**Math 441** **Topology** **Autumn 2004**

# Sample problem writeup

## §13, Problem 4(a,b):

When confronted with a family of different topologies on the same set, one might wonder whether there is a topology that is contained in all of them, or that contains all of them. Of course, the trivial topology is contained in every other topology, and the discrete topology contains every other, but these are not very interesting. A much more compelling question is whether there is a "largest" topology that is contained in every topology in the given family, or a "smallest" topology that contains them all. The purpose of this note is to show that both of these do exist, and in fact are uniquely determined by the given collection of topologies.

If a topology $\mathcal{T}$ is contained in each of the topologies in the given collection, then it must necessarily be contained in their intersection. The most obvious candidate for a *largest* such topology, therefore, would be that intersection itself, provided it is a topology. Luckily, as the following lemma shows, that is always the case.

**Lemma 1.** *Let $X$ be a set. If $\{\mathcal{T}_\alpha\}_{\alpha\in J}$ is a family of topologies on $X$, then $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology on $X$.*

*Proof.* For brevity, let us denote the intersection $\bigcap_\alpha \mathcal{T}_\alpha$ by $\mathcal{T}$. Note that each $\mathcal{T}_\alpha$ is a collection of subsets of $X$, so $\mathcal{T}$ is itself a collection of subsets of $X$. We need to show that $\mathcal{T}$ satisfies the three defining properties of a topology.

First, because $\varnothing \in \mathcal{T}_\alpha$ and $X \in \mathcal{T}_\alpha$ for each $\alpha \in J$, it follows immediately that $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

Second, to show that $\mathcal{T}$ is closed under arbitrary unions, suppose $\{U_\beta\}_{\beta\in K}$ is an arbitrary collection of elements of $\mathcal{T}$. This means by definition that for each choice of $\alpha$, $U_\beta \in \mathcal{T}_\alpha$ for every $\beta$. Since each $\mathcal{T}_\alpha$ is closed under arbitrary unions, it follows that $\bigcup_\beta U_\beta \in \mathcal{T}_\alpha$. This is true for every $\alpha$, so $\bigcup_\beta U_\beta$ is an element of the intersection of all the sets $\mathcal{T}_\alpha$, which is $\mathcal{T}$.

Finally, we need to show that $\mathcal{T}$ is closed under finite intersections. The argument goes just as in the preceding paragraph: If $U_1, \dots, U_n$ are elements of $\mathcal{T}$, then each $U_i$ is in $\mathcal{T}_\alpha$ for every $\alpha$, and thus so is the intersection $U_1 \cap \cdots \cap U_n$ by virtue of the fact that each $\mathcal{T}_\alpha$ is a topology. □

Reasoning by analogy with the preceding lemma, one might be tempted to predict that the union of a family of topologies is again a topology, but that turns out to be false. To see why, consider two simple topologies on a 3-element set $X = \{a, b, c\}$:

$$\mathcal{T}_1 = \{\varnothing, X, \{a, b\}\},$$
$$\mathcal{T}_2 = \{\varnothing, X, \{a, c\}\}.$$

It straightforward to check that $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies. But now let $\mathcal{U}$ denote their union:

$$\mathcal{U} = \mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{a, b\}, \{a, c\}\}.$$

This is not a topology because it is not closed under finite intersections: The subsets $\{a, b\}$ and $\{a, c\}$ are elements of $\mathcal{U}$, but their intersection $\{a\}$ is not.

To correct this problem, we can "fill out" the topology by throwing in all unions of finite intersections of arbitrary elements from any of the topologies in the collection. The next theorem, which is the main result of this note, shows how this is done.

**Theorem 2.** *Let $\{\mathcal{T}_\alpha\}_{\alpha \in J}$ be a family of topologies on a set $X$, and define*

$$\mathcal{T} = \bigcap_{\alpha \in J} \mathcal{T}_\alpha,$$
$$\mathcal{U} = \bigcup_{\alpha \in J} \mathcal{T}_\alpha,$$
$$\widetilde{\mathcal{T}} = \{\text{unions of finite intersections of elements of } \mathcal{U}\}.$$

*Then $\mathcal{T}$ is the unique largest topology on $X$ that is contained in $\mathcal{T}_\alpha$ for every $\alpha \in J$, and $\widetilde{\mathcal{T}}$ is the unique smallest topology on $X$ that contains $\mathcal{T}_\alpha$ for every $\alpha$.*

*Proof.* It is easy to dispense with the uniqueness issue. If two topologies $\mathcal{T}$ and $\mathcal{T}'$are both largest topologies that are contained in each $\mathcal{T}_\alpha$, then the fact that $\mathcal{T}$ is a largest such topology implies that $\mathcal{T}' \subset \mathcal{T}$, and the fact that $\mathcal{T}'$ is largest implies that $\mathcal{T} \subset \mathcal{T}'$. Therefore, $\mathcal{T} = \mathcal{T}'$. The same argument, with "largest" and "smallest" interchanged and containments reversed, applies to smallest topologies containing each $\mathcal{T}_\alpha$. Thus the largest and smallest such topologies, if they exist, are unique, so we need only prove existence.

We begin with $\mathcal{T}$. Lemma 1 shows that $\mathcal{T}$ is a topology, and by definition it is contained in $\mathcal{T}_\alpha$ for every $\alpha$. It remains only to show that it is the largest such topology. To see this, suppose $\mathcal{T}'$ is *any* topology that is contained in $\mathcal{T}_\alpha$ for each $\alpha$. Then, by definition of the intersection, $\mathcal{T}'$ is contained in $\mathcal{T}$, which shows that $\mathcal{T}$ is larger than or equal to $\mathcal{T}'$. This completes the proof that $\mathcal{T}$ is the largest topology contained in every $\mathcal{T}_\alpha$.

Turning now to the other case, note first that the definition of $\widetilde{\mathcal{T}}$ means that it is precisely the topology generated by the subbasis $\mathcal{U}$. Munkres [M, page 82] proves that this is indeed a topology. Because an element $B \in \mathcal{U}$ is, in particular, a union of finite intersections of elements of $\mathcal{U}$ (for example, $B = (B \cap B) \cup (B \cap B)!$), it follows that $\widetilde{\mathcal{T}}$ contains $\mathcal{U}$ and therefore contains $\mathcal{T}_\alpha$ for every $\alpha$.

It remains only to show that $\widetilde{\mathcal{T}}$ is smaller than any other topology that contains every $\mathcal{T}_\alpha$. Suppose $\widetilde{\mathcal{T}}'$ is any such topology. Since it contains each $\mathcal{T}_\alpha$, it contains their union $\mathcal{U}$. Because every topology is closed under arbitrary unions and finite intersections, it follows that all unions of finite intersections of elements of $\mathcal{U}$ are in $\widetilde{\mathcal{T}}'$, which is to say that $\widetilde{\mathcal{T}} \subset \widetilde{\mathcal{T}}'$. Thus $\widetilde{\mathcal{T}}$ is the smallest such topology. □

## References

[M] J. Munkres, *Topology, Second Edition.* Prentice Hall, Upper Saddle River, NJ, 2000.