Preface

Manifolds are the mathematical generalizations of curves and surfaces to arbitrary numbers of dimensions. This book is an introduction to the topological properties of manifolds at the beginning graduate level. It contains the essential topological ideas that are needed for the further study of manifolds, particularly in the context of differential geometry, algebraic topology, and related fields. Its guiding philosophy is to develop these ideas rigorously but economically, with minimal prerequisites and plenty of geometric intuition. Here at the University of Washington, for example, this text is used for the first third of a year-long course on the geometry and topology of manifolds; the remaining two-thirds of the course focuses on smooth manifolds using the tools of differential geometry.

There are many superb texts on general and algebraic topology available. Why add another one to the catalog? The answer lies in my particular vision of graduate education: it is my (admittedly biased) belief that every serious student of mathematics needs to be intimately familiar with the basics of manifold theory, in the same way that most students come to know the integers, the real numbers, vector spaces, functions of one real or complex variable, groups, rings, and fields. Manifolds play a role in nearly every major branch of mathematics (as I illustrate in Chapter 1), and specialists in many fields find themselves using concepts and terminology from topology and manifold theory on a daily basis. Manifolds are thus part of the basic vocabulary of mathematics, and need to be part of basic graduate education. The first steps must be topological, and are embodied in this book; in most cases, they should be complemented by material on smooth manifolds, vector fields, differential forms, and the like, as developed, for example, in [Lee02], which is designed to be a sequel to this book. (After all, few of the really interesting applications of manifold theory are possible without using tools from calculus.)

Of course, it is not realistic to expect all graduate students to take full-year courses in general topology, algebraic topology, and differential geometry. Thus, although this book touches on a generous portion of the material that is typically included in much longer courses, the coverage is selective and relatively concise, so that most of the book can be covered in a single quarter or semester, leaving time in a year-long course for further study in whatever direction best suits the instructor and the students. At UW, we follow it with a two-quarter sequence on smooth manifold theory based on [Lee02]; but it could equally well lead into a full-blown course on algebraic topology.

It is easy to describe what this book is not. It is not a course on general topology—many of the topics that are standard in such a course are ignored here, such as metrization theorems, the Tychonoff theorem for infinite product spaces, a comprehensive treatment of separation axioms, and function spaces. Nor is it a course in algebraic topology—although I treat the fundamental group in detail, there is barely a mention of the higher homotopy groups, and the treatment of homology theory is extremely brief, meant mainly to give the flavor of the theory and to lay some groundwork for the later introduction of de Rham cohomology. It certainly is not a comprehensive course on topological manifolds, which would have to include such topics as PL structures and maps, transversality, surgery, Morse theory, intersection theory, cobordism, bundles, characteristic classes, and low-dimensional geometric topology. (Perhaps a more accurate title for the book would have been *Introduction to Topology with an Emphasis on Manifolds*.) Finally, it is not intended as a reference book, because few of the results are presented in their most general or most complete forms.

Perhaps the best way to summarize what this book is would be to say that it represents, to a good approximation, my conception of the ideal amount of topological knowledge that should be possessed by beginning graduate students who are planning to go on to study smooth manifolds and differential geometry. Experienced mathematicians will probably observe that my choices of material and approach have been influenced by the fact that I am a differential geometer by training and predilection, not a topologist. Thus I give special emphasis to topics that will be of importance later in the study of smooth manifolds, such as paracompactness, group actions, and degree theory. But despite my prejudices, I have tried to make the book useful as a precursor to algebraic topology courses as well, and it could easily serve as a prerequisite to a more extensive course in homology and homotopy theory.

A textbook writer always has to decide how much detail to spell out, and how much to leave to the reader. It can be a delicate balance. When you dip into this book, you will quickly see that my inclination, especially in the early chapters, is toward writing more detail rather than less. This might not appeal to every reader, but I have chosen this path for a reason. In my experience, most beginning graduate students appreciate seeing many proofs written out in careful detail, so that they can get a clear idea of what goes into a complete proof and what lies behind many of the common "hand-waving" and "standard-argument" moves. There is plenty of opportunity for students to fill in details for themselves—in the exercises and problems—and the proofs in the text tend to become a little more streamlined as the book progresses.

When details are left for the student to fill in, whether as formal exercises or simply as arguments that are not carried out in complete detail in the text, I often try to give some indication about how elaborate the omitted details are. If I characterize some omitted detail as "obvious" or as an "easy exercise," you should take that as an indication that the proof, once you see how to do it, should require only a few steps and use only techniques that are probably familiar from other similar proofs, so if you find yourself constructing a long and involved argument you are probably missing something. On the other hand, if I label an omitted argument "straightforward," it might not be short, but it should be possible to carry it out using familiar techniques without requiring new ideas or tricky arguments. In any case, please do not fall into the trap of thinking that just because I declare something to be easy, you should see instantly why it is true; in fact, nothing is easy or obvious when you are first learning a subject. Reading mathematics is not a spectator sport, and if you really want to understand the subject you will have to get involved by filling in some details for yourself.

Prerequisites

The prerequisite for studying this book is, briefly stated, a solid undergraduate degree in mathematics; but this probably deserves some elaboration.

Traditionally, "general topology" has been seen as a separate subject from "algebraic topology," and most courses in the latter begin with the assumption that the students have already completed a course in the former. However, the sad fact is that for a variety of reasons, many undergraduate mathematics majors in the United States never take a course in general topology. For that reason I have written this book without assuming that the reader has had any exposure to topological spaces.

On the other hand, I do assume several essential prerequisites that are, or should be, included in the background of most mathematics majors.

The most basic prerequisite is a thorough grounding in advanced calculus and elementary linear algebra. Since there are hundreds of books that treat these subjects well, I simply assume familiarity with them, and remind the reader of important facts when necessary. I also assume that the reader is familiar with the terminology and rules of ordinary logic.

The other prerequisites are basic set theory such as what one would encounter in any rigorous undergraduate analysis or algebra course; real analysis at the level of Rudin's *Principles of Mathematical Analysis* [Rud76] or Apostol's *Mathematical Analysis* [Apo74], including, in particular, an acquaintance with metric spaces and their continuous functions; and group theory at the level of Hungerford's *Abstract Algebra: An Introduction* [Hun97] or Herstein's *Abstract Algebra* [Her96].

Because it is vitally important that the reader be comfortable with this prerequisite material, in three appendices at the end of the book I have collected a summary of the main points that are used throughout the book, together with a representative collection of exercises. Students can use the exercises to test their knowledge, or to brush up on any aspects of the subject on which they feel their knowledge is shaky. Instructors may wish to assign the appendices as independent reading, and to assign some of the exercises early in the course to help students evaluate their readiness for the material in the main body of the book, and to make sure that everyone starts with the same background. Of course, if you have not studied this material before, you cannot hope to learn it from scratch here; but the appendices and their exercises can serve as a reminder of important concepts you may have forgotten, as a way to standardize our notation and terminology, and as a source of references to books where you can look up more of the details to refresh your memory.

Organization

The book is divided into thirteen chapters, which can be grouped into an introduction and five major substantive sections.

The introduction (Chapter 1) is meant to whet the student's appetite and create a "big picture" into which the many details can later fit.

The first major section, Chapters 2 through 4, is a brief and highly selective introduction to the ideas of general topology: topological spaces; their subspaces, products, disjoint unions, and quotients; and connectedness and compactness. Of course, manifolds are the main examples and are emphasized throughout. These chapters emphasize the ways in which topological spaces differ from the more familiar Euclidean and metric spaces, and carefully develop the machinery that will be needed later, such as quotient maps, local path connectedness, and locally compact Hausdorff spaces.

The second major section, comprising Chapters 5 and 6, explores in detail some of the main examples that motivate the rest of the theory. Chapter 5 introduces *cell complexes*, which are spaces built up from pieces homeomorphic to Euclidean balls. The focus is *CW complexes*, which are by far the most important type of cell complexes; besides being a handy tool for analyzing topological spaces and building new ones, they play a central role in algebraic topology, so any effort invested in understanding their topological properties will pay off in the long run. The first application of the technology is to prove a classification theorem for 1-dimensional manifolds. At the end of the chapter, I briefly introduce *simplicial complexes*, viewing them as a special class of CW complexes in which all the topology is encoded in combinatorial information. Chapter 6 is devoted to a detailed study of 2-manifolds. After exploring the basic examples of surfaces—the sphere, the torus, the projective plane, and their connected sums—I give a proof of the classification theorem for compact surfaces, essentially following the treatment in [Mas77]. The proof is complete except for the triangulation theorem for surfaces, which I state without proof.

The third major section, Chapters 7 through 10, is the core of the book. In it, I give a fairly complete and traditional treatment of the fundamental group. Chapter 7 introduces the definitions and proves the topological and homotopy invariance of the fundamental group. At the end of the chapter I insert a brief introduction to category theory. Categories are not used in a central way anywhere in the book, but it is natural to introduce them after having proved the topological invariance of the fundamental group, and it is useful for students to begin thinking in categorical terms early. Chapter 8 gives a detailed proof that the fundamental group of the circle is infinite cyclic. Because the circle is the precursor and motivation for the entire theory of covering spaces, I introduce some of the terminology of the latter subject—evenly covered neighborhoods, local sections, lifting—in the special case of the circle, and the proofs here are phrased in such a way that they will apply verbatim to the more general theorems about covering spaces in Chapter 11. Chapter 9 is a brief digression into group theory. Although a basic acquaintance with group theory is an essential prerequisite, most undergraduate algebra courses do not treat free products, free groups, presentations of groups, or free abelian groups, so I develop these subjects from scratch. (The material on free abelian groups is included primarily for use in the treatment of homology in Chapter 13, but some of the results play a role also in classifying the coverings of the torus in Chapter 12.) The last chapter of this section gives the statement and proof of the Seifert–Van Kampen theorem, which expresses the fundamental group of a space in terms of the fundamental groups of appropriate subsets, and describes several applications of the theorem including computation of the fundamental groups of graphs, CW complexes, and surfaces.

The fourth major section consists of two chapters on covering spaces. Chapter 11 defines covering spaces, develops properties of the monodromy action, and introduces homomorphisms and isomorphisms of covering spaces and the universal covering space. Much of the early development goes rapidly here, because it is parallel to what was done earlier in the concrete case of the circle. Chapter 12 explores the relationship between group actions and covering maps, and uses it to prove the classification theorem for coverings: there is a one-to-one correspondence between isomorphism classes of coverings of X and conjugacy classes of subgroups of the fundamental group of X. This is then specialized to *proper* covering space actions on manifolds, which are the ones that produce quotient spaces that are also manifolds. These ideas are applied to a number of important examples, including classifying coverings of the torus and the lens spaces, and proving that surfaces of higher genus are covered by the hyperbolic disk.

The fifth major section of the book consists of one chapter only, Chapter 13, on homology theory. In order to cover some of the most important applications of homology to manifolds in a reasonable time, I have chosen a "low-tech" approach to the subject. I focus mainly on singular homology because it is the most straightforward generalization of the fundamental group. After defining the homology groups, I prove a few essential properties, including homotopy invariance and the Mayer–Vietoris theorem, with a minimum of homological machinery. Then I introduce just enough about the homology of CW complexes to prove the topological invariance of the Euler characteristic. The last section of the chapter is a brief introduction to cohomology, mainly with field coefficients, to serve as background for a treatment of de Rham theory in a later course. In keeping with the overall philosophy of focusing only on what is necessary for a basic understanding of manifolds, I do not even mention relative homology, homology with arbitrary coefficients, simplicial homology, or the axioms for a homology theory.

Although this book grew out of notes designed for a one-quarter graduate course, there is clearly too much material here to cover adequately in ten weeks. It should be possible to cover all or most of it in a semester with well-prepared students.

The book could even be used for a full-year course, allowing the instructor to adopt a much more leisurely pace, to work out some of the problems in class, and to supplement the book with other material.

Each instructor will have his or her own ideas about what to leave out in order to fit the material into a short course. At the University of Washington, we typically do not cover simplicial complexes, homology, or some of the more involved examples of covering maps. Others may wish to leave out some or all of the material on covering spaces, or the classification of surfaces. With students who have had a solid topology course, the first four chapters could be skipped or assigned as independent reading.

Exercises and Problems

As is the case with any new mathematical material, and perhaps even more than usual with material like this that is so different from the mathematics most students have seen as undergraduates, it is impossible to learn the subject without getting one's hands dirty and working out a large number of examples and problems. I have tried to give the reader ample opportunity to do so throughout the book. In every chapter, and especially in the early ones, there are questions labeled as *exercises* woven into the text. Do not ignore them; without their solutions, the text is incomplete. The reader should take each exercise as a signal to stop reading, pull out a pencil and paper, and work out the answer before proceeding further. The exercises are usually relatively easy, and typically involve proving minor results or working out examples that are essential to the flow of the exposition. Some require techniques that the student probably already knows from prior courses; others ask the student to practice techniques or apply results that have recently been introduced in the text. A few are straightforward but rather long arguments that are more enlightening to work through on one's own than to read. In the later chapters, fewer things are singled out as exercises, but there are still plenty of omitted details in the text that the student should work out before going on; it is my hope that by the time the student reaches the last few chapters he or she will have developed the habit of stopping and working through most of the details that are not spelled out without having to be told.

At the end of each chapter is a selection of questions labeled as *problems*. These are, for the most part, harder, longer, and/or deeper than the exercises, and give the student a chance to grapple with more significant issues. The results of a number of the problems are used later in the text. There are more problems than most students could do in a quarter or a semester, so the instructor will want to decide which ones are most germane and assign those as homework.

You will notice that there are no solutions to any of the exercises or problems in the back of the book. This is by design: in my experience, if written solutions to problems are available, then most students (even the most conscientious ones) tend to be irresistibly tempted to look at the solutions as soon as they get stuck. But it is exactly at that stage of being stuck that the deepest learning occurs. It is all too easy for students to read someone else's solution and immediately think "Oh, now I understand that," when in fact they do not understand it nearly as well as they would have if they had struggled through it for themselves. A much more effective strategy for getting unstuck is to talk the problem over with an instructor or a fellow student. Getting suggestions from other people and turning them into an argument of your own are much more useful than reading someone else's complete and polished proof. If you are studying the book on your own, and cannot find any nearby kindred spirits to discuss the problems with, try looking for Internet sites that foster discussions among people studying mathematics, such as *math.stackexchange.com*.

About the Second Edition

Although the basic structure of the book has changed little since the first edition, I have rewritten, rearranged, and (hopefully) improved the text in thousands of small ways and a few large ones; there is hardly a page that has not been touched in one way or another. In some places, I have streamlined arguments and eliminated unnecessary verbiage; in others, I have expanded arguments that were insufficiently clear in the original.

The change that is most noticeable is in Chapter 5: I have eliminated most of the material on simplicial complexes, and replaced it with an introduction to CW complexes. I have come to believe that, totally apart from their central role in homotopy theory, CW complexes are wonderful tools for constructing and analyzing topological spaces in general and manifolds in particular, due to their extreme flexibility and the ease of doing explicit computations with them. Besides, they have the added virtue of giving an early introduction to one of the most important tools of algebraic topology. This change has ramifications throughout the rest of the book, especially in Chapter 10, where the computation of fundamental groups of surfaces is streamlined by considering them as special cases of CW complexes, and in Chapter 13, where the exposition of simplicial homology has been replaced by a much simpler treatment of homology properties of CW complexes.

Apart from the addition of CW complexes, the main substantive changes are expanded treatments of manifolds with boundary, local compactness, group actions, and proper maps; and a new section on paracompactness. I have also reworked the treatment of covering maps in Chapters 11 and 12 in order to use the monodromy action to simplify and unify the classification of coverings (I am indebted to Steve Mitchell for suggesting this). And, of course, I have corrected all the errors in the first edition that I know about. I hope that all of the changes will make the book more useful for future topologists and geometers alike.

There are also a few typographical improvements in this edition. Most important, official definitions of mathematical terms are now typeset in ***bold italics***; this reflects the fact that they are just as important as the theorems and proofs and need to be easy to find, but they fit better into the flow of paragraphs rather than being called out with special headings. In addition, the exercises in the text are now indicated more clearly with a special symbol (►), and numbered consecutively with the theorems to make them easier to find. There also is a new notation index just before the subject index.

Although I have tried hard to find and eradicate mistakes in this edition, sad experience teaches that there will probably be plenty of errors left in the final version of the book. For the sake of future readers, I hope every reader will take the time to keep notes of any mistakes or passages that are awkward or unclear, and let me know about them as soon as it is convenient for you. I will keep an up-to-date list of corrections on my website, whose address is listed below.

Acknowledgments

Those of my colleagues with whom I have discussed this material—Judith Arms, Ethan Devinatz, Tom Duchamp, Steve Mitchell, and Scott Osborne here at UW, and Tracy Payne at Idaho State—have provided invaluable help in sorting out what should go into this book and how it should be presented. Each has had a strong influence on the way the book has come out, for which I am deeply grateful. (On the other hand, it is likely that none of them would wholeheartedly endorse all my choices regarding which topics to treat and how to treat them, so they are not to be blamed for any awkwardnesses that remain.) The students at the University of Washington who have used the book have also been especially thoughtful and generous with their suggestions.

I would particularly like to thank Ethan Devinatz for having had the courage to use a draft first edition of the book as a course text when it was still in an inchoate state, and for having the grace and patience to wait while I prepared chapters at the last minute for his course. And most of all, I owe a huge debt of gratitude to Judith Arms, who has taught several times from various editions of this book, and has given me more good suggestions than everyone else put together. To whatever extent this edition of the book is an improvement over its predecessor, it is due in very large part to her thoughtful assistance.

Beyond those who have helped in person, there are the countless readers all over the world who have sent me their suggestions and corrections over the Internet. I hope each of them will be able to see the ways in which their contributions have improved the book.

Thanks are due also to Mary Sheetz, who did an excellent job producing some of the illustrations for the first edition of the book under the pressures of time and a finicky author.

My debt to the authors of several other textbooks will be obvious to anyone who knows those books: Allan Hatcher's *Algebraic Topology* [Hat02], James Munkres's *Topology* [Mun00] and *Elements of Algebraic Topology* [Mun84], William Massey's *Algebraic Topology: An Introduction* [Mas77], Allan Sieradski's *An Introduction to Topology and Homotopy* [Sie92], and Glen Bredon's *Topology and Geometry* [Bre93] are foremost among them.

Finally, I would like to thank my family once again for their support and patience. Revising a book turns out to take just about as big a toll on personal time as writing one from scratch, and my wife and sons have been ever generous.

Seattle,
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