Math 548 **Geometric Structures** **Winter 2008**

Assignment #3 (CORRECTED)

Due March 19, 2008

For full credit, do any seven of the following problems.

1. Suppose M is a smooth manifold and $E \to M$ is a smooth (real or complex) vector bundle. Prove that there is a vector bundle $E' \to M$ such that $E \oplus E'$ is trivial.

2. Determine classifying spaces for the cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/\langle n \rangle$.

3. Suppose $G \to V \to B$ and $G' \to V' \to B'$ are principal bundles.

   (a) Show that the Cartesian product bundle $G \times G' \to V \times V' \to B \times B'$ is a principal $G \times G'$-bundle.

   (b) If both V and V' are universal, show that the Cartesian product bundle $V \times V'$ is also universal.

   (c) Using the fact that every finitely generated abelian group is a direct sum of cyclic groups, determine a classifying space for each finitely generated abelian group.

4. Recall that a fiber bundle $V \to B$ is said to be ***n-universal*** if every bundle with the same group and fiber over a CW complex (or manifold) of dimension at most n is isomorphic to a pullback of V. I showed in class that $\mathbb{RP}^2$ is 1-universal for real line bundles, and $\mathbb{CP}^1$ is 2-universal for complex line bundles. Given a bundle $E \to M$ over a manifold or CW complex of dimension at most n, we extend the notion of ***classifying map for E*** to include a map from M into the base of an n-universal bundle V that pulls V back to E.

   (a) Show that the map $f\colon \mathbb{S}^1 \to \mathbb{RP}^2$ given by

   $$f(e^{i\theta}) = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, 0\right]$$

   is a classifying map for the Möbius bundle. (Be sure to verify that it is well-defined and continuous.)

   (b) Let $U \to \mathbb{CP}^1$ denote the tautological complex line bundle over $\mathbb{CP}^1$. For $k > 0$, show that the map $p_k\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ given by

   $$p_k[z, w] = [z^k, w^k]$$

   is a classifying map for $U^k = U \otimes \ldots \otimes U$; and that

   $$p_{-k}[z, w] = [\bar{z}^k, \bar{w}^k]$$

   is a classifying map for $\overline{U}^k$.

5. Given a topological space X and a basepoint $x_0 \in X$, define $\pi_0(X, x_0)$ to be the set of path components of X, considered as a pointed set with the path component containing x_0 as its distinguished point. (In general, this set does not have a group structure.)

   (a) Given a fiber bundle $F \to E \to B$, show that there is a map $\partial\colon \pi_1(B, b_0) \to \pi_0(F, f_0)$ such that the homotopy sequence of the bundle extends to an exact sequence

   $$\ldots \to \pi_1(E, e_0) \to \pi_1(B, b_0) \to \pi_0(F, f_0) \to \pi_0(E, e_0) \to \pi_0(B, b_0) \to \{0\},$$

   where exactness is interpreted to mean that the image of each map is equal to the preimage of the basepoint under the next map.

   (b) If $\pi\colon E \to B$ is a fiber bundle with path connected fiber, show that the induced homomorphism $\pi_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is surjective.

6. (a) If G is a topological group, show that $\pi_0(G, e)$ has a group structure such that the map $G \to \pi_0(G, e)$ sending each point to the path component containing it is a surjective homomorphism.
   (b) Suppose $G \to P \to B$ is a principal G-bundle, with the identity e chosen as a basepoint in G. Show that the connecting map $\partial\colon \pi_1(B, b_0) \to \pi_0(G, e)$ whose existence you proved in Problem 5 is a group homomorphism.
   (c) If G is a topological group and $H \subseteq G$ is a closed subgroup, show that the induced map $\pi_0(H, e) \to \pi_0(G, e)$ is a group homomorphism.
   (d) If G is a Lie group and $H \subseteq G$ is a closed normal subgroup, show that every map in the extended homotopy sequence of the principal bundle $H \to G \to G/H$ is a group homomorphism.

7. Let $\mathscr{F} \to M$ be a sheaf, and let $\mathscr{U}, \mathscr{V}$ be open covers of M such that $\mathscr{V}$ refines $\mathscr{U}$. For any refining map $\rho\colon \mathscr{V} \to \mathscr{U}$, define the induced cochain map $\rho^{\#}\colon \check{C}^k(\mathscr{U}; \mathscr{F}) \to \check{C}^k(\mathscr{V}; \mathscr{F})$ by
$$(\rho^{\#}\gamma)(V_0, \dots, V_k) = \gamma(\rho V_0, \dots, \rho V_k).$$
   (a) Prove that $\rho^{\#} \circ \delta = \delta \circ \rho^{\#}$. Thus we can define an induced cohomology map $\rho^*\colon \check{H}^k(\mathscr{U}; \mathscr{F}) \to \check{H}^k(\mathscr{V}; \mathscr{F})$ by $\rho^*[\gamma] = [\rho^{\#}\gamma]$.
   (b) Complete the proof that ρ^* depends only on the covers $\mathscr{U}$ and $\mathscr{V}$, and not on the refining map ρ, as follows. Given two refining maps $\rho, \widetilde{\rho}\colon \mathscr{U} \to \mathscr{V}$, define a map $\theta\colon \check{C}^k(\mathscr{U}; \mathscr{F}) \to \check{C}^{k-1}(\mathscr{V}; \mathscr{F})$ by
$$(\theta c)(V_0, \dots, V_{k-1}) = \sum_{i=0}^{k-1} (-1)^i c(\rho V_0, \dots, \rho V_i, \widetilde{\rho} V_i, \dots, \widetilde{\rho} V_{k-1})$$
   and show that $\theta \circ \delta + \delta \circ \theta = \widetilde{\rho}^{\#} - \rho^{\#}$.
   (c) For any sheaf homomorphism $\varphi\colon \mathscr{E} \to \mathscr{F}$, show that $\varphi_* \circ \rho^* = \rho^* \circ \varphi_*$.

8. Suppose $\mathscr{F} \to M$ is a sheaf. If $\mathscr{U}$ and $\mathscr{V}$ are open covers of M such that $\mathscr{V}$ refines $\mathscr{U}$, show that the induced map $\rho^*_{\mathscr{U}\mathscr{V}}\colon \check{H}^1(\mathscr{U}; \mathscr{F}) \to \check{H}^1(\mathscr{V}; \mathscr{F})$ is injective. Conclude that $\check{H}^1(\mathscr{U}; \mathscr{F})$ injects into $\check{H}^1(M; \mathscr{F})$ for every cover $\mathscr{U}$.

9. Let M be a smooth manifold, $E \to M$ a smooth vector bundle, and $\mathscr{E}$ the sheaf of germs of smooth sections of E. For any open subset $U \subseteq M$, we have used the notation $\mathscr{E}(U)$ to denote both the space of continuous sections of $\mathscr{E}$ over U and the space of smooth sections of E over U.
   (a) Show that these spaces are isomorphic $C^\infty(M)$-modules via the map that sends a smooth section $f : U \to E$ to the section $\sigma_f\colon U \to \mathscr{E}$ defined by $\sigma_f(x) = [f]_x$.
   (b) Let f be a smooth section of E over an open set $U \subseteq M$. Show that the set of points where σ_f is nonzero is closed in M, while the set of points where f is nonzero is open in M. If M is connected, does this imply σ_f is constant if it vanishes somewhere? Explain.

10. Suppose $\mathscr{R}$ is a ring, and $\{A_j\}_{j\in J}$, $\{B_j\}_{j\in J}$, and $\{C_j\}_{j\in J}$ are directed systems of $\mathscr{R}$-modules. Suppose also that for each $j \in J$, we are given an exact sequence of module homomorphisms
$$A_j \xrightarrow{\alpha_j} B_j \xrightarrow{\beta_j} C_j$$
such that, whenever $j, k \in J$ satisfy $j \leq k$, the following diagram commutes:
$$\begin{array}{ccccc} A_j & \xrightarrow{\alpha_j} & B_j & \xrightarrow{\beta_j} & C_j \\ \downarrow & & \downarrow & & \downarrow \\ A_k & \xrightarrow[\alpha_k]{} & B_k & \xrightarrow[\beta_k]{} & C_k, \end{array}$$
where the vertical maps are the ones associated with the three directed systems of modules. (Such a system is called a ***directed system of exact sequences***.) Show that there are module homomorphisms α and β such that the following sequence is exact.
$$\varinjlim A_j \xrightarrow{\alpha} \varinjlim B_j \xrightarrow{\beta} \varinjlim C_j.$$