Math 444/445 **Geometry for Teachers** **Summer 2008**

Handout #3: Conventions for Writing Mathematical Proofs

Here are some more-or-less standard conventions for mathematical proof-writing.

- **Write in paragraph form:** First and foremost, remember always that a mathematical proof is designed to communicate the truth of a mathematical statement, and the correctness of your argument, to a *human reader*. There is an overwhelming consensus that ordinary prose is much better suited to this purpose than formal symbolic statements. Although you might initially construct your proof as a sequence of terse symbolic statements, when you write it up you should use complete sentences organized into paragraphs. As you read more and more complicated proofs, you will find that paragraph-style proofs are much easier to read and comprehend than symbolic ones or the two-column proofs of high school geometry.

- **Use proper English:** All mathematical writing should follow the same conventions of grammar, usage, punctuation, and spelling as any other writing. In addition to writing complete sentences organized into paragraphs, you must use correct punctuation (including a period at the end of every sentence), avoid run-on sentences, pay attention to subject-verb agreement and parallel structure, and use correct spelling and capitalization.

- **Identify your audience:** Before you begin writing any proof, be sure you're aware who your audience is and what they already know. For this course, you should always write your proofs as if you were trying to convince a fellow student in this class of the truth of the theorem and the correctness of your argument. You may assume the reader knows the same background material as you do, but doesn't know the proof of this particular theorem.

- **Include the right amount of detail:** A clear awareness of your audience will help you to answer the perennial question, "How much detail do I need to include?" The first thing that must be said is this: *If you think you probably know roughly how an argument would go but it seems too tedious to work through in detail, then you need to work through it!* It's only after you know exactly what's involved in writing out the details that you can make a good judgment about whether those details need to be included in the proof or not. If you're sure that it would be obvious to your audience how to fill in the omitted details, then the proof might be clearer if you leave them out. But if they weren't obvious to you at first, then something probably needs to be said – it might not be necessary to write down every step, but you should include just enough to give the reader the "Aha!" experience that makes the rest obvious (and, just as importantly, makes it clear to the grader that you've figured out the details yourself!). Deciding how much detail to include is one of the most subtle and difficult aspects of writing, and one where experience and artistry are most evident.

- **State what you're proving and label your theorems clearly:** Suppose you are assigned the following problem:

  If x is a real number, show that $x^2 \geq 0$.

  When you write up your solution, it isn't necessary to copy out the problem statement verbatim. Instead, it's much better to state the theorem you're proving, as in:

  Theorem: *If x is a real number, then $x^2 \geq 0$.*

  Each theorem you prove should be clearly labeled as such, and stated clearly and precisely in one or more English sentences. With computer typesetting programs like TeX, the usual convention is to set the word "Theorem" in boldface, with the statement of the theorem itself italicized. In handwritten proofs, just underline the word "Theorem."

In some contexts, the word Theorem might be replaced by Proposition, Corollary, or Lemma. Logically, these all mean the same thing (a mathematical statement to be proved from assumptions and previously proved results), but your choice of label can alert the reader about the role that the result plays in the current context. A *proposition* is a result that is interesting in its own right, but not as important as a theorem; a *lemma* is a result that might not be interesting in itself, but is useful for proving another theorem; and a *corollary* is a result that follows easily from a previously proved theorem.

- **Show where your proofs begin and end.** Each proof should begin with the word *Proof*, and end either with the letters QED (*quod erat demonstrandum*, Latin for "that which was to be proved") or with a symbol such as the square at the end of this paragraph. □

- **Write with precision:** In mathematical writing more than any other kind, precision is of paramount importance. For each step of a mathematical proof, and for every claim you write, ask yourself these two key questions:

  – *What does it mean?*
  Every mathematical statement you make must have a precise mathematical meaning. In particular, every term you use must be well defined, and used properly according to its definition; and every symbol you mention must either be previously defined or quantified in some appropriate way. If you write $f(a) > 0$, do you mean that this is true for every $a \in \mathbb{R}$, or that there exists some $a \in \mathbb{R}$ for which it's true, or that it's true for a particular a that you introduced earlier in the proof?

  – *Why is it true?*
  Every mathematical conclusion you reach must be justified in one or more of the following six ways: By an axiom; by a previously proved theorem; by a definition; by hypothesis; by a previous step in the current proof; or by the rules of logic.

- **Write clearly:** Just as important as mathematical precision is making sure your writing is clear enough to be easily comprehensible to your intended audience. Don't be stingy with intuitive explanations of what's going on and why. For any argument that's longer than a few sentences, it's good to begin by describing informally what you're going to do and why this is a sensible approach, then do it, then say what you've done. If the structure of your proof is anything other than a simple direct proof, state at the beginning what type of proof you're using. ("We will prove the contrapositive," or "We will prove this by contradiction.")

  It's all too easy to write a sequence of mathematical statements that are entirely precise and mathematically correct, and yet that are incomprehensible to a human being. If you have to write a long series of formulas, intersperse them at carefully chosen places with some words about of what you're doing and why, or reasons why one step follows from another.

- **Distinguish formal vs. informal mathematical writing:** Many proofs include both formal and informal parts. The *formal* part lays out the precise mathematical definitions and describes the logical steps of the proof. The *informal* part might describe the motivation for why the theorem should be true, or the intuition behind the proof, or a brief sketch of how the proof will go. Be sure it is easy for the reader to distinguish which parts are formal and which are informal.

- **Use the first person singular sparingly:** Most authors avoid using the word "I" in mathematical writing. It is standard practice to use "we" whenever it can reasonably be interpreted as referring to "the writer and the reader." Thus: "We will prove the theorem by induction on n," and "Because f is injective, we see that $x_1 = x_2$." But if you're really referring only to yourself, it's better to go ahead and use "I": "I learned this technique from Richard Melrose."

- **Avoid most abbreviations:** There are a host of abbreviations that we use frequently in informal mathematical communication: "s.t." (such that), "w.r.t." (with respect to), and "w.l.o.g." (without loss of generality) are some of the most common. These are indispensable for writing on the blackboard and taking notes, but should *never* be used in written mathematical exposition. The only exceptions are abbreviations that would be acceptable in any formal writing, such as “i.e.” (that is) or “e.g.” (for example); but if you use these, be sure you know the difference between them!

One abbreviation that deserves special mention is “iff” (if and only if). Some mathematical writers use this routinely, even in quite formal writing. But my opinion is that, like the other abbreviations mentioned above, it actually acts as a hindrance to understanding in formal writing, and should be left on the blackboard and in your notes.

- **Proofread:** Be sure to read your proofs from beginning to end after you’ve finished writing them. You’ll be amazed how many silly mistakes you can catch that way.

Mathematical formulas

The feature that most clearly distinguishes mathematical writing from other kinds is the extensive use of symbols and formulas. Used appropriately, formulas are absolutely indispensable to clarity and ease of reading. The sentence “Let f be the function whose value at a particular number is equal to the square of the number added to the number itself” is far less clear than “Let f be the function defined by $f(x) = x^2+x$.” On the other hand, formulas must be used judiciously, because their excessive use can lead to writing that is just as obscure as writing without formulas.

Here are some guidelines for using mathematical symbols and formulas in your writing. In this handout, the use of the word “symbol” includes variable names such as x, y, P, Q, α, β; function names such as f, sin, log; as well as all the special mathematical symbols that we use to refer to operators and relations. The word “formula” refers to any expression built up out of one or more mathematical symbols.

- Single symbols and short, simple formulas should usually be included right in your paragraphs; these are called *in-line formulas*. But a formula that is large or especially important should be centered on a line by itself; this is called a *displayed formula*.

- Every mathematical symbol or formula, whether in-line or displayed, must have a definite grammatical function as *part of* a sentence; a formula cannot stand on its own as an entire sentence. Formulas should almost always have one of the following two grammatical functions: (1) An expression representing a particular mathematical object can be used as a noun; and (2) a complete symbolic mathematical statement can be used as a clause. For example, consider the following sentence:

  If $x > 2$, we see that $x^2 + x$ must be greater than 6.

  Here the formula “$x > 2$” is a mathematical statement functioning as a clause (whose verb is “$>$”), while “$x^2 + x$” and “6” are mathematical expressions (representing real numbers) that function as nouns.

  The best way to ensure that your formulas function grammatically correctly is to read each sentence aloud. When you do so, bear in mind that many symbols can be read in several different ways—for example, the symbol “=” can be read as “equals,” “equal to,” “be equal to,” or “is equal to,” depending on context.

- If a displayed formula ends a sentence, it must be followed by a period.

- That last one is easy to forget, so let me say it again with emphasis: *If a displayed formula ends a sentence, it must be followed by a period.* Similarly, if it would have required any other punctuation such as a comma or semicolon had it been written in-line, that punctuation must appear at the end of the displayed equation.

- Symbols representing mathematical relations (like $=$, $>$, $\in$, or $\subseteq$) or operators (like $+$, $-$, or $\cap$) should be used only to connect other mathematical symbols, not to connect words with symbols or with each other. For example, do *not* write:

  If x is a real number that is > 2, then $x^2 + x$ must be > 6. (BAD)

  Either of the following is much better:

  If x is a real number such that $x > 2$, then we must have $x^2 + x > 6$. (GOOD)
  If x is a real number that is greater than 2, then $x^2 + x$ must be greater than 6. (GOOD)

- Built-up expressions like summations, integrals, matrices, or fractions should be either displayed or written in such a way that they fit easily on a line without forcing extra spacing between lines. In particular, if a fraction or fractional expression is included in the text, it should be written with a slash, as in "$x/(y + 2)$". If a fraction is so large or complicated that it needs to be written using a horizontal bar, it should be displayed. The only common exception is small numerical fractions such as $\frac{1}{2}$, which can be included in text as long as they are written small enough to fit naturally on a line.

- It's bad form to begin a sentence with a mathematical symbol, because it makes it hard for the reader to recognize that a new sentence has begun. (You can't capitalize a symbol to indicate the beginning of a sentence!) It's usually easy to avoid this by minor rewording—for example, if you find yourself wanting to write a sentence that begins "f is a continuous function," you could write instead "The function f is continuous."

- Avoid writing two formulas separated only by a comma or other punctuation mark, because they will look like one long formula. For example, the sentence "If $x \neq 0$, $x^2 > 0$" can be confusing; it would be easier to read if a word were interposed between the two formulas, as in "If $x \neq 0$, then $x^2 > 0$."

- Symbols for logical terms, such as $\exists$ (there exists), $\forall$ (for all), $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\Rightarrow$ (implies), or $\Leftrightarrow$ (if and only if), should *never* be used to replace the corresponding words in an English sentence. The only time these symbols have any place in formal mathematical writing is as part of complete symbolic logic formulas. In fact, unless the subject you are writing about is mathematical logic, it is better to write out the statements in English.

  The only possible exceptions to the advice in the preceding sentence are the symbols $\Rightarrow$ and $\Leftrightarrow$, which are not uncommon in ordinary mathematical writing. But if you do use them, be sure to use them only to connect formulas or letters representing mathematical statements, not to connect English statements. Thus the first two statements below are acceptable, but the third is not:

  We will prove that (a) $\Leftrightarrow$ (b) by first showing that
  (a) $\Rightarrow$ (b) and then showing that (b) $\Rightarrow$ (a). (GOOD)

  Therefore, $x \neq 0 \Rightarrow x^2 > 0$. (GOOD)

  The fact that x is nonzero $\Rightarrow$ x^2 is positive. (BAD)