Math 310 B & C **Introduction to Mathematical Reasoning** **Autumn 2005**

Handout #6 (11/28/05)

About Variables

One topic that often vexes students who are beginning to learn to write proofs is the proper use of variables. Variable names are ubiquitous in mathematics, and of course every student of algebra and calculus has seen plenty of them. But the use of variables in proof writing, especially when quantifiers are involved, presents new opportunities for confusion.

This handout is meant to clarify some of the conventions regarding the use of variables in mathematical writing, particularly in proofs. You will not see many of these conventions written down anywhere, because most mathematicians absorb them "by osmosis" after reading and writing a great number of proofs themselves. With practice and careful attention to these guidelines, you might be able to streamline the process a bit.

Variables used in mathematical writing can be classified into four types: ***dummy variables***, ***defined variables***, ***assumed variables***, and ***chosen variables***. They serve different logical functions, and consequently they are treated quite differently. With the exception of dummy variables, every variable must be ***introduced*** before it is used. The descriptions below explain how to introduce each kind of variable. Dummy variables are implicitly introduced by their position in an expression.

Dummy Variables

A ***dummy variable*** is characterized by the following three properties:

- it is confined to one sentence or expression;
- it has no meaning outside that expression;
- if every occurrence of it in that expression is replaced by some other symbol that does not appear elsewhere in the expression, the meaning of the expression is unchanged.

Examples of dummy variables:

Expression	**Dummy Variable**
Let $f(x) = x^2 + 5$ for $x \in \mathbb{R}$	x
For each $n \in \mathbb{Z}$, set $b_n = \dfrac{n(n+1)}{2}$	n
$\forall x \in \mathbb{R},\ x^2 \geq 0$	x
$\exists k \in \mathbb{Z},\ n = 2k+1$	k
$\int_0^1 f(t)\,dt$	t
$\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta}$	θ
$\sum_{i=1}^{10} a_i$	i
$\{z \in \mathbb{R} : z \geq 0\}$	z
$\{f(y) : y \in \mathbb{R}\}$	y
$\bigcup_{\alpha \in I} A_\alpha$	α
$\bigcap_{\beta \in I} B_\beta$	β

Defined Variables

A ***defined variable*** is any variable that is introduced by setting it equal to a specific mathematical object. This is the most straightforward way to introduce a new variable.

Here are typical patterns for introducing defined variables:

- ***Define*** $x =$ [mathematical expression].
- ***Set*** $x =$ [mathematical expression].
- ***Let*** $x =$ [mathematical expression].

In this form of introduction, the right-hand side must describe a specific, well-defined mathematical object, and every non-dummy variable that appears on the right-hand side must have previously been introduced.

Examples of introducing defined variables:

- Define $y = x + 1$ (*assuming x has already been introduced*).
- Set $x = \sqrt{2}$.
- Set $A = \{a \in \mathbb{R} : f(a) = 0\}$ (*assuming f has already been introduced*).
- Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2 + 2$.
- Let x denote the positive fourth root of 11.

Assumed Variables

An ***assumed variable*** is one that is introduced to represent an arbitrary object of a certain type with some assumed property, whether or not any such object is known to exist. This form of variable is typically introduced at the beginning of a proof (or a step in a proof), and can be used *only* to prove a statement of the form "for all x with such-and-such property, ..." Any other use is likely to lead to a circular argument.

Here are some typical patterns for introducing assumed variables. In each case, the variable is introduced in order to prove a statement of the form "For all x [of a certain type with certain properties], ..."

- ***Assume*** x [is an object of a certain type with certain properties].
- ***Suppose*** x [is an object of a certain type with certain properties].
- ***Let*** x [be an object of a certain type with certain properties].
- ***Given*** x [an object of a certain type with certain properties], ...

Examples of introducing assumed variables:

- Assume x is a positive real number.
- Assume n is an integer such that n^2 is odd.
- Assume $a \in S$.
- Suppose x is a real number such that $x^2 = -1$.
- Suppose $x > 0$.
- Suppose x is an arbitrary element of B.
- Let $y \in \mathbb{R}$ be arbitrary.

- Let x be a real solution to $x^3 - 2x - 5 = 0$.
- Let n be an integer.
- Let z denote a negative number.
- Let $w \in \mathbb{Q}$ be given.
- Given a real number x, ...

Chosen Variables

A ***chosen variable*** is one that is introduced to represent a mathematical object that is known to exist, without having a definite formula to specify it. Before such a variable is introduced, it is necessary to establish that such an object exists.

Here are some typical patterns for introducing chosen variables.

- ***Choose*** x [of a certain type with certain properties].
- ***Let*** x [be an object of a certain type with certain properties].

This form of introduction can be used anywhere, provided that a statement of the form "there exists x [of a certain type with certain properties]" has been previously proved (or assumed).

Examples of introducing chosen variables:

- Choose a positive real number x.
- Choose an integer k such that $n = 2k + 1$ (*assuming n is known to be odd*).
- Choose $x \in \mathbb{R}$ such that $x^2 = 2$.
- Choose $y \in B$ (*assuming B is known to be nonempty*).
- Let x be a real solution to $x^3 - 2x - 5 = 0$ (*assuming it is known that such a solution exists*).
- Let n be an integer.
- Let z denote a negative number.

When the "choose" statement immediately follows the existence assertion, the variable introduced by the "choose" statement may or may not have the same name as the dummy variable in the existence statement. For example, assuming neither x nor y has any prior meaning in the current context, the following two excerpts have exactly the same effect, namely to introduce x as a real number such that $f(x) = 0$:

- By the argument above, there exists $x \in \mathbb{R}$ such that $f(x) = 0$. Choose $x \in \mathbb{R}$ such that $f(x) = 0$.
- By the argument above, there exists $y \in \mathbb{R}$ such that $f(y) = 0$. Choose $x \in \mathbb{R}$ such that $f(x) = 0$.

In fact, both of these sound awkwardly repetitive, so one would be more likely to write something like the following:

- By the argument above, there exists $x \in \mathbb{R}$ such that $f(x) = 0$. Choose such a real number x.
- By the argument above, there exists $y \in \mathbb{R}$ such that $f(y) = 0$. Let x denote such a real number.

In fact, it is quite common to omit the "choose" statement entirely, if it would come directly after the existence assertion, and just continue to use the same variable name that was used as a dummy in the existence statement, as if it had been introduced by a "choose" statement. Thus the following statement could be used as a substitute for any of the four excerpts above:

- By the argument above, there exists $x \in \mathbb{R}$ such that $f(x) = 0$.

From this point on, the variable x stands for an element of $\mathbb{R}$ such that $f(x) = 0$. It is no longer available for use as a dummy variable.

Warning: You probably have observed that some very similar phrases involving "let" can be used to introduce defined, assumed, or chosen variables. When you use the word "let" to introduce a variable, or when you read such an introduction in someone else's writing, make sure you know which type of variable is being introduced. The simplest way to check is to try to replace the word "let" by "define," "assume," or "choose"; usually only one of these will make sense in the given context, and that will tell you which kind of variable it is.

The Scope of a Variable

Sometimes a variable is introduced once and retains the same meaning throughout a given document. However, more frequently, variable names are given a particular meaning only temporarily within a certain context, after which they are free to be reused for other purposes. The section of a document or proof in which a particular meaning of a variable is in effect is called the ***scope*** of that variable (or, more properly, the scope of that particular meaning of the variable). You should never refer to a variable outside of its scope, because it literally has no meaning there.

The scope of a dummy variable is easy to describe: by its very definition, a dummy variable has a particular meaning only within the statement or expression in which it is used. Outside of that expression, it has no meaning unless it is reintroduced with a different meaning.

The scope of a defined, assumed, or chosen variable begins where it is introduced. Where the scope ends is not always obvious; but the usual assumption is that, unless otherwise specified, the scope of a variable does not extend beyond the end of the proof or section of the document in which it is introduced. The scope will end sooner, of course, if the variable is explicitly redefined. If there is any ambiguity about how long a variable is to retain its meaning, you can always clarify by using such phrases as "throughout this section, J will denote a closed interval," or "for this proof only, we let n denote an odd integer."

From a strictly logical point of view, there is nothing wrong with reusing the same variable name multiple times in a given document, as long as the scopes of the different uses do not overlap. But to make your writing easy to understand, it is usually a good idea to make sure that a given variable name refers to objects of the same type and with similar functions every time it is used. For example, if you are writing about real-valued functions of a real variable, you might consistently use x for elements of the domain and y for elements of the range.