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Introduction to Riemannian Manifolds

Second Edition

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Preface

Riemannian geometry is the study of manifolds endowed with *Riemannian metrics*, which are, roughly speaking, rules for measuring lengths of tangent vectors and angles between them. It is the most "geometric" branch of differential geometry. Riemannian metrics are named for the great German mathematician Bernhard Riemann (1826–1866).

This book is designed as a textbook for a graduate course on Riemannian geometry for students who are familiar with the basic theory of smooth manifolds. It focuses on developing an intimate acquaintance with the geometric meaning of curvature, and in particular introducing many of the fundamental results that relate the local geometry of a Riemannian manifold to its global topology (the kind of results I like to call "local-to-global theorems," as explained in Chapter 1). In so doing, it introduces and demonstrates the uses of most of the main technical tools needed for a careful study of Riemannian manifolds.

The book is meant to be introductory, not encyclopedic. Its coverage is reasonably broad, but not exhaustive. It begins with a careful treatment of the machinery of metrics, connections, and geodesics, which are the indispensable tools in the subject. Next comes a discussion of Riemannian manifolds as metric spaces, and the interactions between geodesics and metric properties such as completeness. It then introduces the Riemann curvature tensor, and quickly moves on to submanifold theory in order to give the curvature tensor a concrete quantitative interpretation.

The first local-to-global theorem I discuss is the Gauss–Bonnet theorem for compact surfaces. Many students will have seen a treatment of this in undergraduate courses on curves and surfaces, but because I do not want to assume such a course as a prerequisite, I include a complete proof.

From then on, all efforts are bent toward proving a number of fundamental local-to-global theorems for higher-dimensional manifolds, most notably the Killing–Hopf theorem about constant-curvature manifolds, the Cartan–Hadamard theorem about nonpositively curved manifolds, and Myers's theorem about positively curved ones. The last chapter also contains a selection of other important local-to-global theorems.

Many other results and techniques might reasonably claim a place in an introductory Riemannian geometry book, but they would not fit in this book without drastically broadening its scope. In particular, I do not treat the Morse index theorem, Toponogov's theorem, or their important applications such as the sphere theorem; Hodge theory, gauge theory, minimal surface theory, or other applications of elliptic partial differential equations to Riemannian geometry; or evolution equations such as the Ricci flow or the mean curvature flow. These important topics are for other, more advanced, books.

When I wrote the first edition of this book twenty years ago, a number of superb reference books on Riemannian geometry were already available; in the intervening years, many more have appeared. I invite the interested reader, after reading this book, to consult some of those for a deeper treatment of some of the topics introduced here, or to explore the more esoteric aspects of the subject. Some of my favorites are Peter Petersen's admirably comprehensive introductory text [Pet16]; the elegant introduction to comparison theory by Jeff Cheeger and David Ebin [CE08] (which was out of print for a number of years, but happily has been reprinted by the American Mathematical Society); Manfredo do Carmo's much more leisurely treatment of the same material and more [dC92]; Barrett O'Neill's beautifully integrated introduction to pseudo-Riemannian and Riemannian geometry [O'N83]; Michael Spivak's classic multivolume tome [Spi79], which can be used as a textbook if plenty of time is available, or can provide enjoyable bedtime reading; the breathtaking survey by Marcel Berger [Ber03], which richly earns the word "panoramic" in its title; and the "Encyclopaedia Britannica" of differential geometry books, *Foundations of Differential Geometry* by Shoshichi Kobayashi and Katsumi Nomizu [KN96]. At the other end of the spectrum, Frank Morgan's delightful little book [Mor98] touches on most of the important ideas in an intuitive and informal way with lots of pictures—I enthusiastically recommend it as a prelude to this book. And there are many more to recommend: for example, the books by Chavel [Cha06], Gallot/Hulin/Lafontaine [GHL04], Jost [Jos17], Klingenberg [Kli95], and Jeffrey Lee [LeeJeff09] are all excellent in different ways.

It is not my purpose to replace any of these. Instead, I hope this book fills a niche in the literature by presenting a selective introduction to the main ideas of the subject in an easily accessible way. The selection is small enough to fit (with some judicious cutting) into a single quarter or semester course, but broad enough, I hope, to provide any novice with a firm foundation from which to pursue research or develop applications in Riemannian geometry and other fields that use its tools.

This book is written under the assumption that the student already knows the fundamentals of the theory of topological and smooth manifolds, as treated, for example, in my two other graduate texts [LeeTM, LeeSM]. In particular, the student should be conversant with general topology, the fundamental group, covering spaces, the classification of compact surfaces, topological and smooth manifolds, immersions and submersions, submanifolds, vector fields and flows, Lie brackets and Lie derivatives, tensors, differential forms, Stokes's theorem, and the basic theory of Lie groups. On the other hand, I do not assume any previous acquaintance with Riemannian metrics, or even with the classical theory of curves and surfaces in $\mathbb{R}^3$. (In this subject, anything proved before 1950 can be considered "classical"!)

Although at one time it might have been reasonable to expect most mathematics students to have studied surface theory as undergraduates, many current North American undergraduate math majors never see any differential geometry. Thus the fundamentals of the geometry of surfaces, including a proof of the Gauss–Bonnet theorem, are worked out from scratch here.

The book begins with a nonrigorous overview of the subject in Chapter 1, designed to introduce some of the intuitions underlying the notion of curvature and to link them with elementary geometric ideas the student has seen before. Chapter 2 begins the course proper, with definitions of Riemannian metrics and some of their attendant flora and fauna. Here I also introduce pseudo-Riemannian metrics, which play a central role in Einstein's general theory of relativity. Although I do not attempt to provide a comprehensive introduction to pseudo-Riemannian geometry, throughout the book I do point out which of the constructions and theorems of Riemannian geometry carry over easily to the pseudo-Riemannian case and which do not.

Chapter 3 describes some of the most important "model spaces" of Riemannian and pseudo-Riemannian geometry—those with lots of symmetry—with a great deal of detailed computation. These models form a sort of leitmotif throughout the text, serving as illustrations and testbeds for the abstract theory as it is developed.

Chapter 4 introduces connections, together with some fundamental constructions associated with them such as geodesics and parallel transport. In order to isolate the important properties of connections that are independent of the metric, as well as to lay the groundwork for their further study in arenas that are beyond the scope of this book, such as the Chern–Weil theory of characteristic classes and the Donaldson and Seiberg–Witten theories of gauge fields, connections are defined first on arbitrary vector bundles. This has the further advantage of making it easy to define the induced connections on tensor bundles. Chapter 5 investigates connections in the context of Riemannian (and pseudo-Riemannian) manifolds, developing the Levi-Civita connection, its geodesics, the exponential map, and normal coordinates. Chapter 6 continues the study of geodesics, focusing on their distance-minimizing properties. First, some elementary ideas from the calculus of variations are introduced to prove that every distance-minimizing curve is a geodesic. Then the Gauss lemma is used to prove the (partial) converse—that every geodesic is locally minimizing.

Chapter 7 unveils the first fully general definition of curvature. The curvature tensor is motivated initially by the question whether all Riemannian metrics are "flat" (that is, locally isometric to the Euclidean metric). It turns out that the failure of parallel transport to be path-independent is the primary obstruction to the existence of a local isometry. This leads naturally to a qualitative interpretation of curvature as the obstruction to flatness. Chapter 8 is an investigation of submanifold theory, leading to the definition of sectional curvatures, which give curvature a more quantitative geometric interpretation.

The last four chapters are devoted to the development of some of the most important global theorems relating geometry to topology. Chapter 9 gives a simple moving-frames proof of the Gauss–Bonnet theorem, based on a careful treatment of Hopf's rotation index theorem (often known by its German name, the *Umlaufsatz*). Chapter 10 has a largely technical nature, covering Jacobi fields, conjugate points, the second variation formula, and the index form for later use in comparison theorems. Chapter 11 introduces comparison theory, using a simple comparison theorem for matrix Riccati equations to prove the fundamental fact that bounds on curvature lead to bounds (in the opposite direction) on the size of Jacobi fields, which in turn lead to bounds on many fundamental geometric quantities, such as distances, diameters, and volumes. Finally, in Chapter 12 comes the denouement: proofs of some of the most important local-to-global theorems illustrating the ways in which curvature and topology affect each other.

Exercises and Problems

This book contains many questions for the reader that deserve special mention. They fall into two categories: “exercises,” which are integrated into the text, and “problems,” grouped at the end of each chapter. Both are essential to a full understanding of the material, but they are of somewhat different characters and serve different purposes.

The exercises include some background material that the student should have seen already in an earlier course, some proofs that fill in the gaps from the text, some simple but illuminating examples, and some intermediate results that are used in the text or the problems. They are, in general, elementary, but they are *not optional*—indeed, they are integral to the continuity of the text. They are chosen and timed so as to give the reader opportunities to pause and think over the material that has just been introduced, to practice working with the definitions, and to develop skills that are used later in the book. I recommend that students stop and do each exercise as it occurs in the text, or at least convince themselves that they know what is involved in the solution of each one, before going any further.

The problems that conclude the chapters are generally more difficult than the exercises, some of them considerably so, and should be considered a central part of the book by any student who is serious about learning the subject. They not only introduce new material not covered in the body of the text, but they also provide the student with indispensable practice in using the techniques explained in the text, both for doing computations and for proving theorems. If the result of a problem is used in an essential way in the text, or in a later problem, the page where it is used is noted at the end of the problem statement. Instructors might want to present some of these problems in class if more than a semester is available.

At the end of the book there are three appendices that contain brief reviews of background material on smooth manifolds, tensors, and Lie groups. I have omitted most of the proofs, but included references to other books where they may be found. The results are collected here in order to clarify what results from topology and smooth manifold theory this book will draw on, and also to establish definitions and conventions that are used throughout the book. I recommend that most readers at least glance through the appendices *before* reading the rest of the book, and consider consulting the indicated references for any topics that are unfamiliar.

About the Second Edition

This second edition, titled *Introduction to Riemannian Manifolds*, has been adapted from my earlier book *Riemannian Manifolds: An Introduction to Curvature*, Graduate Texts in Mathematics 176, Springer 1997.

For those familiar with the first edition, the first difference you will notice about this edition is that it is considerably longer than the first. To some extent, this is due to the addition of more thorough explanations of some of the concepts. But a much more significant reason for the increased length is the addition of many topics that were not covered in the first edition. Here are some of the most important ones: a somewhat expanded treatment of pseudo-Riemannian metrics, together with more consistent explanations of which parts of the theory apply to them; a more detailed treatment of which homogeneous spaces admit invariant metrics; a new treatment of general distance functions and semigeodesic coordinates; introduction of the Weyl tensor and the transformation laws for various curvatures under conformal changes of metric; derivation of the variational equations for hypersurfaces that minimize area with fixed boundary or fixed enclosed volume; an introduction to symmetric spaces; and a treatment of the basic properties of the cut locus. Most importantly, the entire treatment of comparison theory has been revamped and expanded based on Riccati equations, and a handful of local-to-global theorems have been added that were not present in the first edition: Cartan's torsion theorem, Preissman's theorem, Cheng's maximal diameter theorem, Milnor's theorem on polynomial growth of the fundamental group, and Synge's theorem. I hope these will make the book much more useful.

I am aware, though, that one of the attractions of the first edition for some readers was its brevity. For those who would prefer a more streamlined path toward the main local-to-global theorems in Chapter 12, here are topics that can be omitted on a first pass through the book without essential loss of continuity.

- *Chapter 2*: Other generalizations of Riemannian metrics
- *Chapter 3*: Other homogeneous Riemannian manifolds and model pseudo-Riemannian manifolds
- *Chapter 5*: Tubular neighborhoods, Fermi coordinates, and Euclidean and non-Euclidean geometries
- *Chapter 6*: Distance functions and semigeodesic coordinates
- *Chapter 7*: The Weyl tensor and curvatures of conformally related metrics
- *Chapter 8*: Computations in semigeodesic coordinates, minimal hypersurfaces, and constant-mean-curvature hypersurfaces
- *Chapter 9*: The entire chapter
- *Chapter 10*: Locally symmetric spaces and cut points
- *Chapter 11*: Günther's volume comparison theorem and the Bishop–Gromov volume comparison theorem
- *Chapter 12*: All but the theorems of Killing–Hopf, Cartan–Hadamard, and Myers

In addition to the major changes listed above, there are thousands of minor ones throughout the book. Of course, I have attempted to correct all of the mistakes that I became aware of in the first edition. Unfortunately, I surely have not been able to avoid introducing new ones, so if you find anything that seems amiss, please let me know by contacting me through the website listed below. I will keep an updated list of corrections on that website.

I have also adjusted my notation and terminology to be consistent with my two other graduate texts [LeeSM, LeeTM] and hopefully to be more consistent with commonly accepted usage. Like those books, this one now has a notation index just before the subject index, and it uses the same typographical conventions: mathematical terms are typeset in ***bold italics*** when they are officially defined; exercises in the text are indented, numbered consecutively with the theorems, and marked with the special symbol ► to make them easier to find; the ends of numbered examples are marked with the symbol //; and the entire book is now set in Times Roman, supplemented by the MathTime Professional II mathematics fonts created by Personal TEX, Inc.

Acknowledgements

I owe an unpayable debt to the authors of the many Riemannian geometry books I have used and cherished over the years, especially the ones mentioned above—I have done little more than rearrange their ideas into a form that seems handy for teaching. Beyond that, I would like to thank my Ph.D. advisor, Richard Melrose, who many years ago introduced me to differential geometry in his eccentric but thoroughly enlightening way; my colleagues Judith Arms, Yu Yuan, and Jim Isenberg, who have provided great help in sorting out what topics should be included; and all of the graduate students at the University of Washington who have suffered with amazing grace through the many flawed drafts of both editions of this book and have provided invaluable feedback, especially Jed Mihalisin, David Sprehn, Collin Litterell, and Maddie Burkhart. And my deepest gratitude goes to Ina Mette of Springer-Verlag (now at the AMS), who first convinced me to turn my lecture notes into a book; without her encouragement, I would never have become a textbook author.

Finally, I would like to dedicate this book to the memory of my late colleague Steve Mitchell, who by his sparkling and joyful example taught me more about teaching and writing than anyone.

Seattle, Washington, USA
June 2018

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