 she buys ladybugs and releases them into the cherry trees. Ladybugs feast on aphids and the gardener observes that the number of aphids exponentially decreases by 15% every two day. Set up an exponential function $f_2(t)$ for the declining population of aphids. The independent variable t should be with respect to day 0, when the gardener first noticed the aphids so that the domain of $f_2(t)$ is $t \geq 8$.

$f_1(8) \approx 580$

$$f_2(t) = 580 \cdot \sqrt{0.85}^{\,t-8} \approx \boxed{580 \cdot 0.921954^{t-8}}$$

(c) Sketch the multi-part function that describes the aphid population over at least 15 days with at least 5 precise points. Use the provided coordinate system.





Problem 4. Professor Naehrig runs counter-clockwise on a circular track of radius 45 meters. She needs 72 seconds to complete a complete lap.

(a) What is Prof. Naehrig's linear speed? Round to one decimal place

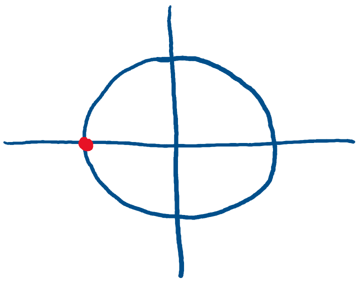
$\omega = \frac{2\pi}{72} = \frac{\pi}{36} \ \frac{\text{rad}}{\text{sec}}$

(a) What is Prof. Naehrig's linear speed? Round to one decimal place

$$\omega = \frac{2\pi}{72} = \frac{\pi}{36}\ \frac{\text{rad}}{\text{sec}}$$

$$v = \omega \cdot r = \frac{\pi}{36} \cdot 45\ \frac{\text{m}}{\text{sec}} \approx \boxed{3.9\ \frac{\text{m}}{\text{sec}}}$$

(b) Impose a coordinate system with the center as the origin. At $t = 0$ seconds she starts running at the westernmost point on the track. Write parametric equations for the position of Professor Naehrig after t seconds.



$$x = x_c + r\cos(\omega t + \theta)$$

$$y = y_c + r\sin(\omega t + \theta)$$

$$x_c = y_c = 0$$

$$\omega = \frac{\pi}{36}\ \frac{\text{rad}}{\text{sec}}$$

$$r = 45\text{ m}$$

$$\theta = \pi$$

$$\boxed{\begin{aligned} x &= 45\cos\left(\tfrac{\pi}{36}t + \pi\right) \\ y &= 45\sin\left(\tfrac{\pi}{36}t + \pi\right) \end{aligned}}$$

(c) When will her x-coordinate be $x = 22.5\sqrt{2}$ for the first, second, and **third** time? Round your answer to the nearest second.

$$45\cos\left(\frac{\pi}{36}t + \pi\right) = 22.5\sqrt{2}$$

$$\cos\left(\frac{\pi}{36}t + \pi\right) = \frac{1}{2}\sqrt{2}$$

princ

symm.

princ

$$\frac{\pi}{36}t+\pi=\frac{\pi}{4}+2\pi k$$

$$\frac{\pi}{36}t=-\frac{3}{4}\pi+2\pi k$$

$$t=-27+72k$$

symm.

$$\frac{\pi}{36}t+\pi=-\frac{\pi}{4}+2\pi k$$

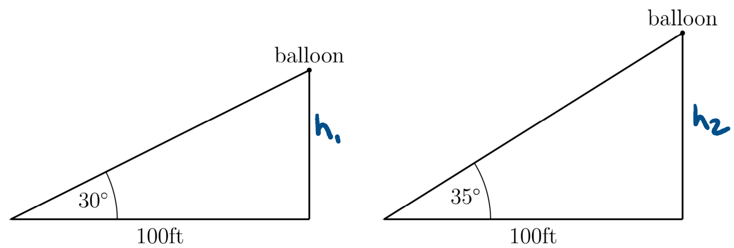
$$\frac{\pi}{36}t=-\frac{5\pi}{4}+2\pi k$$

$$t=-45+72k$$

k	princ	symm
1	45	27
2	117	99

$$\boxed{\begin{array}{l}\text{first time: } 27\,s\\ \text{second time: } 45\,s\\ \text{third time: } 99\,s\end{array}}$$

Problem 5. A person observes a balloon that rises vertically at a constant speed. The horizontal distance between the person and the balloon is constant at 100 ft. At the first observation, the person measures an angle of observation of 30°, 10 seconds later an angle of 35°. How fast is the balloon rising in feet per second? Round to one decimal place and make sure your calculator is in degree mode.



first: $\tan 30° = \frac{h_1}{100}$

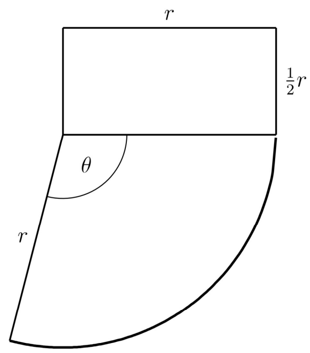
$h_1 = 57.7$ ft

second: $\tan 35° = \frac{h_2}{100}$

$h_2 = 70$ ft

$$\boxed{\text{speed}: \frac{70-57.7}{10} = 1.2 \frac{\text{ft}}{\text{sec}}}$$

Problem 6. Design two enclosures for two different types of animals that need to be kept apart as illustrated in the sketch. The enclosures consist of a rectangle and a circle sector. The sector radius must be equal to the length of the rectangle and the width of the rectangle is half the length. Moreover, you want to use exactly 480 meters of fencing. How do you have to choose r and θ to obtain a maximal area for the enclosure? **Don't forget units in your final answer.**



To maximize:

$$A = \frac{1}{2}\cdot r\cdot r + \frac{1}{2}\theta r^2 = \frac{1}{2}r^2 + \frac{1}{2}\theta r^2$$

Given:

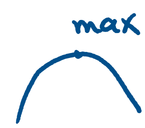
$$480 = r + r + r + 2\cdot\frac{1}{2}r + r\cdot\theta$$

$$480 = 4r + r\theta \rightarrow \theta = \frac{480-4r}{r} = \frac{480}{r} - 4$$

Substitute: $A = \frac{1}{2}r^2 + \frac{1}{2}r^2\cdot\left(\frac{480}{r} - 4\right)$

$$A = \frac{1}{2}r^2 - 2r^2 + 240r$$

$$A = -\frac{3}{2}r^2 + 240r$$



max @ vertex

$$\boxed{\begin{array}{l} r = \frac{-240}{-3} = 80 \text{ meters} \\ \theta = \frac{480-4\cdot 80}{80} = \frac{160}{80} = 2 \text{ rad} \end{array}}$$