Math 549 **Complex Manifolds** **Spring 2004**

ASSIGNMENT #4 (CORRECTED AGAIN, 6/4/04)
DUE 6/7/04

1. Let $E \to M$ be a smooth complex vector bundle, and let $\overline{E}$ be the complex vector bundle whose fiber $\overline{E}_x$ at each point $x \in M$ is equal to E_x, but with complex multiplication defined by $(a, v) \mapsto \overline{a}v$. Show that $\overline{E}$ is isomorphic to E^* but not necessarily to E.

2. Let M be a complex manifold, and let $\pi\colon E \to M$ be a smooth complex vector bundle. A *Cauchy-Riemann operator* on E is a $\mathbb{C}$-linear map $\overline{\partial}\colon \Gamma(E) \to \Gamma(\Lambda^{0,1}M \otimes E)$ satisfying

   (i) $\overline{\partial}(f\sigma) = (\overline{\partial}f) \otimes \sigma + f\overline{\partial}\sigma$ for all smooth complex-valued functions f.

   (ii) $\overline{Z}(\overline{W}\sigma) - \overline{W}(\overline{Z}\sigma) = [\overline{Z}, \overline{W}]\sigma$ for all $\overline{Z}, \overline{W} \in T''M$.

   (In part (ii), we define $\overline{Z}\sigma$ as in Problem 8 on Assignment 3. It follows from that problem that every holomorphic vector bundle admits a Cauchy-Riemann operator.) If E is endowed with a Cauchy-Riemann operator, show that E has a unique structure as a holomorphic vector bundle such that the holomorphic sections of E are exactly those in the kernel of $\overline{\partial}$. [Hint: If (e_k) is a smooth local frame for E over $U \subset M$, show that the $(0,1)$-forms θ_k^j on U defined by $\overline{\partial}e_k = \theta_k^j \otimes e_j$ satisfy $\overline{\partial}\theta_k^j + \theta_l^j \wedge \theta_k^l = 0$. Let (z^j) be local holomorphic coordinates for U and let (z^j, b^k) be the (complex-valued) coordinates on $\pi^{-1}(U) \subset E$ defined by the local frame (e_k), via the correspondence $(z^j, b^k) \leftrightarrow b^k e_k|_z$. Show that there is a unique integrable complex structure on the total space of E such that $\Lambda^{1,0}E$ is locally spanned by $(\pi^* dz^j, db^j + b^k \pi^* \theta_k^j)$, and apply the Newlander-Nirenberg theorem.]

3. Let Σ be a Riemann surface and let g be a Kähler metric on Σ. If z is any local holomorphic coordinate on Σ, show that the holomorphic sectional curvature of g is equal to its Gaussian curvature, and both are equal to

   $$-\frac{1}{u}\frac{\partial^2}{\partial z \partial \overline{z}} \log u,$$

   where $u = g_{\mathbb{C}}(\partial/\partial z, \partial/\partial \overline{z})$. Use this formula to compute the Gaussian curvatures of the 1-dimensional Fubini-Study and complex hyperbolic metrics.

4. Let $Q \subset \mathbb{CP}^2$ be the quadric curve defined by the homogeneous polynomial $z^1z^2 - (z^3)^2$. Compute the Gaussian curvature and the area of Q in the metric obtained by restricting the Fubini-Study metric to Q.

5. Let $E \to M$ be a smooth complex vector bundle of rank k. Show that $c_1^{\mathbb{R}}(E) = c_1^{\mathbb{R}}(\Lambda_k E)$, where $\Lambda_k E$ denotes the bundle of antisymmetric contravariant k-tensors on E and $c_1^{\mathbb{R}}$ denotes the real first Chern class.

6. ****PROBLEM DELETED****

7. Let $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ be smooth complex vector bundles of rank k, and let $F\colon E \to E'$ be a smooth bundle map covering $f\colon M \to M'$. (Recall that this means $\pi' \circ F = f \circ \pi$, and for each $x \in M$, the map $F_x = F|_{E_x}\colon E_x \to E'_x$ is a linear isomorphism.)

   (a) If (e'_j) is a smooth frame for E' over an open set $U' \subset M'$, show that there is a unique smooth frame (e_j) for E over $f^{-1}(U')$ such that $F \circ e_j = e'_j \circ f$ for each j.

   (b) If ∇' is a connection on E', show that there is a unique connection ∇ on E, called the *pullback connection*, with the property that

   $$\nabla_X e_j = F_x^{-1} \nabla'_{f_* X}(e'_j)$$

   whenever the frames (e_j) and (e'_j) are related as in part (a).

   (c) For each $j = 1, \ldots, k$, show that $c_j^{\mathbb{R}}(E) = f^* c_j^{\mathbb{R}}(E')$.

8. (a) Show that $\mathrm{U}(n+1)$ acts transitively on $\mathbb{CP}^n$ by projective transformations.

   (b) Show that the Fubini-Study metric is $\mathrm{U}(n+1)$-invariant, and is, up to a constant multiple, the unique $\mathrm{U}(n+1)$-invariant metric on $\mathbb{CP}^n$.

9. (a) Let $\mathrm{U}(n,1)$ be the subgroup of $\mathrm{GL}(n+1,\mathbb{C})$ leaving invariant the following hermitian bilinear form:

   $$H = dz^1 \otimes \overline{dz^1} + \cdots + dz^n \otimes \overline{dz^n} - dz^{n+1} \otimes \overline{dz^{n+1}}.$$

   Considering the unit ball $\mathbb{B}^{2n} \subset \mathbb{C}^n \subset \mathbb{CP}^n$ as a subset of projective space, show that $\mathrm{U}(n,1)$ acts transitively on $\mathbb{B}^{2n}$ by projective transformations.

   (b) Let g be the complex hyperbolic metric on $\mathbb{B}^{2n}$, defined by the Kähler form $\omega = \frac{i}{2}\partial\overline{\partial}\log(|z|^2 - 1)$. Show that g is, up to a constant multiple, the unique $\mathrm{U}(n,1)$-invariant metric on $\mathbb{B}^{2n}$.

   (c) Show that g has constant holomorphic sectional curvature equal to -4.

10. Let M be a complex manifold of dimension n, and let g be a Kähler metric on M with constant holomorphic sectional curvature C.

    (a) Let $X, Y \in T_x M$ be a pair of orthonormal vectors. Show that the (ordinary) sectional curvature of g in the direction of the plane spanned by (X,Y) is given by

    $$K(X,Y) = \tfrac{1}{4} C \left(1 + 3\langle X, JY\rangle^2\right).$$

    (b) If $n \geq 2$, show that at each point of M, the (ordinary) sectional curvatures of g take on all values between $\frac{1}{4}C$ and C, inclusive.