

# CORRECTIONS TO Introduction to Riemannian Manifolds (Second Edition)

BY JOHN M. LEE

MAY 9, 2024

- (7/31/19) **Page ix, near the middle of the page:** “Preissmann” is misspelled.
- (6/14/23) **Page 12, second sentence under the heading *Isometries*:** After “shows that,” insert “when  $\partial M = \emptyset$ .”
- (6/16/21) **Page 15, line 3:** After “*connected if and only if  $M$  is connected,*” insert “(when  $\dim M > 1$ ).”
- (11/17/21) **Page 16, line after Example 2.13:** Change “next lemma” to “next proposition.”
- (7/29/19) **Page 20, Exercise 2.23:** Change “Exercise 2.21” to “Example 2.21.”
- (1/25/21) **Page 27, third-to-last displayed equation:** In the line below that equation, change  $\alpha_{k+1}$  to  $\alpha_{k+l}$ . [The numeral 1 should be a letter  $l$ .] Then in the second and third lines below the equation, change  $g^{kl}$  to  $g^{pq}$  twice [to avoid conflict with the notation  $(k, l)$  for the type of  $F$ ].
- (1/25/21) **Page 27, third line from the bottom:** Change  $g_{kl}$  to  $g_{pq}$ .
- (11/14/22) **Page 29, two lines above equation (2.16):** Change  $T^{(k,l)}(T_p M)$  to  $T^{(k,l)}TM$ .
- (9/4/22) **Page 33, second paragraph under “Lengths and Distances,” next-to-last sentence:** Replace that sentence by “(If  $\partial M \neq \emptyset$ , this is still true provided we embed  $M$  in a smooth manifold  $\tilde{M}$  without boundary such as the double of  $M$  [LeeSM, Example 9.32], and consider an extension of  $\gamma$  as a map into  $\tilde{M}$ .)”
- (10/19/21) **Page 41, lines 4 & 3 from the bottom:** Change “inner products of pairs of elements of  $S$ ” to “scalar products of pairs of elements of  $S^\perp$ .”
- (1/23/21) **Page 56, third line from the bottom:** Change “Chapter 1” to “Chapter 2.”
- (3/6/21) **Page 62, last line:** Change “Chapter 1” to “Chapter 2.”
- (6/24/22) **Page 63, Fig. 3.3:** The map  $\kappa$  should be going in the opposite direction.
- (6/24/22) **Page 63, second paragraph:** Change  $\kappa: \mathbb{B}^n(R) \rightarrow \mathbb{U}^n(R)$  to  $\kappa: \mathbb{U}^n(R) \rightarrow \mathbb{B}^n(R)$ .
- (4/12/23) **Page 65, third line from the bottom:** The formula should read  $|\pi(\xi, \tau)|^2 = R^2(\tau - R)/(\tau + R) < R^2$ .
- (1/10/21) **Page 70, lines 4 & 5:** Replace the clause beginning “let  $\mu$  be” with the following: “let  $\mu$  be a right-invariant density on  $K$  (for example, the Riemannian density of some right-invariant metric on  $K$ ).”
- (11/1/19) **Page 71, first paragraph, last line:** Change  $\text{Ad}(\varphi)X$  to  $\text{Ad}(\varphi)X_0$ .
- (5/17/20) **Page 80, last line:** Change  $z$  to  $w$  (twice).
- (1/10/21) **Page 81, Problem 3-9:** Add the hypothesis that  $G$  is connected.
- (6/14/19) **Page 82, Problem 3-20:** Delete the word “isotropic”; it should just say “homogeneous and symmetric.”
- (5/3/20) **Page 82, Problem 3-21:** In the third line, change  $M$  to  $\tilde{M}$ .
- (7/8/20) **Page 87, last two lines:** The second occurrence of  $\frac{\partial}{\partial x}$  in each line should be  $\frac{\partial}{\partial y}$ .

(9/4/22) **Page 97, second line:** After the last sentence of the proof, add the following: “It is then a straightforward computational exercise to show that the resulting connection satisfies conditions (i)–(iii). To prove (iv), first observe that every  $(k, l)$ -tensor field can be written locally as a sum of tensor fields of the form  $Z_1 \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \zeta^l$ , and for such a tensor field the trace on the  $i$ th contravariant index and the  $j$ th covariant one satisfies

$$\operatorname{tr}(Z_1 \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \zeta^l) = \zeta^j(Z_i)Z_1 \otimes \cdots \otimes \widehat{Z}_i \otimes \cdots \otimes Z_k \otimes \zeta^1 \otimes \cdots \otimes \widehat{\zeta}^j \otimes \cdots \otimes \zeta^l.$$

Then (iv) follows by applying (4.12) and (4.13) to this formula.”

(4/22/19) **Page 100, first displayed equation:** Change the last expression on the right to  $X(Yu) - (\nabla_X Y)u$ .

(9/4/22) **Page 102, proof of Theorem 4.24, second paragraph, last sentence:** Replace that sentence by “(If  $t_0$  is an endpoint of  $I$ , extend  $\gamma$  to a slightly bigger open interval, prove the lemma there, and then restrict back to  $I$ . If  $M$  has nonempty boundary, we can do this after first embedding  $M$  into a smooth manifold  $\widetilde{M}$  without boundary and extending  $\nabla$  arbitrarily to a connection on  $\widetilde{M}$ .)”

(4/19/24) **Page 102, equation (4.15), first line:** Remove the “ $|_{\gamma(t)}$ ” from the second term. [The covariant derivative  $\nabla_{\gamma'(t)} \partial_j$  is already a tangent vector at  $\gamma(t)$ .]

(11/7/19) **Page 105, first paragraph under the section heading:** Delete the sentence beginning “As we did with geodesics . . . .”

(5/11/23) **Page 105, last paragraph, first line:** Change “manifold” to “manifold with or without boundary.”

(7/8/20) **Page 106, last line:** Change  $\eta(t)$  to  $\eta^0(t)$ .

(5/19/23) **Page 106, Theorem 4.31:** For later use, we also need to generalize this theorem to the case in which  $I$  is not open and  $t_0$  is an endpoint of  $I$ . Existence and smoothness of the solution are easily proved by extending  $A_j^k$  to an open interval containing  $I$ , so only uniqueness is at issue. To prove uniqueness, suppose for simplicity that  $t_0 = 0$  is the left endpoint of  $I$ . (Other cases are easily handled by suitable changes of independent variable.) If  $V$  and  $\widetilde{V}$  are solutions to the same initial-value problem corresponding to different extensions, let  $W(t) = V(t) - \widetilde{V}(t)$ , so that  $\dot{W}^k(t) = A_j^k(t)W^j(t)$  for  $t \in [0, \varepsilon]$  and  $W^k(0) = 0$ . Choose  $b > 0$  such that  $|A(t)| \leq b$  for all  $t \in [0, \varepsilon]$ . A simple computation shows that  $(d/dt)|e^{-bt}W(t)|^2 \leq 0$ , so  $W(t) = 0$  for  $t \in [0, \varepsilon]$ . Then the usual uniqueness result with  $t_0 = \varepsilon$  shows that  $V = \widetilde{V}$  on all of  $I$ .

(5/19/23) **Page 107, Theorem 4.32, 4th line:** Change  $\gamma(t)$  to  $\gamma(t_0)$ .

(7/5/20) **Page 109, Corollary 4.33, third and fourth lines:** Replace  $T_{\gamma(t_0)}M$  and  $T_{\gamma(t)}M$  by  $T_{\gamma(a)}M$ .

(5/9/24) **Page 112, Problem 4-6, first line:** After “smooth manifold,” insert “with or without boundary.”

(6/27/19) **Page 113, Problem 4-11(b):** Replace “ $G$  is abelian” by “the identity component of  $G$  is abelian.”

(1/23/21) **Page 119, just above the last display:** Change  $V^i, W^j: I \rightarrow \mathbb{R}$  to  $V^i, W^j: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}$ .

(5/9/24) **Page 121, second paragraph:** After “on the tangent bundle of a smooth manifold  $M$ ,” insert “(with or without boundary).”

(8/26/20) **Page 132, proof of Proposition 5.23, fourth and fifth lines:** Change  $d\varphi_p^{-1}$  to  $(d\varphi_p)^{-1}$  (twice).

(11/23/20) **Page 147, Problem 5-8(c):** Change “geodesics of  $g$ ” to “maximal geodesics of  $g$ .”

(2/11/21) **Page 155, just below equation (6.1):** Change “admissible partition for  $V$ ” to “admissible partition for  $\Gamma$ .”

(8/26/20) **Page 160, line 4:** Change  $\partial_i|_p$  to  $\partial_i|_p$ .

(5/29/20) **Page 163, second paragraph, third line:** Delete the repeated word “metric.”

- (8/13/21) **Page 164, line 8 from the bottom:** Change  $\varepsilon/c$  to  $c\varepsilon$  (twice).
- (12/5/21) **Page 165, proof of Theorem 6.15:** Replace the two sentences beginning “If  $a, b \in I_0$ ” with the following: “If  $a, b \in I_0$  with  $a < b$ , then the definition of uniformly normal neighborhood implies that the image of  $\gamma|_{[a,b]}$  is contained in a geodesic ball centered at  $\gamma(a)$ . Proposition 5.24 shows that every geodesic segment lying in that ball and starting at  $\gamma(a)$  is part of a radial geodesic, and Proposition 6.11 shows that each radial geodesic segment is minimizing.”
- (11/16/20) **Page 173, proof of Proposition 6.25, last line:** Change “no longer than” to “no shorter than” (twice).
- (1/25/21) **Page 175, two lines above Theorem 6.31:** Change “it it” to “it is.”
- (1/25/21) **Page 177, line 7 from the bottom:** Change  $B_\varepsilon(p)$  to  $B_{\varepsilon_p}(p)$ .
- (2/27/21) **Page 179, second paragraph, first line:** Change  $NS$  to  $NP$ .
- (12/15/23) **Page 183, Example 6.43, third-to-last line:** Change  $\pm\partial/\partial v$  to  $\pm\partial/\partial v$ .
- (3/3/21) **Page 197, Proposition 7.5, fourth line of the proof:** Change “Formula (4.15)” to “The product rule for covariant derivatives along curves.” Then two lines below that, change “(4.15)” to “the product rule.”
- (10/19/19) **Page 199, Proof of Lemma 7.8, second paragraph, last sentence:** Replace the phrase “an inductive application of the theorem concerning smooth dependence of solutions to linear ODEs on initial conditions (Thm. 4.31)” by “an inductive application of Theorem A.42 to vector fields of the form  $W_k|_{(x,v)} = \partial/\partial x^k - v^i \Gamma_{ki}^j(x) \partial/\partial v^j$  on  $C_\varepsilon \times \mathbb{R}^n$ .”
- (6/15/20) **Page 201, third line from the bottom:** Change “application of Theorem 4.31” to “application of Theorem A.42 as in the proof of Lemma 7.8.”
- (11/1/19) **Page 214, middle displayed equation:** The indices on the left-hand side should be  $jl$  (not  $jk$ ).
- (11/1/19) **Page 217, last line:** Change  $\langle df, df \rangle_g^2$  to  $\langle df, df \rangle_g$ .
- (11/1/19) **Page 218, proof of Theorem 7.30, last displayed equation:** In the first line of that display, the indices in the last two terms are wrong. The first line should read
- $$\tilde{R}_{ijkl} = e^{2f} \left( R_{ijkl} - (f_{;il} g_{jk} + f_{;jk} g_{il} - f_{;ik} g_{jl} - f_{;jl} g_{ik}) \dots \right)$$
- (2/1/19) **Page 218, proof of Theorem 7.30, last line:** The last formula in the proof should read  $\tilde{g}^{ij} = e^{-2f} g^{ij}$  (swap  $g$  and  $\tilde{g}$ ).
- (2/9/21) **Page 223, Problem 7-6:** Replace  $\pi_1^* S_2$  with  $\pi_2^* S_2$  in the last displayed equation.
- (9/17/23) **Page 226, line 10:** Change  $F(U) \subseteq M$  to  $F(U) \subseteq \tilde{M}$ .
- (6/9/19) **Page 226, middle of the page:** The reference [dC92] should be [O’N83].
- (6/21/23) **Page 228, third line after the end of the proof:** Change  $W_p = W$  to  $W_p = w$ .
- (6/21/23) **Page 234, statement of Proposition 8.12:** Change the comma before “The” to a period.
- (4/22/19) **Page 234, proof of Proposition 8.12, third-to-last line:** Change  $\tilde{D}_t \gamma$  and  $D_t \gamma$  to  $\tilde{D}_t \gamma'$  and  $D_t \gamma'$ , respectively.
- (1/23/21) **Page 245, statement of Proposition 8.21(a):** Change  $\gamma(t_0)$  to  $\gamma(t_0)$  [fixing the misplaced parenthesis].
- (1/31/20) **Page 257, Problem 8-11:** Change “a smooth surface” to “an embedded smooth surface.”

(8/7/20) **Page 262, Problem 8-33(a):** Change the problem statement to read as follows:

(a) Show that there is a unique fiber metric on  $\Lambda^2(TM)$  that satisfies

$$\langle w \wedge x, y \wedge z \rangle = \langle w, y \rangle \langle x, z \rangle - \langle w, z \rangle \langle x, y \rangle$$

for all tangent vectors  $w, x, y, z$  at every point  $q \in M$ .

(8/29/21) **Page 282, Problem 9-9(b), third line:** Change  $\gamma_2(2)$  to  $\gamma_2(t)$ .

(3/15/21) **Page 288, statement of Corollary 10.8:** At the beginning of the last sentence, insert the phrase “Provided  $g$  is Riemannian or  $|\gamma'(t)|_g \neq 0$  for some  $t \in I$ .”

(3/24/24) **Page 298, last paragraph, second line:** After “some  $a, b \in I$ ,” insert “with  $a \neq b$ .”

(4/22/24) **Page 299, statement of Proposition 10.20:** At the end of the proposition statement, add “(in the sense that there is a nontrivial Jacobi field  $J$  along  $\gamma$  such that  $J(0) = J(1) = 0$ ).”

(2/1/19) **Page 304, statement of Theorem 10.28:** At the end of the statement, after “ $I(V, V) > 0$ ,” insert “unless  $V \equiv 0$ .”

(7/8/20) **Page 305, line below equation (10.25):** Change  $M$  to  $\mathbb{R}$ .

(2/1/19) **Page 307, statement of Corollary 10.29(c):** At the end of the statement, after “for all sufficiently small nonzero  $s$ ,” insert “unless the variation field of  $\Gamma$  is identically zero.”

(7/5/21) **Page 311, proof of Theorem 10.34, first paragraph:** Replace  $\exp_p(t_{\text{cut}}(p, v))$  by  $\exp_p(t_{\text{cut}}(p, v)v)$  in the next-to-last line of the paragraph.

(12/12/20) **Page 311, statement of Corollary 10.35:** Add the hypothesis that the manifold is nonempty.

(6/15/20) **Page 315, last displayed equation:** Replace that equation by the following:

$$J(a) \in T_{\gamma(a)}P \quad \text{and} \quad D_t J(a) + W_{\gamma'(a)}(J(a)) \perp T_{\gamma(a)}P.$$

(12/5/20) **Page 317, Problem 10-23:** At the end of the first sentence, add the hypothesis “and  $q \in \text{Cut}(p)$ .”

(12/11/20) **Page 317, Problem 10-23(b):** Change “closed geodesic” to “unit-speed closed geodesic.”

(5/31/22) **Page 326, second paragraph, 4th line from the end of the paragraph:** Change “ $f(a, x) = 0$ ” to “ $f(a, x) \leq 0$ .”

(8/19/23) **Page 338, next-to-last paragraph, second line:** Change “nonnegative” to “nonpositive.”

(7/31/19) **Page 345, third paragraph:** “Preissmann” is misspelled.

(10/27/19) **Page 350, near the middle of the page, in the statements of the Bieberbach theorems:** Change “Euclidean space form(s)” to “compact Euclidean space form(s)” (twice).

(9/19/21) **Page 354, Corollary 12.14:** Change the statement of the corollary to “*If  $M$  is a connected smooth manifold that admits a complete metric of nonpositive sectional curvature, then  $M$  is aspherical.*” Then in the first sentence of the proof, change “a nonpositively curved metric” to “a complete nonpositively curved metric.”

(7/31/19) **Page 357, subheading and first three paragraphs:** “Preissmann” is misspelled (four times).

(3/15/21) **Page 357, statement of Lemma 12.21:** Replace “every covering automorphism” by “every nontrivial covering automorphism.”

(7/31/19) **Page 360, near the top and near the bottom:** “Preissmann” is misspelled (twice).

- (9/4/22) **Page 374, paragraph above Prop. A.10, first line:** Change “every coordinate chart” to “every coordinate chart  $(U, \varphi)$ .”
- (9/4/22) **Page 383, third paragraph, last sentence:** Change “*component functions of  $\sigma$* ” to “*component functions of  $\tau$* .”
- (1/3/22) **Page 397, three lines above equation (B.8):** Change  $T^{(k,l)}(T^*M)$  to  $T^{(k,l)}TM$ .
- (3/12/18) **Page 415, reference [BBBMP]:** “Zürich” is misspelled.
- (3/12/18) **Page 415, reference [Cha06]:** The phrase “A modern introduction” should be part of the title: *Riemannian Geometry: A Modern Introduction*.
- (7/31/19) **Page 418, reference [Pre43]:** “Preissmann” is misspelled.
- (7/31/19) **Page 433:** “Preissmann” is misspelled (twice).