Math 549 **Complex Manifolds** **Spring 2004**

Assignment #2

Due 5/10/04

1. Let M be a smooth manifold and let J be an almost complex structure on M. Define $N\colon \mathscr{T}(M) \times \mathscr{T}(M) \to \mathscr{T}(M)$ by

$$N(X,Y) = J[X,JY] + J[JX,Y] + [X,Y] - [JX,JY].$$

   (a) Show that N is bilinear over $C^\infty(M)$, and therefore defines a $\binom{2}{1}$-tensor field on M.

   (b) Show that J is integrable if and only if $N \equiv 0$.

2. *An almost complex structure on* $\mathbb{S}^6$*:* Let $\mathbb{O}$ denote the algebra of *octonions* (see [ITM, Problem 8-21]). For $P, Q \in \mathbb{O}$, define $P^* = (p^*, -q)$ where $P = (p,q) \in \mathbb{O} = \mathbb{H} \times \mathbb{H}$. Let $\mathbb{R} = \{P \in \mathbb{O} : P^* = P\}$ denote the set of real octonions, identified with the real numbers in the natural way, and $\mathbb{E} = \{P \in \mathbb{O} : P^* = -P\}$ the set of imaginary octonions. We can define an inner product on $\mathbb{O}$ by $\langle P, Q\rangle = \frac{1}{2}(P^*Q + Q^*P)$. Let $\mathbb{S} = \{P \in \mathbb{E} : |P| = 1\}$ be the unit sphere in $\mathbb{E}$, and for each $P \in \mathbb{S}$, define a map $J_P\colon T_P\mathbb{S} \to \mathbb{O}$ by $J_P(Q) = QP$, where we identify $T_P\mathbb{S}$ with the subspace $P^\perp \cap \mathbb{E} \subset \mathbb{O}$.

   (a) Show that J_P maps $T_P\mathbb{S}$ to itself, and defines an almost complex structure on $\mathbb{S}$.

   (b) Show that this almost complex structure is not integrable.

3. Let M be a complex manifold. A *meromorphic function* on M is a function $f\colon M \backslash V \to \mathbb{C}$, where $V \subset M$ is an analytic hypersurface (not necessarily smooth), such that in a neighborhood U of each point f can be written $f|_U = g/h$, where $g, h \in \mathscr{O}(U)$ with $h^{-1}(0) = V \cap U$. The set V is called the *polar divisor* of f, and the closure in M of the set $f^{-1}(0)$ is called the *zero divisor* of f. Suppose V, V' are smooth analytic hypersurfaces in M. Show that the line bundles L_V and $L_{V'}$ are isomorphic if and only if there exists a meromorphic function on M whose polar divisor is V and whose zero divisor is V'.

4. Prove the *Local* $\partial\overline{\partial}$*-Lemma:* Suppose ω is a smooth, real, closed $(1,1)$-form on a complex manifold M. Then in a neighborhood of each point of M, there exists a real-valued smooth function u such that $\omega = i\partial\overline{\partial}u$.

5. Let $H \to \mathbb{CP}^n$ denote the hyperplane bundle. For $k \neq l \in \mathbb{Z}$, show that H^k is not isomorphic to H^l.

6. Let $K \to \mathbb{CP}^n$ denote the canonical bundle of $\mathbb{CP}^n$ (i.e., the bundle of $(n,0)$-forms). Show that $K \cong H^{-(n+1)}$.

7. Show that $T'\mathbb{CP}^1 \cong H^2$.

8. Let M be a complex manifold. A *holomorphic vector field* on M is a holomorphic section of $T'M$. Let Z be a smooth section of $T'M$ and let θ_t denote the flow of $\operatorname{Re} Z$. Show that Z is holomorphic if and only if θ_t is a holomorphic map (where it's defined) for each t.

9. Let M be a complex manifold, and let $\mathscr{C}$ and $\mathscr{O}$ denote the sheaves of continuous and holomorphic functions on M, respectively. Show that $\mathscr{O}$ is Hausdorff but $\mathscr{C}$ is not.

10. Let M be a complex manifold, let $\mathscr{O}$ be its sheaf of holomorphic functions, and let W be a connected component of $\mathscr{O}$. Show that W has a unique complex manifold structure such that $\pi|_W$ is a local biholomorphism.