Math 583B **Research in Geometric Analysis** **Spring 2005**

SOME NOTES ON WRITING MATHEMATICS

Here are some observations about the writing of mathematics that I hope will be useful as you work on the writing assignment for this course.

Goals and audience: As with any written piece, mathematical exposition must be written with a particular audience and specific goals in mind. Be sure you have a clear sense of what these are before you start writing.

The process: It is important to bear in mind that writing is a *process*, just like proving a theorem. No one pours forth a well-organized, clear, and error-free exposition the first time they sit down to write, just as no one produces a complete, well-structured proof the first time they think about a problem. Most good expository prose has been thoroughly rewritten at least once or twice before it reaches the reader, with key sections undergoing perhaps three to five major revisions. To some people, this thought makes the prospect of writing seem daunting or even overwhelming, but it needn't: The idea that much of what you write will eventually be replaced or discarded can be liberating. Just sit down and write, knowing that anything that doesn't measure up can later be fixed. You might well find yourself throwing away the first several pages you write—this is not wasted time, since the trial and error process helps you immensely in clarifying what you really want to say.

When you begin writing a draft, the introduction may not be the best place to start, since the structure of the paper may not become completely clear until later in the process. Try starting somewhere in the middle, with whichever part of the paper is clearest in your mind. As soon as you have a section or more in relatively coherent form, sit back and read it. Put yourself in the mind of your audience, and see if it makes complete sense. Then rewrite.

When you have something you think is close to acceptable, give it to someone else to read and comment on. Then rewrite again.

After you think the paper is finished, go through it with a fine-toothed comb and a sharp razor. Sharpen your definitions, statements of theorems, and proofs. Clarify your logic and your intuitive descriptions. Make sure your spelling, punctuation, and grammar are absolutely correct. Omit needless words, terminology, and symbols.

Note that "rewriting" usually means much more than simply correcting errors. It means looking critically at what you've written both locally and globally, figuring out what works well and what doesn't, and doing whatever is necessary to make the whole thing work perfectly.

Conventions: Although you might not believe it after reading some of the mathematical writing that has made it into print, mathematical writing should follow the same conventions of grammar, usage, punctuation, and spelling as any other writing. This means, in particular, that you must write complete sentences organized into paragraphs. While many mathematical terms have technical meanings that are different from their usage in ordinary English, you should still be careful to observe the usual rules regarding parts of speech and subject-verb agreement. Although you will run across (all too many) mathematicians who write ungrammatical sentences like "Suppose f is an onto map," *don't you do it!*

If you are not a native English speaker, it would be a good idea to cultivate the habit of asking a native speaker to look over your writing before you submit it.

Precision: In mathematical writing more than any other kind, precision is of paramount importance. Every mathematical statement you make must have a precise mathematical meaning. Specifically, every term you use must be well defined, and used properly according to its definition; every mathematical conclusion you reach must be justified; and every symbol you mention must be either previously defined or quantified in some appropriate way. If you write $f(a) > 0$, do you mean that this is true for every $a \in X$, or that there exists some $a \in X$ for which it's true, or that it's true for a particular a that you introduced earlier in the argument?

Ask yourself these two key questions about each mathematical sentence you write:

- *What exactly does this mean?*
- *Why exactly is this true?*

You don't necessarily have to include the entire answer to the second question in your paper, but make sure you let the reader know where the answer can be found.

Clarity: Just as important as mathematical precision is making sure your writing is easily comprehensible to your intended audience. Don't be stingy with *motivation*—intuitive explanations of what the result means, what led up to it historically, why it's interesting, why we should expect it to be true, what are the main ideas in the proof, why the proof is done this way and not some other way.

In addition to providing motivation, you should think carefully about how to make your proofs themselves easily understandable. The point of writing a mathematical proof is to convince your reader that something is true and why. This is not the same as writing a formal-logic proof that could be checked by a computer. It's all too easy to write a sequence of mathematical statements that are entirely precise and mathematically correct, and yet that are incomprehensible to a human being. For example, if a substantial part of your argument consists of a series of equations, be sure to introduce them with a clear explanation of what is to follow, and intersperse them at carefully chosen places with some words about what you're doing and why, or reasons why one step follows from another.

The first person: Most authors avoid using the word "I" in mathematical writing. It is standard practice to use "we" whenever it can reasonably be interpreted as referring to "the writer and the reader." Thus: "We will prove the theorem by induction on n," and "Because f is injective, we see that $x_1 = x_2$." But if you're really referring only to yourself, it's better to go ahead and use "I": "I learned this technique from Richard Melrose."

Abbreviations: There are a host of abbreviations that we use frequently in informal mathematical communication: "s.t." or "$\ni$" (such that), "iff" (if and only if), "w.r.t." (with respect to), "WLOG" (without loss of generality), "$\therefore$" (therefore). These are indispensable for writing on the blackboard and taking notes, but should never be used in written mathematical exposition. The only exceptions are abbreviations that would be acceptable in any formal writing, such as "i.e." (that is) or "e.g." (for example); but if you use these, be sure you know the difference between them!

Mathematical symbols: The feature that most clearly distinguishes mathematical writing from other kinds is the extensive use of symbols and formulas. If used appropriately, these are indispensable for communicating mathematical ideas precisely, and you should have no hesitation about employing them if they are carefully chosen, clearly defined, and

correctly used. The sentence "Let f be the function whose value at a particular number is equal to the square of that number added to the number itself" is far less clear than "Let f be the function defined by $f(x) = x^2 + x$." On the other hand, excessive use of symbols can lead to writing that is just as obscure.

Here are some guidelines for using mathematical symbols in your writing:

- Keep the number of named symbols to the minimum necessary for clarity. Ideally, each symbol should refer to an object whose role in the paper, or in a particular section of the paper, is important enough that you want the reader to "remember it by name." This is especially true in statements of theorems. Which of the following statements is clearer?

  Theorem 1. *If G is any Lie group, there exists a Lie group $\widetilde{G}$ that is the universal covering group of G.*

  Theorem 1′. *Every Lie group has a universal covering group.*

  Of course, even if you use the second version, in the proof of the theorem you will probably want to introduce symbols such as G and $\widetilde{G}$ to refer to the given group and its universal cover.
- Single symbols and short, simple formulas should be included right in your paragraphs; but a formula that is large or especially important should be centered on a line by itself (this is called a "displayed formula").
- Every mathematical symbol or formula, whether included in the text or displayed, must have a definite grammatical function in a sentence, usually as a noun or a clause. Consider the following sentence:

  If $x > 2$, we see that $x^2 + x$ must be greater than 6.

  Here the formula "$x > 2$" is a clause whose verb is ">," while "$x^2 + x$" functions as a noun.
- If a displayed formula ends a sentence, it must be followed by a period.
- The best way to ensure that your formulas function grammatically correctly is to read each sentence aloud. When you do so, bear in mind that many symbols can be read in several different ways—for example, the symbol "=" can be read as "equals," "equal to," "be equal to," or "is equal to," depending on context.
- Symbols representing mathematical relations (like $=$, $>$, or $\in$) or operators (like $+$, $-$, or $\cap$) should be used only to connect other mathematical symbols, not words. For example, do *not* write

  Let v be a vector of length < 1 that is $\in T_xM$.

  Instead, this sentence could be rewritten as follows:

  Let $v \in T_xM$ be a vector such that $|v| < 1$.
- Fractions and fractional expressions included in the text should be written with a slash, as in "$x/(y + 2)$". If a fraction is so large or complicated that it needs to be written using a horizontal bar, it should be displayed. The only exceptions are small numerical fractions such as $\frac{1}{2}$, which can be included in text as long as they are typeset small enough to fit naturally on a line.
- Do not begin a sentence with a symbol. This might take a bit of rewriting: For example, if you find yourself wanting to begin a sentence with "f is a continuous function," you could write instead "The function f is continuous."

- The symbols of symbolic logic, such as $\exists$, $\forall$, $\wedge$, $\vee$, $\neg$, and $\Rightarrow$, should never be used in formal mathematical writing, unless you are writing about symbolic logic and they appear in logical formulas. Otherwise, write out the words instead.

Citing sources: When you write a mathematics paper, you must list in your bibliography every published source from which you obtain ideas, mathematical results, proofs, facts, or specific language. Whenever you write something that you obtained from such a source, you must refer specifically to the source in the text. If you use a large amount of material from one source, as you might do in an expository paper, it's permissible to write something like "all of the results in this section are from [XYZ]," but it's not enough just to list a paper or book in your bibliography. Here are some examples of situations that require citations.

- *Any mathematical result that was first proved by someone other than you.* If you give a proof that is due to someone else, give a reference for it. If you state or use a result that you do not prove, give a reference where the proof can be found. If you give a new proof of an old result, give a reference where the original proof can be found. In general, for results proved in the last 50 years or so, you should refer to the paper in which the original proof was given. For older results, for which the original sources may no longer be easily available, you can refer to any source that gives a proof. There are some exceptions, however: You don't need to give references for very trivial results that the intended reader can easily check for him/herself, or results that are well known to your intended audience.
- *Any important idea that helped you in a central way to work out a proof, if you obtained it from one of your sources.* This is a often a matter of judgment: None of us works in a vacuum, so all of our ideas are influenced in some measure by others. But if you got an idea from someone else's published work that was indispensable to you in working out a proof, you should give credit. This occasionally applies also to ideas that you get in private conversations or correspondence. In such cases, it's common to create a silly bibliographic entry with a reference like "private communication," but I've come to believe that it's better just to give appropriate attribution in the text.
- *Any direct quotation of more than a few words.* A direct quotation must be enclosed in quote marks or, if it's long, indented. In mathematical writing, quotations are rarely appropriate. They should generally be used only when the author's actual choice of words is significant or quaint; for mathematical ideas, it is vastly preferable to digest the ideas and express them in your own words.
- *Any paraphrase of someone else's writing.* Paraphrasing means restating in your own words what someone else said. If you use a substantial portion of someone else's language but change a word here and there, it is not a paraphrase but a quotation. Aim to write in your own voice.
- *Any non-mathematical facts that you obtained from one of your sources.* For example, historical facts or information about applications of mathematical ideas.

If you use information from a published source without referring to it, or if you use someone else's language without quoting or indenting (even if you name the source), you are plagiarizing. *Don't do it!*

The bibliography: The conventions for bibliographic references in mathematical writing are somewhat different from those in other fields. A sample bibliography is shown below; the first entry is for a book, the second is for an unpublished preprint in an online database, the third is for an article in a book, and the fourth is for a journal article.

References

[1] Artuhur L. Besse, *Einstein Manifolds*, Springer-Verlag, New York, 1987.

[2] Xianzhe Dai, Xiaodong Wang, and Guofang Wei, *On the stability of Riemannian manifold with parallel spinors*, preprint (2003), www.arxiv.org/math.DG/0311253.

[3] David G. Ebin, *The space of Riemannian metrics*, Global Analysis (Proc. Symp. Pure Math. Vol. 15), pp. 11–40, Amer. Math. Soc., Providence, 1968.

[4] Norihito Koiso, *Rigidity and infinitesimal deformability of Einstein metrics*, Osaka J. Math. **19** (1982), no. 3, 643–668.

Minor variations on these styles are acceptable, but most mathematical publishers use something very close to this. For a journal article, you need only give the author(s), article title, journal name (using standard abbreviations that you can find on MathSciNet), volume number (usually shown in boldface as above), the year of publication, the issue number (optional), and the page numbers. List your references in alphabetical order by the last name of the first author. You can either number your bibliographic entries as above or refer to them by "keys" consisting of abbreviations of the authors' last names. LaTeX has macros that will help you to create bibliographic entries and keep track of their citation numbers or keys. If you know how to use BibTeX, it will help you to arrange and format your bibliographic entries correctly.

In mathematical writing, footnotes and endnotes are never used for bibliographic references; instead, you refer to an entry in your bibliography by giving the key or reference number in square brackets. (In fact, footnotes should generally be avoided altogether in mathematical writing, with the exception only of a footnote on the first page giving administrative information, such as acknowledging NSF grant support, or giving keywords or subject classification codes for Math Reviews. It is much better to incorporate parenthetical remarks directly into the text.) Here is an example of how the references above might be cited:

> A local slice theorem for the action of the diffeomorphism group on the space of Riemannian metrics was proved in 1968 by D. Ebin [3]. Later, N. Koiso [4] used Ebin's construction to analyze the local structure of the space of Einstein metrics on a compact manifold, and in particular to show that it is finite-dimensional. The classic book [1, pp. 345–352] gives a good expository account. See also [2] for a recent related result.

When you refer to a book, give page numbers in the brackets, as shown above. When you refer to an article, you generally don't need to give page numbers, unless the article is unusually long or you think it might be difficult for the reader to find the fact you are referring to.

Further reading: The booklets below are available in either the Reference or the Reserve section of the Math Research Library. It would be a good idea to look at them before you start writing.

[1] J. R. Dieudonné, P. R. Halmos, M. M. Schiffer, and N. E. Steenrod, *How to write mathematics*, American Mathematical Society, Providence, 1981. Look especially at the essay by Halmos, which is a classic.
[2] L. Gillman, *Writing Mathematics Well*, Mathematical Association of America, 1987.
[3] D. E. Knuth, T. Larrabee, and P. M. Roberts, *Mathematical Writing*, Mathematical Association of America, Washington, 1989. Look especially at pages 1-8.