Corrections to

Riemannian Manifolds: An Introduction to Curvature

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Changes or additions made in the past twelve months are dated.

Page 15, Exercise 2.3, part (a): In the first sentence, change "smooth function on $\widetilde{M}$" to "smooth real-valued function on a neighborhood of M in $\widetilde{M}$."

Page 16, first paragraph, Exercise 2.3(b): Change "vector field on $\widetilde{M}$" "vector field on a neighborhood of M in $\widetilde{M}$." [Remark: the original claims of 2.3(a) and (b)—that a smooth function or vector field on M can be extended smoothly to all of $\widetilde{M}$—are true if we assume in addition that M is closed in $\widetilde{M}$; but for this book we need the local extension properties stated here for embedded submanifolds that are not necessarily closed.]

Page 16, second paragraph, Exercise 2.3(c): Change $\widetilde{X}f = 0$ to $\big(\widetilde{X}f\big)\big|_M = 0$.

Page 17, part (c) in the definition of a vector bundle: Replace the statement of (c) by "With U and φ as above, for each $q \in U$, the restriction $\varphi|_{E_q}\colon E_q \to \mathbf{R}^k$ is a linear isomorphism."

Page 19, paragraph before Lemma 2.3: Insert the following before the last sentence of the paragraph: "A *local frame* for E is a finite sequence $(\sigma_1, \ldots, \sigma_k)$ of smooth sections of E over U such that $(\sigma_1|_p, \ldots, \sigma_k|_p)$ form a basis for E_p at each point $p \in U$."

Page 19, Lemma 2.3: $F^{j_1\ldots j_l}_{i_1\ldots i_k}$ should read F^i; also, change the name of the local frame from $\{E_i\}$ to $\{\sigma_i\}$.

Page 19, Exercise 2.4: Replace the given exercise by:

(a) If $(\sigma_1, \ldots, \sigma_k)$ is a local frame for a vector bundle E over an open set $U \subset M$, let $\psi\colon U \times \mathbf{R}^k \to \pi^{-1}(U)$ be the map $\psi(p, x) = x^i \sigma_i|_p$. Show that ψ^{-1} is a local trivialization of E.

(b) Prove Lemma 2.3.

Page 20, paragraph before Exercise 2.6: Replace the first sentence by "Let $(E_1, \ldots, E_n)$ be any local frame for TM."

Page 21, line 4: Change $F(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l)$ to $F(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k)$.

Page 21, just after Exercise 2.7: Add the following sentence in a paragraph by itself: "Because of the result of Lemma 2.4, it is common to use the same symbol for both a tensor field and the multilinear map on sections that it defines, and to refer either of these objects as a tensor field."

Page 24, five lines above the first displayed equation: Replace "It can be shown . . . " by "If M has finitely many components, it can be shown . . . ." [If M has infinitely many isometric components, then its isometry group contains a discrete subgroup isomorphic to the full group of permutations of a countably infinite set, and thus cannot be second countable.]

Page 25, second full paragraph, third line: After "open subset of M," insert "and which is smooth as a map into M."

Page 27, line 6: Change "smooth map" to "diffeomorphism."

Page 27, paragraph before Exercise 3.6: Replace this paragraph by "The following exercise shows that the converse is true provided we make the additional assumption that π is a *normal* covering, which means that the group of covering transformations acts transitively on each fiber of π."

Page 27, Exercise 3.6: Change "smooth covering map" to "smooth normal covering map."

Page 33, paragraph beginning "One of the first. . . ": Replace the fourth and fifth sentences of that paragraph by the following: "First, M is a *homogeneous Riemannian manifold* if its isometry group $\mathcal{I}(M)$ acts transitively on M. Second, given a point $p \in M$, M is *isotropic at* p if the isotropy subgroup $\mathcal{I}_p(M) \subset \mathcal{I}(M)$ (the subgroup of elements of $\mathcal{I}(M)$ that fix p) acts transitively on the set of unit vectors in T_pM (where $g \in \mathcal{I}_p(M)$ acts on T_pM by $g_* \colon T_pM \to T_pM$)."

Page 41, Exercise 3.11(ii): In the second line, change the domain of σ to $\mathbf{S}^n_R - \{N\}$. Then in the sixth line, change "conformal map" to "conformal equivalence."

Page 41, Exercise 3.11(iii): Replace the last sentence by "In the higher-dimensional case, for any point $p \in \mathbf{B}^n_R$ and any vector $V \in T_p\mathbf{B}^n_R$, first show that $h^3_R(\kappa_* V, \kappa_* V) = h^2_R(V, V)$ if $p \in \mathbf{B}^2_R \subset \mathbf{B}^n_R$ and V is tangent to $\mathbf{B}^2_R$; then show that the same is true if $p \in \mathbf{B}^2_R$ but V is arbitrary (using the fact that h^3_R and h^2_R are multiples of the Euclidean metrics at p and $\kappa(p)$); and finally conjugate κ with a suitable orthogonal transformation in $n-1$ variables to reduce to the case $p \in \mathbf{B}^2_R$."

Page 45, Problem 3-8, line 9: Just after the sentence ending with "*horizontal space*," insert the following sentence: "A vector field on $\widetilde{M}$ is said to be a *horizontal* or *vertical vector field* if its value is in the horizontal or vertical space at each point, respectively."

Page 46, Problem 3-9(a): Change the problem statement to: "Note that the natural action of $U(n+1)$ on $\mathbf{C}^{n+1}$ descends to a transitive action on $\mathbf{CP}^n$. Show that $\mathbf{CP}^n$ can be uniquely given the structure of a smooth, compact, real $2n$-dimensional manifold on which this action is smooth."

Page 58, Exercise 4.8: At the end of the sentence, add "and the constant curves."

Page 63, Problem 4-1: Replace the last sentence of the problem by the following: "Show that the two sets of Christoffel symbols are related by the following formula:

$$\widetilde{\Gamma}^k_{ij} = \left(A^{-1}\right)^k_q \left(A^p_i E_p A^q_j + A^p_i A^r_j \Gamma^q_{pr}\right)."$$

Page 63, problem 4-3(b): Replace the first sentence by "Show that there are vector fields V and W on $\mathbf{R}^2$ such that $V = W = \partial_1$ along the x^1-axis, but the Lie derivatives $\mathcal{L}_V(\partial_2)$ and $\mathcal{L}_W(\partial_2)$ are not equal on the x^1-axis."

Page 66, first full paragraph: Second sentence should read "Any vector field on M can be extended to a smooth vector field on a neighborhood of M in $\mathbf{R}^n$ by the result of Exercise 2.3(b)." The part of the second sentence after the two displayed equations should read "where X and Y are extended arbitrarily to a neighborhood of M, ..."

Page 66, proof of Lemma 5.1, second paragraph: Second sentence should read "Let $f \in C^\infty(M)$ be extended arbitrarily to a neighborhood of M."

Page 76, commutative diagram: Change T_pM and $T_{\varphi(p)}\widetilde{M}$ to $\mathcal{E}_p$ and $\mathcal{E}_{\varphi(p)}$, respectively (the domains of the restricted exponential maps).

Page 78, Proposition 5.11(a): Change "as long as γ_V stays within $\mathcal{U}$" to "as long as t is in some interval J containing 0 such that $\gamma_V(J) \subset \mathcal{U}$."

Page 78, line 6 from the bottom: Change "thought" to "though."

Page 86, first paragraph: Replace (ξ^1, ξ^{n+1}, τ) by (ξ^1, ξ^n, τ).

Page 86, last sentence: Replace the first part of the sentence by "In the higher-dimensional case, we just precede κ with a a suitable orthogonal transformation of the ball, and follow it with a translation and rotation in the x variables (both of which preserve geodesics as well as lines and circles), and apply the usual ...."

Page 88, Problem 5-6(b): In the first displayed equation, replace $\omega \otimes N$ by $\omega \otimes N^\flat$.

Page 89, Problem 5-9: Insert the following sentence after line 3: "(If Z is any vector field on M, we are using the notation $\widetilde{Z}$ to denote its horizontal lift.)" Also, in the hint, change both occurrences of ∇ to $\widetilde{\nabla}$.

Page 93, Exercise 6-2(b): The second occurrence of γ should be $\tilde{\gamma}$.

Page 93, Exercise 6-3(a): After "independent of parametrization," add "in the following sense: If $\varphi\colon [c,d] \to [a,b]$ is a smooth map with smooth inverse, then

$$\int_a^b f(t)\,|\dot{\gamma}(t)|\,dt = \int_c^d \widetilde{f}(u)\,\big|\dot{\widetilde{\gamma}}(u)\big|\,du,$$

where $\widetilde{f} = f \circ \varphi$ and $\widetilde{\gamma} = \gamma \circ \varphi$."

Page 95, second displayed inequality: Delete "$d(p,q) \geq$" from the beginning of the inequality, and replace the next sentence by "It follows that $d(p,q) \geq c\varepsilon > 0$, so d is a metric."

Page 101, proof of Corollary 6.7, third line: Change "second variation formula" to "first variation formula."

Page 102, proof of Theorem 6.8, fifth line: Change "$R = d(p,q)$" to "$R = |V|_g$."

Page 105, first sentence of last paragraph: Change both instances of "$[a,b]$" to "$[0,b]$."

Page 106, just after the proof of Corollary 6.11: Insert the following corollary:

Corollary 6.11a. *Every geodesic ball in M is also a metric ball of the same radius.*

Proof. Let $p \in M$ and let $\mathcal{U} = \exp_p(B_R(0)) \subset M$ be a geodesic ball of radius $R > 0$. Suppose q is an arbitrary point of M. If $q \in \mathcal{U}$, Corollary 6.11 shows that q is also in the metric ball of radius R. Conversely, suppose $q \notin \mathcal{U}$. For any positive number $R' < R$, the complement of the geodesic sphere of radius R' is disconnected, so any admissible curve from p to q must pass through that sphere, and then Corollary 6.19 shows that the length of that curve must be at least R'. Letting $R' \to R$ shows that $d(p,q) \geq R$, so q is not in the metric ball of radius R. □

Page 106, next-to-last paragraph, lines 2 and 3: Change "Corollary 6.11" to "Corollary 6.11a."

Page 106, second-to-last paragraph, last sentence: Delete the last part of this sentence, starting with "which are exactly ... ." [The claim is not true as stated: for example, if M is a bounded convex open subset of $\mathbf{R}^n$, then for large R, the metric ball of radius R around any point is equal to M, but is not in general a geodesic ball.]

(12/5/21) **Page 107, proof of Theorem 6.12:** Replace the two sentences beginning "If $t_1, t_2 \in \mathcal{U}$" with the following: "If $t_1, t_2 \in \mathcal{U}$ with $t_1 < t_2$, the definition of uniformly normal neighborhood implies that the image of $\gamma|_{[t_1,t_2]}$ is contained in a geodesic ball centered at $\gamma(t_1)$. Proposition 5.11 shows that every geodesic segment lying in that ball and starting at $\gamma(t_1)$ is part of a radial geodesic, and Proposition 6.10 shows that each radial geodesic segment is minimizing."

Page 108, next-to-last line: The definition of $\tilde{\gamma}$ is wrong: It should be $\tilde{\gamma}(t) = \sigma(t - t_j)$ for $t \in (t_j - \delta, t_j + \delta)$.

Page 111, Corollary 6.15: Change the statement to *"If M is complete, then any two points in M can be joined by a minimizing geodesic segment."*

Page 112, Problem 6-2: Replace the hint by "[Hint: For the hard direction, proceed as follows. (1) Show that any metric isometry $\varphi\colon (M,g) \to (\widetilde{M},\widetilde{g})$ takes geodesics to geodesics. (2) For any $p \in M$, show that there is an open ball $\mathcal{V} = B_\varepsilon(0) \subset T_pM$ and a map $\psi\colon \mathcal{V} \to T_{\varphi(p)}\widetilde{M}$ satisfying $\exp_{\varphi(p)} \psi(X) = \varphi(\exp_p X)$ for all $X \in \mathcal{V}$. (3) If ε is small enough and $X, Y \in \mathcal{V}$, show that there exists a constant $C > 0$ such that

$$(1 - C|t|)|tX - tY|_g \le d_g(\exp_p tX, \exp_p tY) \le (1 + C|t|)|tX - tY|_g$$

whenever $|t| \le 1$, by comparing g with the Euclidean metric in normal coordinates. (4) Using the result of (3), conclude that

$$\lim_{t\to 0} \frac{d_g(\exp_p tX, \exp_p tY)^2}{t^2} = |X - Y|_g^2 = |X|_g^2 + |Y|_g^2 - 2\langle X, Y\rangle_g,$$

and an analogous formula holds for $\widetilde{g}$. (5) Show that $\langle \psi(X), \psi(Y)\rangle_{\widetilde{g}} = \langle X, Y\rangle_g$ for all $X, Y \in \mathcal{V}$. (6) Show that ψ is the restriction of a linear map. (7) Conclude that φ is smooth and $\varphi_* = \psi$.]"

Page 112, Problem 6-4: In part (a), change the second sentence to "For $\varepsilon > 0$ small enough that $B_{3\varepsilon}(p) \subset \mathcal{W}, \dots$." In part (b), add to the hint "Be careful to verify that ε can be chosen independently of V."

Page 113, Problem 6-8: Delete the word "complete" and add instead "connected." Also, revise the hint as follows: "[Hint: Given $p, q \in M$ sufficiently near each other, consider the midpoint of a geodesic joining p and q. You may use without proof the fact that the isometry group of M is a Lie group acting smoothly on M.]"

Page 113, Problem 6-10: Change "a closed, embedded submanifold" to "an immersed submanifold."

Page 119, Lemma 7.2: Because φ is not a global diffeomorphism, the second displayed equation has to be interpreted pointwise. Replace the equation in the book by the following:

$$\widetilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z = \varphi_*(R(X,Y)Z) \qquad \text{for all } p \in M \text{ and } X, Y, Z \in T_pM.$$

Page 120, line above (7.5): Change x^m to x^n.

Page 125, line 4: Replace the phrase "*where* div *is the divergence operator* (*Problem 3-3*)" by "*where* $\operatorname{div} Rc$ *is the* 1*-tensor obtained from* ∇Rc *by raising the last index and contracting it with the next-to-last one.*"

Page 125, line 2 from the bottom: Change ds to dS.

Page 132, line 9 from the bottom: Change "smooth section of $T\widetilde{M}$" to "smooth section of $T\widetilde{M}$ on a neighborhood of M."

Page 133, last line: Replace the phrase "vector fields on $\widetilde{M}$" by "vector fields on a neighborhood of M in $\widetilde{M}$."

Page 134, just below the second displayed equation: Replace the phrase "to $\widetilde{M}$" by "to a neighborhood of M in $\widetilde{M}$."

Page 134, line 2 from the bottom: Replace the phrase "to M" by "to a neighborhood of M in $\widetilde{M}$."

Page 135, Statement of Theorem 8.2: Replace the phrase "vector fields on $\widetilde{M}$" by "vector fields on a neighborhood of M in $\widetilde{M}$."

Page 135, just below the second displayed equation, and again two lines below the third: Replace the phrase "to $\widetilde{M}$" by "to a neighborhood of M in $\widetilde{M}$."

Page 136, Statement of Lemma 8.3: Replace the phrase "to $\widetilde{M}$" by "to a neighborhood of M in $\widetilde{M}$."

Page 136, proof of Theorem 8.4, second line: Replace the phrase "vector fields on $\widetilde{M}$" by "vector fields on a neighborhood of M in $\widetilde{M}$."

Page 137, last displayed equation: Should be

$$\kappa = \left(\frac{|D_t\dot\gamma(t)|^2}{|\dot\gamma(t)|^4} - \frac{\langle D_t\dot\gamma(t), \dot\gamma(t)\rangle^2}{|\dot\gamma(t)|^6} \right)^{1/2}.$$

Page 139, line before Exercise 8.4: Change "lies entirely in M" to "lies in M at least for some small time interval $(-\varepsilon, \varepsilon)$."

Page 146, proof of Prop. 8.8, second paragraph: Three corrections in this paragraph: (1) Replace "which lies in S_Π" by "which lies in S_Π for t near zero." (2) Replace "V was an arbitrary element of T_pM" by "V was an arbitrary element of $\Pi = T_p(S_\Pi)$." (3) Replace S by S_Π twice in the last line.

Page 150, Problem 8-3: The left-hand side of the displayed equation should be $h(V, V)$ instead of $h(V, W)$, and the denominator on the right-hand side should be $|\operatorname{grad} F|$ instead of $|\operatorname{grad} F|^2$. Also, replace the last sentence with the following: "Show that the mean curvature of M is given by

$$H = -\frac{1}{n} \operatorname{div} \left(\frac{\operatorname{grad} F}{|\operatorname{grad} F|} \right) = -\frac{1}{n} \sum_{i,j=1}^{n+1} \frac{(\partial_i\partial_i F)(\partial_j F)(\partial_j F) - (\partial_i\partial_j F)(\partial_i F)(\partial_j F)}{|\operatorname{grad} F|^3}.$$

[Hint: Use an adapted orthonormal frame.]"

Page 151, Problem 8-6: In the second to last line, change $K\,dV_g$ to $(-1)^n K\,dV_g$.

Page 162, last paragraph: Replace the second sentence by "In this setting, a unit-speed admissible curve $\gamma\colon [a, b] \to M$ is called a curved polygon if the image of γ is the boundary of an open set $\Omega \subset M$ with compact closure, and there is an oriented smooth coordinate chart containing $\overline{\Omega}$ under whose image γ is a curved polygon in the plane (Figure 9.12).

Page 165, fourth display: In some editions, there is a dot over ω in the last integral in the first line of that display. It should not be there.

Page 171, Problem 9-3(b): Change "equal interior angles" to "equal interior angles and proportional corresponding side lengths." [The claim is true without this extra hypothesis, but the proof requires a more detailed analysis of hyperbolic and (especially) spherical geometry than is worth carrying out just for this problem.]

Page 176, Exercise 10.1: This is somewhat harder than most of the other exercises in the book, and needs Proposition 10.4 for its solution, so it should probably be moved to the Problems section, say as Problem 10-4.

Page 180, statement of Proposition 10.9: The first case of formula (10.8) should be $C = 0$, not $K = 0$.

Page 181, statement of Proposition 10.10: Add the hypothesis that $\dim M = \dim \widetilde{M}$.

Page 184, last line of the proof of Proposition 10.11: Change $J_W(q)$ to $J_W(1)$.

Page 188, Figure 10.10: Replace $\gamma(b)$ by $\gamma(a)$.

Page 187, equation (10.16): The summation should be from 1 to $k-1$, to be consistent with the notation in Chapter 6.

Page 187, last displayed equation: There's a missing dt on the right-hand side. It should read

$$\int_{a_{i-1}}^{a_i} \langle D_t V, D_t W\rangle \, dt = -\int_{a_{i-1}}^{a_i} \left\langle D_t^2 V, W\right\rangle dt + \left. \langle D_t V, W\rangle \right|_{a_{i-1}}^{a_i}.$$

Page 188, line 4: Replace $J(q)$ by $J(b)$.

Page 188, last paragraph: Replace b by a in each formula in this paragraph.

Page 189, last two displayed equations: Replace b by a in three places.

Page 189, last line: Delete the reference to [Spi79].

Page 190: In the definitions of *cut locus* and *cut point*, add the hypothesis that M is complete.

Page 194, statement of Theorem 11.1: Change "differentiable" to "continuously differentiable."

Page 195, paragraph beginning "We wish": Replace "arrange that $d|J|/dt = 1$ at $t = 0$" by "arrange that $|J|$ is continuously differentiable and $d|J|/dt = 1$ at $t = 0$."

Page 195, last displayed equation: Replace the displayed equation and the word "Therefore" above it by the following: "Since $W(t)$ does not vanish, $|W(t)|$ is a smooth function of t. Therefore,

$$\frac{d}{dt}|J(t)| = \frac{d}{dt}(t|W(t)|) = |W(t)| + t\frac{\langle D_t W(t), W(t)\rangle}{|W(t)|}.$$

This shows that $|J(t)|$ is a continuously differentiable function of t on some interval $[0,\varepsilon)$, and also that its derivative at $t = 0$ is $|W(0)| = |D_t J(0)| = 1$."

Page 196, Corollary 11.4: At the end of the statement of the corollary, add the following parenthesized sentence: (*If* $C = 1/R^2 > 0$, *this inequality is valid only inside a normal coordinate chart of radius less than or equal to* πR.)

Page 197, first line: Insert the following just after "...local isometry.": "Note that each line $t \mapsto tX$ in T_pM is a $\tilde{g}$-geodesic, so $(T_pM, \tilde{g})$ is complete by Corollary 6.14."

Page 197, proof of Lemma 11.6, fourth line: Change $p \in M$ to $p \in \pi\big(\widetilde{M}\big)$.

Page 197, line 4 from bottom: After "M is complete," insert "by Corollary 6.14."

Page 198, last paragraph, second sentence: Change "π is an isometry" to "π is a local isometry."

Page 201, proof of Theorem 11.7, next-to-last line: Change "infinite discrete set" to "infinite discrete closed set."

Page 203, last line: Change "Walter" to "Wilhelm."

Page 208, Problem 11-2(a): Replace the last sentence by "If $t_1 < t_2$ are zeros of v, then u must have at least one zero in (t_1, t_2), unless $a \equiv b$ on $[t_1, t_2]$ and u and v are constant multiples of each other there."

Page 208, Problem 11-3: In the last sentence, insert "is" before "at least."

Page 213: The index entry for "Bianchi identity/contracted" should be page 125, not 124.

Page 216: The index entry for "escape lemma" should be page 61, not 60.

Page 219: Change "Klingenberg, Walter" to "Klingenberg, Wilhelm."