## Index

30-60-90 triangle, 190, 233
36-72-72 triangle, 226
360 theorem, 96, 97
45-45-90 triangle, 190, 233
60-60-60 triangle, 189

AA congruence theorem for asymptotic triangles, 353
AA similarity theorem, 216
AAA congruence theorem in hyperbolic geometry, 338
AAA construction theorem, 191
AAASA congruence, 197, 354
AAS congruence theorem, 119
AASAS congruence, 179
ABCD property of rigid motions, 441
absolute value, 434
acute angle, 88
acute triangle, 105
adapted coordinate function, 72
adjacency lemma, 98
adjacent angles, 90, 91
adjacent edges of a polygon, 156
adjacent interior angle, 113
admissible decomposition, 201
algebraic number, 317
all-or-nothing theorem, 333
alternate interior angles, 150
alternate interior angles postulate, 323
alternate interior angles theorem, 150
converse, 185, 323
altitude concurrence theorem, 269
altitude of a triangle, 144, 208
intersects interior of a side, 144
to the base of an isosceles triangle, 145
to the hypotenuse, 144
to the longest side, 144
Amtrak model, 29
and (logical conjunction), 385
angle, 83
acute, 88
included between two sides, 104
inscribed in a semicircle, 257
inscribed in an arc, 257
obtuse, 88
of a polygon, 156
of a triangle, 103
of an asymptotic triangle, 351
on a side of a line, 149
opposite a side, 104
proper, 84
right, 88
straight, 84
zero, 84
angle addition theorem, 90
angle bisector, 100,147
angle bisector concurrence theorem, 268
angle bisector proportion theorem, 219
angle bisector theorem, 147
converse, 149
angle construction theorem, 88
angle criterion for convexity, 160
angle measure, 54, 85
between two lines, 357
in taxicab geometry, 138
in the Cartesian plane, 125
in the Poincaré disk, 133
interior, 173
of a regular polygon, 192
properties of, 86
reflex, 87
standard, 87
angle measure postulate, 85
angle measurement postulate (SMSG), 21, 380
angle of parallelism, 367
angle-side-longer-side congruence theorem, 121
angle subtraction theorem, 90
angle sum, 189
of an asymptotic triangle, 351
of two angles of a triangle, 114
angle-sum postulate, 325
weak, 333
angle-sum theorem
for asymptotic triangles, 352
for convex polygons, 191
for convex quadrilaterals, 194
for general polygons, 193
for triangles, 189, 325
hyperbolic, 338
arbitrary, 412
arc, 254
inscribed, 257
intercepted, 257
major, 254
minor, 254
semicircle, 254
arc addition theorem, 261
arc length, 293
arc measure, 257
arccosine, 125
Archimedean property, 438
Archimedes' theorem, 292
area, 199-210
of a circular region, 291-293
of a parallelogram, 209
of a polygon, 200
of a rectangle, 207
of a right triangle, 207
of a sector, 294
of a square, 206
of a trapezoid, 209
of a triangle, 208
area addition property, 200
area congruence property, 200
area function, 200
area paradox, 211
area postulate
Euclidean, 200
hyperbolic, 371
is not independent, 209
area scaling theorem
quadrilateral, 226
triangle, 226
Aristotle, 363
Aristotle's lemma, 363
ASA congruence theorem, 108
ASAAS congruence, 197
ASASA congruence, 197
ASASS congruence, 197
AsS congruence theorem, 121
ASS nontheorem, 120, 122
ASSAS congruence, 197
asymptotic rays, 344
determine parallel lines, 344
endpoint independence, 348
existence and uniqueness, 345
symmetry property, 347
transitive property, 349
asymptotic triangle, 350
AA congruence theorem, 353
angle sum of, 351
angle-sum theorem, 352
copying theorem, 352
defect addition theorem, 353
defect of, 351
exterior angle inequality, 353
exterior angle of, 351
Pasch's theorem for, 353
remote interior angle of, 351
SA congruence theorem, 351
SA inequality, 353
asymptotically parallel lines, 355
distance between, 365
existence and uniqueness, 357
atomic statement, 384
axiom, 13,23
of incidence geometry, 25
axiomatic method, 23
axiomatic system, 13, 23
base
of a parallelogram, 209
of a Saccheri quadrilateral, 341
of a trapezoid, 209
of a triangle, 208
of an isosceles triangle, 105
base angles, 105

Beltrami, Eugenio, 11, 35
Beltrami-Klein disk, 35, 41
betweenness
consistency of, 61, 90
in Hilbert's axioms, 20, 23, 375
of numbers, 59
of points, 60
of rays, 89
symmetry of, 60, 89
vs. betweenness, 99
vs. interior, 97
betweenness axiom, Hilbert's, 61, 375
betweenness theorem
converse, 61, 117
for points, 60
for rays, 89
biconditional statement, 392
bijective function, 57, 429
Birkhoff's axioms, 377
Birkhoff, George D., 20, 53, 377
bisect, 68
bisector
of an angle, 100, 147, 298
perpendicular, 145, 146
Bolyai, János, xiii, 10
boundary of a simple polygonal region, 199
braces in set notation, 424
Cartesian plane, 33-34, 125-132
is a model of Euclidean geometry, 132
is a model of incidence geometry, 33
is a model of neutral geometry, 132
Cartesian product, 426
cases, proof by, 414
categorical axiomatic system, 132, 243
Euclidean geometry, 132, 243
center
of a circle, 70, 247
of a regular polygon, 271
central angle
of a circle, 257
of a regular polygon, 271
centroid, 224
Ceva's theorem, 223
Ceva, Giovanni, 223
cevian, 223
chord
of a circle, 35, 247, 250
of a polygon, 161, 162
circle, 70, 247
area of, 293
circumference of, 289
through three noncollinear points, 264 unit, 289
circular region, 290
area of, 291
circumcenter, 262
circumcircle, 262
for a triangle, 263
circumcircle postulate, 335
circumcircle theorem, 263
circumference, 284
approximation by regular polygons, 285
formula for, 289
scaling theorem, 288
circumscribed circle, 262
constructing, 305
for a triangle, 263
circumscribed polygon, 265
is convex, 266
regular, 273
Clairaut's postulate, 330
Clairaut, Alexis, 330
classification of parallels through a point, 357
closed half-plane, 77
closed interval, 434
closest point, 146
on a line, 147
on a segment, 147
closure properties of numbers, 435
codomain of a function, 428
coefficient of a polynomial, 314
collection (synonym for set), 423
collinear points
in incidence geometry, 25
in plane geometry, 55
collinear rays, 74
common notions
for angles, 90
for segments, 66
of Euclid, 4
common perpendicular, $151,187,359,360$
uniqueness, 359
common perpendicular theorem, 151
converse, 187
compass and straightedge, $3,295,296$
complementary angles, 91
components
of a vector, 124
of an ordered pair, 426
composition of rigid motions, 449
compound statement, 385
concave polygon, 158
concave vertex, 172
concentric circles, 247, 248
conclusion
in Greek proofs, 7
of a rule of inference, 408
of an implication, 388
concurrence theorem
altitude, 269
angle bisector, 268
median, 224
perpendicular bisector, 264
concurrent, 223, 262
conditional statement, 388
congruence
in Hilbert's axioms, 20, 23, 375, 376
of angles, 88
of asymptotic triangles, 351
of convex polygons, 159
of polygons, 173
of segments, 64
of simple polygonal regions, 199
of triangles, 106
via rigid motions, 446
congruent, see congruence
conjugate arcs, 254,257
conjunction, 385
connective, logical, 385
consecutive angles of a polygon, 156
consecutive interior angles, 150
consecutive interior angles theorem, 151
converse, 186
consecutive vertices of a polygon, 156
consistency
of an axiomatic system, 32
of betweenness of points, 61
of betweenness of rays, 90
of endpoints of arcs, 256
of endpoints of rays, 76
of endpoints of segments, 66
of polygon vertices, 157,159
of triangle vertices, 103
of vertices of proper angles, 84
relative, 32
constant of proportionality, 214
constructible angle, 313
constructible circle, 308
constructible line, 308
constructible number, 308
characterization of, 311
constructible point, 308
constructing a perpendicular, 101
constructing a rectangle, 195
constructing a square, 196
construction
compass and straightedge, 296-319
in Greek proofs, 3, 6, 7
constructive proof, 417
contains (a point)
in incidence geometry, 24
in plane geometry, 55
contrapositive, 391
proof by, 411
converse, 391
convex decomposition lemma, 201
convex polygon, 157-166
angle criterion, 160
characterizations of, 173
diagonal criterion, 176
interior of, 164, 171
semiparallel criterion, 161, 176
vertex criterion, 160
convex quadrilateral, 175-183
diagonal criterion, 176
semiparallel criterion, 176
convex set, 80
intersection of, 80
convex vertex, 172
coordinate
of a point, 57
of a ray, 86
coordinate function
adapted to a ray, 72
for a half-rotation, 86
for a line, 57
starting at a ray, 86
coordinate representation
of a ray, 72
of a segment, 67
copying theorem
asymptotic triangle, 352
quadrilateral, 179, 300
triangle, 111, 300
corollary, 6, 24
correspondence, one-to-one, 429
corresponding angles, 150
corresponding angles theorem, 151
converse, 186
cosine, 238, 239
is injective, 239
cosines, law of, 240
cross lemma, 177
crossbar theorem, 106
cubic polynomial, 314
cut by a transversal, 149
cutting off a segment, 74, 297
cyclic polygon, 262, 263
is convex, 262
cyclic quadrilateral, 264, 276
cyclic triangle theorem, 263
dart, 183
decagon, 157
decomposition lemma
convex, 201
parallelogram, 203
defect
of a polygon, 328,371
of a triangle, 328
of an asymptotic triangle, 351
defect addition theorem, 328
for asymptotic triangles, 353
defect formula, 372
defined terms, 23
definition
descriptive, 2
logical, 2
mathematical, 401
degree
in angle measure, 85
of a polynomial, 314
density of real and rational numbers, 438
Descartes, René, 33, 311
descriptive definition, 2
diagonal
of a polygon, 156
of a square, 234
diagonal criterion for convex quadrilaterals, 176
diagonal scaling theorem, 226
diameter of a circle, 247
length of, 248
difference of sets, 425
different (meaning not equal), 25, 393
differential geometry, 135, 373-374
direct proof, 410
disjoint sets, 425
disjunction, 386
proof of, 415
displacement vector, 124
distance
as a primitive term, 54
between real numbers, 57
from a point to a line, 147
from a point to a segment, 147
from a point to a set, 146
in plane geometry, 54,56
in taxicab geometry, 138
in the Cartesian plane, 125
in the Poincaré disk, 132
properties of, 59
distance postulate, 56
distance-preserving function, 57
distinct (meaning not equal), 25, 393
divides, 434
divisible, 434
dodecagon, 157
domain
of a function, 428
of a variable, 394
dot product, 125
double angle formula
for cosine, 242
for sine, 242
double elliptic geometry, 35
doubling a cube, 308,317
doubling a square, 305
dropping a perpendicular, 142
edge
finite, of an asymptotic triangle, 351
of a polygon, 156
of a triangle, 103
edge line, 158
edge-line lemma, 158
element of a set, 423
Elements of Euclid, 1
elliptic geometry, 11
double, 35
single, 35, 41, 136
elliptic parallel postulate, 38
empty set, 424
endpoint
of a ray, 71
of a segment, 65
of an arc, 254
endpoint independence of asymptotic rays, 348
enunciation (in Greek proofs), 6
equality
in Euclid's Elements, 3
in modern mathematics, 25, 393
of functions, 428
of ordered pairs, 426
of sets, 424
equiangular polygon, 159
equiangular quadrilateral, 181, 182, 194
equiangular triangle, 105,110
equidistance postulate, 323
equidistance theorem, 152
converse, 187
equidistant
from a line, 151
from two lines, 147
from two points, 68
equidistant lines, $151,152,187,362$
symmetry of, 188
equilateral polygon, 159
equilateral triangle, 104, 110
construction theorem, 191, 296
equivalence, 392
proof of, 413
equivalence relation, 393
equivalent postulates, 191, 321
Erchinger, Johannes, 318
Erlangen program, 373
Euclid, 1
Euclid's Elements, 1
Euclid's Postulate 1, 3
Euclid's Postulate 2, 3
Euclid's Postulate 3, 3, 71
Euclid's Postulate 4, 3, 88
Euclid's Postulate 5, 3-4, 8-10, 322
implied by Euclidean parallel postulate, 190
implies Euclidean parallel postulate, 322
is independent, 134
Euclid's segment cutoff theorem, 74
Euclidean area postulate, 200
Euclidean geometry, 53, 123, 185
is categorical, 132, 243
is consistent, 132
postulates of, 123, 382
Euclidean parallel postulate, 38, 185
implied by Euclid's Postulate 5, 322
implies Euclid's Postulate 5, 190
is independent, 133
Euclidean postulates, 321-333
Euler line theorem, 276
even integer, 434
even parity, 167
every triangle has two equal sides (fake theorem), 18
everywhere defined function, 428
exhaustion, method of, 279
existence
and uniqueness, 398, 419
of parallels, 153
of two distinct points, 55
proof of, 417-419
statement of, 397
existence postulate, 55
existential quantifier, 397
existential statement, 397
extension of a field, 309
exterior
of a circle, 248
of a polygon, 164, 167
of an angle, 95
exterior angle
formed by a transversal, 150
of a convex polygon, 192
of a triangle, 113, 189
of an asymptotic triangle, 351
exterior angle inequality, 17, 113
for asymptotic triangles, 353
exterior angle sum for a convex polygon, 192
extreme and mean ratio, 225
extreme point, 65
weak, 174
fake theorem (every triangle has two equal sides), 18
family (synonym for set), 423
Fano plane, 28
Fermat prime, 318
Fermat, Pierre de, 318
field, 309
fifteen-sided polygon, 307
finite edge of an asymptotic triangle, 351
five-point plane, 27-29, 39
fixed point of a rigid motion, 443
foot of a perpendicular, 143
for all (universal quantifier), 394
four-point plane, 27-29, 38
four right angles theorem, 101
fourth angle of a Lambert quadrilateral, 342
fourth vertex of a Lambert quadrilateral, 342
free variable, 394
function, 24, 428
gaps in Euclid's arguments, 13-18
Garfield, James A., 245
Gauss, Carl Friedrich, xiii, 10, 318
general polygonal region, 200
general triangle inequality, 116
geometric mean, 236
constructing, 301
Gergonne point, 276

Gergonne, Joseph, 276
glide reflection, 442, 449
golden ratio, 224
constructing, 302
golden rectangle, 224
golden triangle, 225, 226
gravitational lensing, 12
great circle, 11, 34
great hyperbola, 37
greatest lower bound, 435
gremlin, 409, 412, 417
group, 373
half-plane
closed, 77
is convex, 80
open, 77
Poincaré, 36
half-rotation, 85
height
of a parallelogram, 209
of a trapezoid, 209
of a triangle, 208
height scaling theorem, 226
heptadecagon, 318
heptagon, 157, 319
Heron of Alexandria, 243
Heron's formula, 243
hexagon, 157
constructing, 307
Hilbert's axioms, 375-376
Hilbert's betweenness axiom, 61, 375
Hilbert, David, 20, 375
hinge theorem, 117
HL congruence theorem, 121
horizontal line, 33
hyperbolic angle-sum theorem, 338
hyperbolic geometry, 11, 53, 123, 337-367
is consistent, 133
is not categorical, 339
postulates of, 123, 382
hyperbolic parallel postulate, 38, 337
hyperboloid model, 37, 41
hyperboloid, two-sheeted, 37
hypotenuse, 105
is the longest side, 116
hypothesis, 388, 406
if and only if, 392
if-then statement, 388
iff, 392
implication, 388
implicit universal quantifier, 397
implies (logical connective), 388
incenter, 265
incidence axioms, 25
in neutral geometry, 57
incidence geometry, 24-50
incidence theorems, 42-50
incircle, 265
for a triangle, 268
incircle theorem, 268
included angle, 104
included side, 104
inconsistent axiomatic system, 32
independence
of a statement, 32
of axioms, 32, 139
of Euclid's Postulate 5, 134
of the Euclidean parallel postulate, 133
of the neutral postulates, 139
indirect proof, 416
induction, mathematical, 420-422
injective function, 56, 429
inscribed angle in a circle, 257
inscribed angle theorem, 259
inscribed arc, 257
inscribed circle, 265
constructing, 305
for a triangle, 268
inscribed polygon, 262
is convex, 262
regular, 272
integer, 424, 433
intercepted arc, 257
interior
of a circle, 248
of a convex polygon, 164, 171
of a polygon, 164, 167
of a ray, 71
of a segment, 65
of a simple polygonal region, 199
of an angle, 94,95
of an arc, 254
vs. betweenness, 97
interior angle
adjacent, 113
alternate, 150
formed by a transversal, 149
of a triangle, 113
of an asymptotic triangle, 351
remote, 113
interior angle measure, 159,173
interior lemma, 97
interpretation of an axiomatic system, 26
intersecting chords theorem, 261
intersecting lines
in incidence geometry, 24
in plane geometry, 55
intersecting secants theorem, 261
intersection of sets, 425
inward-pointing ray, 172
irrational number, 434
irrationality of $\sqrt{2}, 5$
isomorphic models, 29
isomorphism between models, 29, 132, 243
isosceles right triangle, 190, 233
isosceles triangle, 104
base, 105
base angles, 105
isosceles triangle altitude theorem, 145
isosceles triangle theorem, 109
converse, 110
Pappus's proof, 110
iterated quadratic extension, 311
Jordan polygon theorem, 167
justifications for steps in a proof, 406
Khayyam, Omar, 9, 10, 341
kite, 183, 210
Klein geometry, 373
Klein, Felix, 35, 373
Lambert quadrilateral, 341-343
Lambert, Johann Heinrich, 341
law of cosines, 240
law of sines, 241
least upper bound, 280, 435, 438
least upper bound property, 280
leg
of a right triangle, 105
of a Saccheri quadrilateral, 341
Legendre, Adrien-Marie, 326
lemma, 6, 24
length
of a diameter, 248
of a segment, 64
of a vector, 125
of an arc, 293
let, in mathematical proofs, 411
lies on
as a primitive term, 24
in incidence geometry, 24
in plane geometry, 55
line
as a primitive term, 24,54
contains infinitely many points, 57
Euclid's definition, 2
in incidence geometry, 24
in plane geometry, 54
in single elliptic geometry, 35, 136
in spherical geometry, 34,135
in taxicab geometry, 138
in the Beltrami-Klein disk, 35
in the Cartesian plane, 33, 125
in the Poincaré disk, 35, 132
in the Poincaré half-plane, 36
in the rational plane, 136
is convex, 80
straight, 2
line segment, see segment
line-circle theorem, 250
linear pair, 92
linear pair theorem, 92
converse, 93
linear polynomial, 314
linear triple, 94,101
linear triple theorem, 94
Lobachevskian geometry, 337
Lobachevsky, Nikolai, xiii, 10
logic, laws of, 408
logical connective, 385
logical definition, 2
lower bound, 435
lowest terms, 438
major arc, 254
map, 428
mapping, 428
mathematical definition, 401
mathematical induction, 420-422
mathematical object, 24, 383, 423
mathematical relation, 384
mathematical statement, 383
mean proportional, 236
measure
of an angle, see angle measure
of an arc, see arc measure
median
of a triangle, 145
of an isosceles triangle, 145
median concurrence theorem, 224
meet
in incidence geometry, 24
in plane geometry, 55
member of a set, 423
membership criterion for a set, 423
Menalaus of Alexandria, 221
Menelaus's theorem, 221
meridian circle, 35
method of exhaustion, 279
method of superposition, 15
midpoint, 68
existence and uniqueness, 69
midsegment
of a Saccheri quadrilateral, 342
of a triangle, 196
midsegment theorem, 196
minor arc, 254
model of an axiomatic system, 26
monic polynomial, 314
multiple quantifiers, 399
multiplicative inverse, 434
$n$-gon, 157
$n$-point plane, 27-29
necessary condition, 389
negation, 387
negative number, 434
neutral geometry, 53
is consistent, 132
postulates of, 123, 381
nine-point circle theorem, 277
non-Euclidean geometry, 10-13
nonagon, 157
noncollinear, 25
nonconvex polygon, 166-174
nonexistence statement, 399, 420
nonnegative, 434
nonoverlapping regions, 199
nonpositive, 434
nonvertical line, 33, 125, 136
nonzero, 434
north pole, 35
not (logical negation), 387
object, mathematical, 24, 383, 423
obtuse angle, 88
obtuse triangle, 105
octagon, 157
constructing, 307
odd integer, 434
odd parity, 167
one-dimensional quantity, 226
one-point geometry, 30,135
one-to-one correspondence, 429
one-to-one function, 429
one-two geometry, 31
onto, 429
open half-plane, 77
open interval, 434
open sentence, 394
opposite angle
of a quadrilateral, 175
of a triangle, 104
opposite ray theorem, 75
opposite rays, 74,75
opposite side
of a line, 76
of a quadrilateral, 175
of a triangle, 104
opposite vertices of a quadrilateral, 175
or (logical disjunction), 386
ordered $n$-tuple, 426
ordered pair, 426
equality of, 426
ordering lemma
for points, 73
for rays, 98
origin in the Cartesian plane, 124
orthocenter, 269
outward-pointing ray, 172
overlapping regions, 199
$\pi$, definition of, 289
$\mathcal{P}$-side of an edge line, 158
pair, ordered, 426
Pappus of Alexandria, 110
parallel lines
are equidistant, 187
constructing, 300
existence of, 153
in incidence geometry, 25
in plane geometry, 55, 149-154
parallel postulate
elliptic, 38
Euclidean, 38, 185
hyperbolic, 38, 337
in single elliptic geometry, 41
in spherical geometry, 41
in the Beltrami-Klein disk, 41
in the Cartesian plane, 40
in the hyperboloid model, 41
in the Poincaré disk, 41
in the Poincaré half-plane, 41
independence, in incidence geometry, 39
independence, in neutral geometry, 133
parallel projection theorem, 220
parallelism, transitivity of, 188, 323
parallelogram, 175, 194
area of, 209
congruent opposite angles, 180, 194
congruent opposite sides, 181, 194
diagonals bisect each other, 182, 194
is convex, 177
properties, 194
parallelogram decomposition lemma, 203
parallelogram lemma, 178
parity
of a point, 167
of a ray, 167
Pascal's mystic hexagon, 277
Pasch's axiom, 105
Pasch's theorem, 105
for asymptotic triangles, 353
Pasch, Moritz, 105
passing point, 65
strong, 174
path, polygonal, 167
pentagon, 157
constructing, 306
perimeter of a polygon, 226
perimeter scaling theorem, 226
perpendicular bisector, 145
constructing, 299
perpendicular bisector concurrence theorem, 264
perpendicular bisector theorem, 145
converse, 146
perpendicular lines, 101, 141
constructing, 101, 299, 300
dropping, 142, 300
Euclid's definition, 2
perpendicular rays, 141
perpendicular segments, 141
pi, definition of, 289
pizza lemma, 202
plane geometry, 53
plane separation postulate, 76
plane, the, 55
Playfair's postulate, 10, 323
Playfair, John, 10
Poincaré disk, 35-36, 41, 132-135
is a model of hyperbolic geometry, 133
is a model of incidence geometry, 36
is a model of neutral geometry, 133
Poincaré half-plane, 36,41
is a model of incidence geometry, 36
point
as a primitive term, 24,54
Euclid's definition, 2
in incidence geometry, 24
in plane geometry, 54
in single elliptic geometry, 35, 136
in spherical geometry, 34,135
in taxicab geometry, 138
in the Beltrami-Klein disk, 36
in the Cartesian plane, 33, 125
in the Poincaré disk, 35, 132
in the Poincaré half-plane, 36
in the rational plane, 136
point of tangency, 249
polygon, 155
area of, 200
concave, 158
convex, 157-166, 173
interior of, 164, 171
nonconvex, 166-174
regular, 159, 271-273
polygon splitting theorem, 163
polygonal path, 167
polygonal region, 200
general, 200
simple, 199
polynomial, 314
positive number, 434
postulate, 13, 23
angle measure, 85
area, 209
distance, 56
elliptic parallel, 38
Euclidean area, 200
Euclidean parallel, 38, 185
existence, 55
hyperbolic parallel, 38, 337
of Euclid, 3
of Euclidean geometry, 123, 382
of hyperbolic geometry, 123, 382
of incidence geometry, 25
of neutral geometry, 123, 381
plane separation, 76
protractor, 85
reflection, 443
ruler, 57
SAS, 108
set, 55
unique line, 56
predicate, 394
premise of a rule of inference, 408
prime number, 434
primitive term, 23, 54
Proclus, 6, 9, 186

Proclus's lemma, 186
Proclus's postulate, 323
product, Cartesian, 426
projection of a leg, 236
proof, 405
by cases, 414
by contradiction, 416
by contrapositive, 411
by induction, 420
constructive, 417
direct, 410
indirect, 416
justifications in, 406-410
of existence, 417,418
of existence and uniqueness, 419
stages of writing, 42
structure of, 405
templates for, 405-422
proper angle, 84
proper subset, 424
proper superset, 424
proportion, 213
proportion theorem
angle bisector, 219
right triangle, 237
triangle area, 209
proportional, 213
proposition, 6, 24
protractor postulate, 85
Pythagorean identity, 240
Pythagorean postulate, 335
Pythagorean theorem, 229-233, 237
converse, 233
Euclid's first proof, 230
Euclid's second proof, 237
Garfield's proof, 245
proof by decomposition, 232
proof by similar triangles, 237
q.e.d. (quod erat demonstrandum), 7
q.e.f. (quod erat faciendum), 7
quadratic extension of a field, 310
iterated, 311
quadratic polynomial, 314
quadrilateral, 156, 175
area scaling theorem, 226
convex, 175-183
copying theorem, 179
cyclic, 264
tangential, 269
quantifier, 394
existential, 397
multiple, 399
negating, 399
universal, 394, 395
quod erat demonstrandum, 7
quod erat faciendum, 7
radii, see radius
radius, 70, 247
range of a function, 428
rating system for proofs, 54
ratio, 213
rational number, 434
rational plane, 136
ray, 71
collinear, 74
coordinate representation of, 72
is convex, 81
lying in the interior of an angle, 95
lying on a side of a line, 78, 149
opposite, 74, 75
with the same endpoint, 74
real number, 53, 424, 433-439
reciprocal, 434
rectangle, $175,194,195$
area of, 207
construction theorem, 195, 300
golden, 224
is a parallelogram, 175
is convex, 177
nonexistent in hyperbolic geometry, 337
rectangular region, 199
reflection across a line, 146, 442, 443
reflection postulate, 443
reflex angle, 88
reflex measure of an angle, 87
reflexive property of congruence
of angles, 90
of segments, 66
reflexive property of equality, 393
region
circular, 290
determined by a polygon, 164, 199
polygonal, 199, 200
regular heptagon, 308
regular polygon, 159, 271, 272
angle measures of, 192
center of, 271
central angle of, 271
circumscribed, 273
constructing, 317-319
fifteen-sided, 307
inscribed in a circle, 272
is cyclic and tangential, 271
regular quadrilateral, 182, 183, 194
relation, mathematical, 24, 384
relative consistency, 32
remote interior angle
in a triangle, 113
in an asymptotic triangle, 351
rhombus, 175, 182
is a parallelogram, 182
Riemann, Bernhard, 11
Riemannian geometry, 11
right angle
Euclid's definition, 2
in plane geometry, 88,92
in the Cartesian plane, 126
right triangle, 105, 229-243
area of, 207
hypotenuse, 105
isosceles, 233
leg, 105
proportion theorem, 237
similarity theorem, 236
rigid motion, 441
root of a polynomial, 314
rotation, 442, 449
rule of inference, 408
ruler flipping lemma, 58
ruler placement theorem, 58
ruler postulate, 57
SMSG, 21, 379
ruler sliding lemma, 58
SA congruence theorem, 351
SA inequality for asymptotic triangles, 353
SAAAS congruence, 197
SAASS congruence, 197
Saccheri quadrilateral, 340, 342, 343
Saccheri's repugnant theorem, 365
Saccheri, Giovanni, 9, 10, 326, 341, 367
Saccheri-Legendre theorem, 327
for convex polygons, 330
SAS congruence theorem of Euclid, 107
SAS postulate, 108
SAS similarity postulate, 335
SAS similarity theorem, 218
SASAS congruence, 179
SASSS congruence, 197
scale factor, 214
scalene inequality, 115
for quadrilaterals, 342
scalene triangle, 104
scaling theorem
circumference, 288
diagonal, 226
height, 226
perimeter, 226
quadrilateral area, 226
triangle area, 226
School Mathematics Study Group, 21, 379
secant line, 249
sector of a circle, 294
area of, 294
segment, 64
contains infinitely many points, 70
coordinate representation of, 67
is convex, 80
lying on a side of a line, 149
segment addition theorem, 66
segment construction theorem, 74
segment cutoff theorem, Euclid's, 74
segment extension theorem, 65
segment subtraction theorem, 66
semicircle, 254
semiparallel criterion for convexity, 161
of quadrilaterals, 176
semiparallel segments, 160
semiperimeter, 243
set, 423
defined by a list, 424
defined by specification, 424
membership criterion, 423
set-builder notation, 425
set difference, 425
set equality, 424
set postulate, 55
set theory, 423-431
setting out (in Greek proofs), 6
side
included between two angles, 104
of a line, 76
of a polygon, 156
of a triangle, 103
of an angle, 83
opposite an angle, 104
side-angle-side congruence theorem of Euclid, 107
side-angle-side postulate, 108
side-side-side congruence theorem, 107
side-splitter theorem, 215
converse, 219
similar polygons, 213
similar triangle construction theorem, 217
similarity theorem

AA, 216
right triangle, 236
SAS, 218
SSS, 218
similarity, transitive property of, 214
simple polygonal region, 199
simple statement, 384
sine, 238
sines, law of, 241
single elliptic geometry, 35, 41, 136
singleton, 424
slope of a line, 33
small angle lemma, 325
SMSG (School Mathematics Study Group), 21, 379
SMSG postulates, 379-380
south pole, 35
specification
in Greek proofs, 7
to define a set, 424
spherical geometry, 34, 41, 135
spherical trigonometry, 371
square, 175,194
area of, 206
constructing, 196, 300, 305
diagonal of, 234
square geometry, 31
square of a number, 434
square region, 199
square root, 435, 438
square root of 2 is irrational, 5
squaring
a circle, 308, 317
a figure, 304
a polygon, 304
a rectangle, 304
SSS congruence theorem, 107, 111
SSS construction theorem, 235
SSS existence theorem, 234
SSS similarity theorem, 218
standard measure of an angle, 87
starting point of a ray, 71
statement
atomic, 384
compound, 385
existential, 397
mathematical, 383
simple, 384
universal, 394
straight angle, 84
straight line, 2
straightedge, 3, 295, 296
strong passing point, 174
subfield, 309
subpolygons cut off by a chord, 162
subset, 424
substitution property of equality, 393
such that
in existence statements, 397
in logical conjunctions, 385
sufficient condition, 389
sum formula
for cosine, 241
for sine, 241
summit angles, 341
summit of a Saccheri quadrilateral, 341
superposition, method of, 15
superset, 424
supplementary angles, 91
supremum, 280
surjective function, 56,429
symmetric property of equality, 393
symmetry, 372
of a Klein geometry, 373
of asymptotic rays, 347
of betweenness of points, 60
of betweenness of rays, 89
of equidistant lines, 188
symmetry group, 373
tangent circles, 253
tangent circles theorem, 254
tangent line, 249
existence and uniqueness, 252,258
exterior to circle, 252
properties of, 252
tangent line theorem, 251
tangent segment to a circle, 265
tangential polygon, 265, 268
is convex, 266
tangential quadrilateral, 269, 276
tangential triangle theorem, 268
taxicab distance, 138
taxicab geometry, 138
Thales of Miletus, 257
Thales' postulate, 335
Thales' theorem, 257
converse, 258
theorem, 6, 23, 405
in an axiomatic system, 23
in Euclid's Elements, 6
there exists (existential quantifier), 397
three-point line, 31
three-point plane, 26, 38, 135
three-ring geometry, 31
three-two geometry, 51
transcendental number, 317
transformation, 108, 441
transitive property of asymptotic rays, 349
transitive property of congruence
of angles, 90
of segments, 66
of triangles, 107
transitive property of equality, 393
transitive property of similarity, 214
transitivity of parallelism, 188
transitivity postulate, 323
translation, 442, 449
transversal
for a triangle, 221
for two lines, 149
trapezoid, 175
area of, 209
is convex, 177
trapezoid lemma, 178
triangle, 103
acute, 105
area of, 207, 208
constructing, 307
equiangular, 105
equilateral, 104
is convex, 160
is cyclic, 263
is tangential, 268
isosceles, 104
obtuse, 105
right, 105
scalene, 104
triangle area proportion theorem, 209
triangle area scaling theorem, 226
triangle copying theorem, 111
triangle inequality, 116
for real numbers, 437
general, 116
triangle sliding theorem, 208
triangular region, 199
trichotomy law, 437
trigonometry, 238
triple angle formula
for cosine, 243
for sine, 243
trisecting an angle, 307, 313-316
truth table, 385
two circles theorem, 252
two-column proof, 405
two-dimensional quantity, 226
two-point equidistance theorem, 152
two-point formula for a line, 34
two-sheeted hyperboloid, 37
two transversals theorem, 218
ultraparallel lines, 355, 360
admit a common perpendicular, 360
distance between, 364
ultraparallel theorem, 360
union, 425
unique line postulate, 56
unique point theorem, 74
unique ray theorem, 89
unique triangle theorem, 111
uniquely defined function, 428
uniqueness, 398
of common perpendicular, 359
proof of, 419
unit circle, 289
universal implication, 412
proof of, 412
universal quantifier, 394, 395
implicit, 397
universal statement, 394
upper bound, 280, 435
least, 280
vacuously true statement, 396
value of a function, 428
variable, 394
free, 394
Varignon's theorem, 197
vector, 124
components, 124
difference, 125
displacement, 124
sum, 125
vertex
of a polygon, 156
of a triangle, 103
of an angle, 83
of an asymptotic triangle, 351
vertex criterion for convexity, 160
vertical angles, 93
vertical angles theorem, 93
converse, 94
vertical line, 33, 125, 136
vertices, see vertex
Wallis's postulate, 324

Wallis, John, 324
Wantzel, Pierre, 316, 319
weak angle-sum postulate, 333
weak extreme point, 174
well defined, 428
well-ordering principle, 438
whole is greater than the part angle, 90
converse, 73, 97
in Euclid's Elements, 4
segment, 66
X-lemma, 79
Y-lemma, 78
zero angle, 84

