Index

30-60-90 triangle, 190, 233 36-72-72 triangle, 226 360 theorem, 96, 97 45-45-90 triangle, 190, 233 60-60-60 triangle, 189

AA congruence theorem for asymptotic triangles, 353 AA similarity theorem, 216 AAA congruence theorem in hyperbolic geometry, 338 AAA construction theorem, 191 AAASA congruence, 197, 354 AAS congruence theorem, 119 AASAS congruence, 179 ABCD property of rigid motions, 441 absolute value, 434 acute angle, 88 acute triangle, 105 adapted coordinate function, 72 adjacency lemma, 98 adjacent angles, 90, 91 adjacent edges of a polygon, 156 adjacent interior angle, 113 admissible decomposition, 201 algebraic number, 317 all-or-nothing theorem, 333 alternate interior angles, 150 alternate interior angles postulate, 323 alternate interior angles theorem, 150 converse, 185, 323 altitude concurrence theorem, 269 altitude of a triangle, 144, 208

intersects interior of a side, 144 to the base of an isosceles triangle, 145 to the hypotenuse, 144 to the longest side, 144 Amtrak model, 29 and (logical conjunction), 385 angle, 83 acute, 88 included between two sides, 104 inscribed in a semicircle, 257 inscribed in an arc, 257 obtuse, 88 of a polygon, 156 of a triangle, 103 of an asymptotic triangle, 351 on a side of a line, 149 opposite a side, 104 proper, 84 right, 88 straight, 84 zero, 84 angle addition theorem, 90 angle bisector, 100, 147 angle bisector concurrence theorem, 268 angle bisector proportion theorem, 219 angle bisector theorem, 147 converse, 149 angle construction theorem, 88 angle criterion for convexity, 160 angle measure, 54, 85 between two lines, 357 in taxicab geometry, 138 in the Cartesian plane, 125

in the Poincaré disk, 133 interior, 173 of a regular polygon, 192 properties of, 86 reflex, 87 standard, 87 angle measure postulate, 85 angle measurement postulate (SMSG), 21, 380 angle of parallelism, 367 angle-side-longer-side congruence theorem, 121 angle subtraction theorem, 90 angle sum, 189 of an asymptotic triangle, 351 of two angles of a triangle, 114 angle-sum postulate, 325 weak, 333 angle-sum theorem for asymptotic triangles, 352 for convex polygons, 191 for convex quadrilaterals, 194 for general polygons, 193 for triangles, 189, 325 hyperbolic, 338 arbitrary, 412 arc, 254 inscribed, 257 intercepted, 257 major, 254 minor, 254 semicircle, 254 arc addition theorem, 261 arc length, 293 arc measure, 257 arccosine, 125 Archimedean property, 438 Archimedes' theorem, 292 area, 199-210 of a circular region, 291-293 of a parallelogram, 209 of a polygon, 200 of a rectangle, 207 of a right triangle, 207 of a sector, 294 of a square, 206 of a trapezoid, 209 of a triangle, 208 area addition property, 200 area congruence property, 200 area function, 200

area paradox, 211 area postulate Euclidean, 200 hyperbolic, 371 is not independent, 209 area scaling theorem quadrilateral, 226 triangle, 226 Aristotle, 363 Aristotle's lemma, 363 ASA congruence theorem, 108 ASAAS congruence, 197 ASASA congruence, 197 ASASS congruence, 197 AsS congruence theorem, 121 ASS nontheorem, 120, 122 ASSAS congruence, 197 asymptotic rays, 344 determine parallel lines, 344 endpoint independence, 348 existence and uniqueness, 345 symmetry property, 347 transitive property, 349 asymptotic triangle, 350 AA congruence theorem, 353 angle sum of, 351 angle-sum theorem, 352 copying theorem, 352 defect addition theorem, 353 defect of, 351 exterior angle inequality, 353 exterior angle of, 351 Pasch's theorem for, 353 remote interior angle of, 351 SA congruence theorem, 351 SA inequality, 353 asymptotically parallel lines, 355 distance between, 365 existence and uniqueness, 357 atomic statement, 384 axiom, 13, 23 of incidence geometry, 25 axiomatic method, 23 axiomatic system, 13, 23 base

of a parallelogram, 209 of a Saccheri quadrilateral, 341 of a trapezoid, 209 of a triangle, 208 of an isosceles triangle, 105 base angles, 105 Beltrami, Eugenio, 11, 35 Beltrami-Klein disk, 35, 41 betweenness consistency of, 61, 90 in Hilbert's axioms, 20, 23, 375 of numbers, 59 of points, 60 of rays, 89 symmetry of, 60, 89 vs. betweenness, 99 vs. interior, 97 betweenness axiom, Hilbert's, 61, 375 betweenness theorem converse, 61, 117 for points, 60 for rays, 89 biconditional statement, 392 bijective function, 57, 429 Birkhoff's axioms, 377 Birkhoff, George D., 20, 53, 377 bisect, 68 bisector of an angle, 100, 147, 298 perpendicular, 145, 146 Bolyai, János, xiii, 10 boundary of a simple polygonal region, 199 braces in set notation, 424 Cartesian plane, 33-34, 125-132 is a model of Euclidean geometry, 132 is a model of incidence geometry, 33 is a model of neutral geometry, 132 Cartesian product, 426 cases, proof by, 414 categorical axiomatic system, 132, 243 Euclidean geometry, 132, 243 center of a circle, 70, 247 of a regular polygon, 271 central angle of a circle, 257 of a regular polygon, 271 centroid. 224 Ceva's theorem, 223 Ceva, Giovanni, 223 cevian. 223 chord of a circle, 35, 247, 250 of a polygon, 161, 162 circle, 70, 247 area of. 293 circumference of, 289

through three noncollinear points, 264 unit, 289 circular region, 290 area of, 291 circumcenter, 262 circumcircle, 262 for a triangle, 263 circumcircle postulate, 335 circumcircle theorem, 263 circumference, 284 approximation by regular polygons, 285 formula for, 289 scaling theorem, 288 circumscribed circle, 262 constructing, 305 for a triangle, 263 circumscribed polygon, 265 is convex, 266 regular, 273 Clairaut's postulate, 330 Clairaut, Alexis, 330 classification of parallels through a point, 357 closed half-plane, 77 closed interval, 434 closest point, 146 on a line, 147 on a segment, 147 closure properties of numbers, 435 codomain of a function, 428 coefficient of a polynomial, 314 collection (synonym for set), 423 collinear points in incidence geometry, 25 in plane geometry, 55 collinear rays, 74 common notions for angles, 90 for segments, 66 of Euclid, 4 common perpendicular, 151, 187, 359, 360 uniqueness, 359 common perpendicular theorem, 151 converse, 187 compass and straightedge, 3, 295, 296 complementary angles, 91 components of a vector, 124 of an ordered pair, 426 composition of rigid motions, 449 compound statement, 385

concave polygon, 158 concave vertex, 172 concentric circles, 247, 248 conclusion in Greek proofs, 7 of a rule of inference, 408 of an implication, 388 concurrence theorem altitude, 269 angle bisector, 268 median, 224 perpendicular bisector, 264 concurrent, 223, 262 conditional statement, 388 congruence in Hilbert's axioms, 20, 23, 375, 376 of angles, 88 of asymptotic triangles, 351 of convex polygons, 159 of polygons, 173 of segments, 64 of simple polygonal regions, 199 of triangles, 106 via rigid motions, 446 congruent, see congruence conjugate arcs, 254, 257 conjunction, 385 connective, logical, 385 consecutive angles of a polygon, 156 consecutive interior angles, 150 consecutive interior angles theorem, 151 converse, 186 consecutive vertices of a polygon, 156 consistency of an axiomatic system, 32 of betweenness of points, 61 of betweenness of rays, 90 of endpoints of arcs, 256 of endpoints of rays, 76 of endpoints of segments, 66 of polygon vertices, 157, 159 of triangle vertices, 103 of vertices of proper angles, 84 relative, 32 constant of proportionality, 214 constructible angle, 313 constructible circle, 308 constructible line, 308 constructible number, 308 characterization of, 311 constructible point, 308

constructing a perpendicular, 101 constructing a rectangle, 195 constructing a square, 196 construction compass and straightedge, 296-319 in Greek proofs, 3, 6, 7 constructive proof, 417 contains (a point) in incidence geometry, 24 in plane geometry, 55 contrapositive, 391 proof by, 411 converse, 391 convex decomposition lemma, 201 convex polygon, 157-166 angle criterion, 160 characterizations of, 173 diagonal criterion, 176 interior of, 164, 171 semiparallel criterion, 161, 176 vertex criterion, 160 convex quadrilateral, 175-183 diagonal criterion, 176 semiparallel criterion, 176 convex set, 80 intersection of. 80 convex vertex, 172 coordinate of a point, 57 of a ray, 86 coordinate function adapted to a ray, 72 for a half-rotation, 86 for a line, 57 starting at a ray, 86 coordinate representation of a ray, 72 of a segment, 67 copying theorem asymptotic triangle, 352 quadrilateral, 179, 300 triangle, 111, 300 corollary, 6, 24 correspondence, one-to-one, 429 corresponding angles, 150 corresponding angles theorem, 151 converse, 186 cosine, 238, 239 is injective, 239 cosines, law of, 240 cross lemma, 177

crossbar theorem, 106 cubic polynomial, 314 cut by a transversal, 149 cutting off a segment, 74, 297 cyclic polygon, 262, 263 is convex, 262 cyclic quadrilateral, 264, 276 cyclic triangle theorem, 263 dart, 183 decagon, 157 decomposition lemma convex, 201 parallelogram, 203 defect of a polygon, 328, 371 of a triangle, 328 of an asymptotic triangle, 351 defect addition theorem, 328 for asymptotic triangles, 353 defect formula, 372 defined terms, 23 definition descriptive, 2 logical, 2 mathematical, 401 degree in angle measure, 85 of a polynomial, 314 density of real and rational numbers, 438 Descartes, René, 33, 311 descriptive definition, 2 diagonal of a polygon, 156 of a square, 234 diagonal criterion for convex quadrilaterals, 176 diagonal scaling theorem, 226 diameter of a circle, 247 length of, 248 difference of sets, 425 different (meaning not equal), 25, 393 differential geometry, 135, 373-374 direct proof, 410 disjoint sets, 425 disjunction, 386 proof of, 415 displacement vector, 124 distance as a primitive term, 54 between real numbers, 57 from a point to a line, 147

from a point to a segment, 147 from a point to a set, 146 in plane geometry, 54, 56 in taxicab geometry, 138 in the Cartesian plane, 125 in the Poincaré disk, 132 properties of, 59 distance postulate, 56 distance-preserving function, 57 distinct (meaning not equal), 25, 393 divides, 434 divisible, 434 dodecagon, 157 domain of a function, 428 of a variable, 394 dot product, 125 double angle formula for cosine, 242 for sine, 242 double elliptic geometry, 35 doubling a cube, 308, 317 doubling a square, 305 dropping a perpendicular, 142 edge finite, of an asymptotic triangle, 351 of a polygon, 156 of a triangle, 103 edge line, 158 edge-line lemma, 158 element of a set, 423 Elements of Euclid, 1 elliptic geometry, 11 double, 35 single, 35, 41, 136 elliptic parallel postulate, 38 empty set, 424 endpoint of a ray, 71 of a segment, 65 of an arc, 254 endpoint independence of asymptotic rays, 348 enunciation (in Greek proofs), 6 equality in Euclid's Elements, 3 in modern mathematics, 25, 393 of functions, 428 of ordered pairs, 426 of sets, 424 equiangular polygon, 159

equiangular quadrilateral, 181, 182, 194 equiangular triangle, 105, 110 equidistance postulate, 323 equidistance theorem, 152 converse, 187 equidistant from a line, 151 from two lines, 147 from two points, 68 equidistant lines, 151, 152, 187, 362 symmetry of, 188 equilateral polygon, 159 equilateral triangle, 104, 110 construction theorem, 191, 296 equivalence, 392 proof of, 413 equivalence relation, 393 equivalent postulates, 191, 321 Erchinger, Johannes, 318 Erlangen program, 373 Euclid, 1 Euclid's Elements, 1 Euclid's Postulate 1, 3 Euclid's Postulate 2, 3 Euclid's Postulate 3, 3, 71 Euclid's Postulate 4, 3, 88 Euclid's Postulate 5, 3-4, 8-10, 322 implied by Euclidean parallel postulate, 190 implies Euclidean parallel postulate, 322 is independent, 134 Euclid's segment cutoff theorem, 74 Euclidean area postulate, 200 Euclidean geometry, 53, 123, 185 is categorical, 132, 243 is consistent, 132 postulates of, 123, 382 Euclidean parallel postulate, 38, 185 implied by Euclid's Postulate 5, 322 implies Euclid's Postulate 5, 190 is independent, 133 Euclidean postulates, 321-333 Euler line theorem, 276 even integer, 434 even parity, 167 every triangle has two equal sides (fake theorem), 18 everywhere defined function, 428 exhaustion, method of, 279 existence and uniqueness, 398, 419

of parallels, 153 of two distinct points, 55 proof of, 417-419 statement of, 397 existence postulate, 55 existential quantifier, 397 existential statement, 397 extension of a field, 309 exterior of a circle, 248 of a polygon, 164, 167 of an angle, 95 exterior angle formed by a transversal, 150 of a convex polygon, 192 of a triangle, 113, 189 of an asymptotic triangle, 351 exterior angle inequality, 17, 113 for asymptotic triangles, 353 exterior angle sum for a convex polygon, 192 extreme and mean ratio, 225 extreme point, 65 weak. 174 fake theorem (every triangle has two equal sides), 18 family (synonym for set), 423 Fano plane, 28 Fermat prime, 318 Fermat, Pierre de, 318 field, 309 fifteen-sided polygon, 307 finite edge of an asymptotic triangle, 351 five-point plane, 27-29, 39 fixed point of a rigid motion, 443 foot of a perpendicular, 143 for all (universal quantifier), 394 four-point plane, 27-29, 38 four right angles theorem, 101 fourth angle of a Lambert quadrilateral, 342 fourth vertex of a Lambert quadrilateral, 342 free variable, 394 function, 24, 428 gaps in Euclid's arguments, 13-18 Garfield, James A., 245 Gauss, Carl Friedrich, xiii, 10, 318 general polygonal region, 200 general triangle inequality, 116 geometric mean, 236 constructing, 301

Gergonne point, 276

Gergonne, Joseph, 276 glide reflection, 442, 449 golden ratio, 224 constructing, 302 golden rectangle, 224 golden triangle, 225, 226 gravitational lensing, 12 great circle, 11, 34 great hyperbola, 37 greatest lower bound, 435 gremlin, 409, 412, 417 group, 373 half-plane closed, 77 is convex, 80 open, 77 Poincaré, 36 half-rotation, 85 height of a parallelogram, 209 of a trapezoid, 209 of a triangle, 208 height scaling theorem, 226 heptadecagon, 318 heptagon, 157, 319 Heron of Alexandria, 243 Heron's formula, 243 hexagon, 157 constructing, 307 Hilbert's axioms, 375-376 Hilbert's betweenness axiom, 61, 375 Hilbert, David, 20, 375 hinge theorem, 117 HL congruence theorem, 121 horizontal line, 33 hyperbolic angle-sum theorem, 338 hyperbolic geometry, 11, 53, 123, 337-367 is consistent, 133 is not categorical, 339 postulates of, 123, 382 hyperbolic parallel postulate, 38, 337 hyperboloid model, 37, 41 hyperboloid, two-sheeted, 37 hypotenuse, 105 is the longest side, 116 hypothesis, 388, 406 if and only if, 392 if-then statement, 388 iff. 392

implicit universal quantifier, 397 implies (logical connective), 388 incenter, 265 incidence axioms, 25 in neutral geometry, 57 incidence geometry, 24-50 incidence theorems, 42-50 incircle, 265 for a triangle, 268 incircle theorem, 268 included angle, 104 included side, 104 inconsistent axiomatic system, 32 independence of a statement. 32 of axioms, 32, 139 of Euclid's Postulate 5, 134 of the Euclidean parallel postulate, 133 of the neutral postulates, 139 indirect proof, 416 induction, mathematical, 420-422 injective function, 56, 429 inscribed angle in a circle, 257 inscribed angle theorem, 259 inscribed arc, 257 inscribed circle, 265 constructing, 305 for a triangle, 268 inscribed polygon, 262 is convex, 262 regular, 272 integer, 424, 433 intercepted arc, 257 interior of a circle, 248 of a convex polygon, 164, 171 of a polygon, 164, 167 of a ray, 71 of a segment, 65 of a simple polygonal region, 199 of an angle, 94, 95 of an arc, 254 vs. betweenness, 97 interior angle adjacent, 113 alternate, 150 formed by a transversal, 149 of a triangle, 113 of an asymptotic triangle, 351 remote, 113 interior angle measure, 159, 173

implication, 388

interior lemma, 97 interpretation of an axiomatic system, 26 intersecting chords theorem, 261 intersecting lines in incidence geometry, 24 in plane geometry, 55 intersecting secants theorem, 261 intersection of sets, 425 inward-pointing ray, 172 irrational number, 434 irrationality of $\sqrt{2}$, 5 isomorphic models, 29 isomorphism between models, 29, 132, 243 isosceles right triangle, 190, 233 isosceles triangle, 104 base, 105 base angles, 105 isosceles triangle altitude theorem, 145 isosceles triangle theorem, 109 converse, 110 Pappus's proof, 110 iterated quadratic extension, 311

Jordan polygon theorem, 167 justifications for steps in a proof, 406

Khayyam, Omar, 9, 10, 341 kite, 183, 210 Klein geometry, 373 Klein, Felix, 35, 373

Lambert quadrilateral, 341-343 Lambert, Johann Heinrich, 341 law of cosines, 240 law of sines. 241 least upper bound, 280, 435, 438 least upper bound property, 280 leg of a right triangle, 105 of a Saccheri quadrilateral, 341 Legendre, Adrien-Marie, 326 lemma, 6, 24 length of a diameter, 248 of a segment, 64 of a vector, 125 of an arc, 293 let, in mathematical proofs, 411 lies on as a primitive term, 24 in incidence geometry, 24 in plane geometry, 55

line as a primitive term, 24, 54 contains infinitely many points, 57 Euclid's definition, 2 in incidence geometry, 24 in plane geometry, 54 in single elliptic geometry, 35, 136 in spherical geometry, 34, 135 in taxicab geometry, 138 in the Beltrami-Klein disk, 35 in the Cartesian plane, 33, 125 in the Poincaré disk, 35, 132 in the Poincaré half-plane, 36 in the rational plane, 136 is convex. 80 straight, 2 line segment, see segment line-circle theorem, 250 linear pair, 92 linear pair theorem, 92 converse, 93 linear polynomial, 314 linear triple, 94, 101 linear triple theorem, 94 Lobachevskian geometry, 337 Lobachevsky, Nikolai, xiii, 10 logic, laws of, 408 logical connective, 385 logical definition, 2 lower bound, 435 lowest terms, 438 major arc, 254 map, 428 mapping, 428 mathematical definition, 401 mathematical induction, 420-422 mathematical object, 24, 383, 423 mathematical relation, 384 mathematical statement, 383 mean proportional, 236 measure of an angle, see angle measure of an arc, see arc measure median of a triangle, 145 of an isosceles triangle, 145 median concurrence theorem, 224 meet in incidence geometry, 24 in plane geometry, 55 member of a set, 423

membership criterion for a set, 423 Menalaus of Alexandria, 221 Menelaus's theorem, 221 meridian circle, 35 method of exhaustion, 279 method of superposition, 15 midpoint, 68 existence and uniqueness, 69 midsegment of a Saccheri quadrilateral, 342 of a triangle, 196 midsegment theorem, 196 minor arc. 254 model of an axiomatic system, 26 monic polynomial, 314 multiple quantifiers, 399 multiplicative inverse, 434 n-gon, 157 n-point plane, 27-29 necessary condition, 389 negation, 387 negative number, 434 neutral geometry, 53 is consistent, 132 postulates of, 123, 381 nine-point circle theorem, 277 non-Euclidean geometry, 10-13 nonagon, 157 noncollinear. 25 nonconvex polygon, 166-174 nonexistence statement, 399, 420 nonnegative, 434 nonoverlapping regions, 199 nonpositive, 434 nonvertical line, 33, 125, 136 nonzero, 434 north pole, 35 not (logical negation), 387 object, mathematical, 24, 383, 423 obtuse angle, 88 obtuse triangle, 105 octagon, 157 constructing, 307 odd integer, 434 odd parity, 167 one-dimensional quantity, 226 one-point geometry, 30, 135 one-to-one correspondence, 429 one-to-one function, 429 one-two geometry, 31

onto, 429 open half-plane, 77 open interval, 434 open sentence, 394 opposite angle of a quadrilateral, 175 of a triangle, 104 opposite ray theorem, 75 opposite rays, 74, 75 opposite side of a line, 76 of a quadrilateral, 175 of a triangle, 104 opposite vertices of a quadrilateral, 175 or (logical disjunction), 386 ordered *n*-tuple, 426 ordered pair, 426 equality of, 426 ordering lemma for points, 73 for rays, 98 origin in the Cartesian plane, 124 orthocenter, 269 outward-pointing ray, 172 overlapping regions, 199 π , definition of, 289 \mathcal{P} -side of an edge line, 158 pair, ordered, 426 Pappus of Alexandria, 110 parallel lines are equidistant, 187 constructing, 300 existence of, 153 in incidence geometry, 25 in plane geometry, 55, 149-154 parallel postulate elliptic, 38 Euclidean, 38, 185 hyperbolic, 38, 337 in single elliptic geometry, 41 in spherical geometry, 41 in the Beltrami-Klein disk, 41 in the Cartesian plane, 40 in the hyperboloid model, 41 in the Poincaré disk, 41 in the Poincaré half-plane, 41 independence, in incidence geometry, 39 independence, in neutral geometry, 133 parallel projection theorem, 220 parallelism, transitivity of, 188, 323 parallelogram, 175, 194

area of. 209 congruent opposite angles, 180, 194 congruent opposite sides, 181, 194 diagonals bisect each other, 182, 194 is convex, 177 properties, 194 parallelogram decomposition lemma, 203 parallelogram lemma, 178 parity of a point, 167 of a ray, 167 Pascal's mystic hexagon, 277 Pasch's axiom, 105 Pasch's theorem, 105 for asymptotic triangles, 353 Pasch, Moritz, 105 passing point, 65 strong, 174 path, polygonal, 167 pentagon, 157 constructing, 306 perimeter of a polygon, 226 perimeter scaling theorem, 226 perpendicular bisector, 145 constructing, 299 perpendicular bisector concurrence theorem, 264 perpendicular bisector theorem, 145 converse, 146 perpendicular lines, 101, 141 constructing, 101, 299, 300 dropping, 142, 300 Euclid's definition, 2 perpendicular rays, 141 perpendicular segments, 141 pi, definition of, 289 pizza lemma, 202 plane geometry, 53 plane separation postulate, 76 plane, the, 55 Playfair's postulate, 10, 323 Playfair, John, 10 Poincaré disk, 35-36, 41, 132-135 is a model of hyperbolic geometry, 133 is a model of incidence geometry, 36 is a model of neutral geometry, 133 Poincaré half-plane, 36, 41 is a model of incidence geometry, 36 point as a primitive term, 24, 54 Euclid's definition, 2

in incidence geometry, 24 in plane geometry, 54 in single elliptic geometry, 35, 136 in spherical geometry, 34, 135 in taxicab geometry, 138 in the Beltrami-Klein disk, 36 in the Cartesian plane, 33, 125 in the Poincaré disk, 35, 132 in the Poincaré half-plane, 36 in the rational plane, 136 point of tangency, 249 polygon, 155 area of, 200 concave, 158 convex, 157-166, 173 interior of, 164, 171 nonconvex, 166-174 regular, 159, 271-273 polygon splitting theorem, 163 polygonal path, 167 polygonal region, 200 general, 200 simple, 199 polynomial, 314 positive number, 434 postulate, 13, 23 angle measure, 85 area, 209 distance, 56 elliptic parallel, 38 Euclidean area, 200 Euclidean parallel, 38, 185 existence, 55 hyperbolic parallel, 38, 337 of Euclid, 3 of Euclidean geometry, 123, 382 of hyperbolic geometry, 123, 382 of incidence geometry, 25 of neutral geometry, 123, 381 plane separation, 76 protractor, 85 reflection, 443 ruler, 57 SAS, 108 set, 55 unique line, 56 predicate, 394 premise of a rule of inference, 408 prime number, 434 primitive term, 23, 54 Proclus, 6, 9, 186

Proclus's lemma, 186 Proclus's postulate, 323 product, Cartesian, 426 projection of a leg, 236 proof, 405 by cases, 414 by contradiction, 416 by contrapositive, 411 by induction, 420 constructive, 417 direct, 410 indirect. 416 justifications in, 406-410 of existence, 417, 418 of existence and uniqueness, 419 stages of writing, 42 structure of, 405 templates for, 405-422 proper angle, 84 proper subset, 424 proper superset, 424 proportion, 213 proportion theorem angle bisector, 219 right triangle, 237 triangle area, 209 proportional, 213 proposition, 6, 24 protractor postulate, 85 Pythagorean identity, 240 Pythagorean postulate, 335 Pythagorean theorem, 229-233, 237 converse, 233 Euclid's first proof, 230 Euclid's second proof, 237 Garfield's proof, 245 proof by decomposition, 232 proof by similar triangles, 237 q.e.d. (quod erat demonstrandum), 7 q.e.f. (quod erat faciendum), 7 quadratic extension of a field, 310 iterated. 311 quadratic polynomial, 314 quadrilateral, 156, 175 area scaling theorem, 226 convex, 175-183 copying theorem, 179 cyclic, 264 tangential, 269 quantifier, 394 existential, 397

multiple, 399 negating, 399 universal, 394, 395 quod erat demonstrandum, 7 quod erat faciendum, 7 radii, see radius radius, 70, 247 range of a function, 428 rating system for proofs, 54 ratio, 213 rational number, 434 rational plane, 136 ray, 71 collinear, 74 coordinate representation of, 72 is convex, 81 lying in the interior of an angle, 95 lying on a side of a line, 78, 149 opposite, 74, 75 with the same endpoint, 74 real number, 53, 424, 433-439 reciprocal, 434 rectangle, 175, 194, 195 area of, 207 construction theorem, 195, 300 golden, 224 is a parallelogram, 175 is convex, 177 nonexistent in hyperbolic geometry, 337 rectangular region, 199 reflection across a line, 146, 442, 443 reflection postulate, 443 reflex angle, 88 reflex measure of an angle, 87 reflexive property of congruence of angles, 90 of segments, 66 reflexive property of equality, 393 region circular, 290 determined by a polygon, 164, 199 polygonal, 199, 200 regular heptagon, 308 regular polygon, 159, 271, 272 angle measures of, 192 center of, 271 central angle of, 271 circumscribed, 273 constructing, 317-319 fifteen-sided, 307 inscribed in a circle, 272

is cyclic and tangential, 271 regular quadrilateral, 182, 183, 194 relation, mathematical, 24, 384 relative consistency, 32 remote interior angle in a triangle, 113 in an asymptotic triangle, 351 rhombus, 175, 182 is a parallelogram, 182 Riemann, Bernhard, 11 Riemannian geometry, 11 right angle Euclid's definition, 2 in plane geometry, 88, 92 in the Cartesian plane, 126 right triangle, 105, 229-243 area of, 207 hypotenuse, 105 isosceles, 233 leg, 105 proportion theorem, 237 similarity theorem, 236 rigid motion, 441 root of a polynomial, 314 rotation, 442, 449 rule of inference, 408 ruler flipping lemma, 58 ruler placement theorem, 58 ruler postulate, 57 SMSG, 21, 379 ruler sliding lemma, 58 SA congruence theorem, 351 SA inequality for asymptotic triangles, 353 SAAAS congruence, 197 SAASS congruence, 197 Saccheri quadrilateral, 340, 342, 343 Saccheri's repugnant theorem, 365 Saccheri, Giovanni, 9, 10, 326, 341, 367 Saccheri-Legendre theorem, 327 for convex polygons, 330 SAS congruence theorem of Euclid, 107 SAS postulate, 108 SAS similarity postulate, 335 SAS similarity theorem, 218 SASAS congruence, 179 SASSS congruence, 197 scale factor, 214 scalene inequality, 115 for quadrilaterals, 342 scalene triangle, 104 scaling theorem

circumference, 288 diagonal, 226 height, 226 perimeter, 226 quadrilateral area, 226 triangle area, 226 School Mathematics Study Group, 21, 379 secant line, 249 sector of a circle, 294 area of, 294 segment, 64 contains infinitely many points, 70 coordinate representation of, 67 is convex, 80 lying on a side of a line, 149 segment addition theorem, 66 segment construction theorem, 74 segment cutoff theorem, Euclid's, 74 segment extension theorem, 65 segment subtraction theorem, 66 semicircle, 254 semiparallel criterion for convexity, 161 of quadrilaterals, 176 semiparallel segments, 160 semiperimeter, 243 set. 423 defined by a list, 424 defined by specification, 424 membership criterion, 423 set-builder notation, 425 set difference, 425 set equality, 424 set postulate, 55 set theory, 423-431 setting out (in Greek proofs), 6 side included between two angles, 104 of a line, 76 of a polygon, 156 of a triangle, 103 of an angle, 83 opposite an angle, 104 side-angle-side congruence theorem of Euclid, 107 side-angle-side postulate, 108 side-side congruence theorem, 107 side-splitter theorem, 215 converse, 219 similar polygons, 213 similar triangle construction theorem, 217 similarity theorem

AA. 216 right triangle, 236 SAS, 218 SSS, 218 similarity, transitive property of, 214 simple polygonal region, 199 simple statement, 384 sine. 238 sines, law of, 241 single elliptic geometry, 35, 41, 136 singleton, 424 slope of a line, 33 small angle lemma, 325 SMSG (School Mathematics Study Group), 21.379SMSG postulates, 379–380 south pole, 35 specification in Greek proofs, 7 to define a set, 424 spherical geometry, 34, 41, 135 spherical trigonometry, 371 square, 175, 194 area of, 206 constructing, 196, 300, 305 diagonal of, 234 square geometry, 31 square of a number, 434 square region, 199 square root, 435, 438 square root of 2 is irrational, 5 squaring a circle, 308, 317 a figure, 304 a polygon, 304 a rectangle, 304 SSS congruence theorem, 107, 111 SSS construction theorem, 235 SSS existence theorem, 234 SSS similarity theorem, 218 standard measure of an angle, 87 starting point of a ray, 71 statement atomic, 384 compound, 385 existential, 397 mathematical, 383 simple, 384 universal, 394 straight angle, 84 straight line, 2

straightedge, 3, 295, 296 strong passing point, 174 subfield, 309 subpolygons cut off by a chord, 162 subset, 424 substitution property of equality, 393 such that in existence statements, 397 in logical conjunctions, 385 sufficient condition, 389 sum formula for cosine, 241 for sine, 241 summit angles, 341 summit of a Saccheri quadrilateral, 341 superposition, method of, 15 superset, 424 supplementary angles, 91 supremum, 280 surjective function, 56, 429 symmetric property of equality, 393 symmetry, 372 of a Klein geometry, 373 of asymptotic rays, 347 of betweenness of points, 60 of betweenness of rays, 89 of equidistant lines, 188 symmetry group, 373 tangent circles, 253 tangent circles theorem, 254 tangent line, 249 existence and uniqueness, 252, 258 exterior to circle, 252 properties of, 252 tangent line theorem, 251 tangent segment to a circle, 265 tangential polygon, 265, 268 is convex, 266 tangential quadrilateral, 269, 276 tangential triangle theorem, 268 taxicab distance, 138 taxicab geometry, 138 Thales of Miletus, 257 Thales' postulate, 335 Thales' theorem, 257 converse, 258 theorem, 6, 23, 405 in an axiomatic system, 23 in Euclid's Elements, 6 there exists (existential quantifier), 397 three-point line, 31

three-point plane, 26, 38, 135 three-ring geometry, 31 three-two geometry, 51 transcendental number, 317 transformation, 108, 441 transitive property of asymptotic rays, 349 transitive property of congruence of angles, 90 of segments, 66 of triangles, 107 transitive property of equality, 393 transitive property of similarity, 214 transitivity of parallelism, 188 transitivity postulate, 323 translation, 442, 449 transversal for a triangle, 221 for two lines, 149 trapezoid, 175 area of, 209 is convex, 177 trapezoid lemma, 178 triangle, 103 acute, 105 area of, 207, 208 constructing, 307 equiangular, 105 equilateral, 104 is convex, 160 is cyclic, 263 is tangential, 268 isosceles, 104 obtuse, 105 right, 105 scalene, 104 triangle area proportion theorem, 209 triangle area scaling theorem, 226 triangle copying theorem, 111 triangle inequality, 116 for real numbers, 437 general, 116 triangle sliding theorem, 208 triangular region, 199 trichotomy law, 437 trigonometry, 238 triple angle formula for cosine, 243 for sine, 243 trisecting an angle, 307, 313-316 truth table, 385 two circles theorem, 252

two-column proof, 405 two-dimensional quantity, 226 two-point equidistance theorem, 152 two-point formula for a line, 34 two-sheeted hyperboloid, 37 two transversals theorem, 218 ultraparallel lines, 355, 360 admit a common perpendicular, 360 distance between, 364 ultraparallel theorem, 360 union, 425 unique line postulate, 56 unique point theorem, 74 unique ray theorem, 89 unique triangle theorem, 111 uniquely defined function, 428 uniqueness, 398 of common perpendicular, 359 proof of, 419 unit circle, 289 universal implication, 412 proof of, 412 universal quantifier, 394, 395 implicit, 397 universal statement, 394 upper bound, 280, 435 least, 280 vacuously true statement, 396 value of a function, 428 variable, 394 free, 394 Varignon's theorem, 197 vector, 124 components, 124 difference, 125 displacement, 124 sum, 125 vertex of a polygon, 156 of a triangle, 103 of an angle, 83 of an asymptotic triangle, 351 vertex criterion for convexity, 160 vertical angles, 93 vertical angles theorem, 93 converse, 94 vertical line, 33, 125, 136 vertices, see vertex

Wallis's postulate, 324

Wallis, John, 324 Wantzel, Pierre, 316, 319 weak angle-sum postulate, 333 weak extreme point, 174 well defined, 428 well-ordering principle, 438 whole is greater than the part angle, 90 converse, 73, 97 in Euclid's *Elements*, 4 segment, 66

X-lemma, 79

Y-lemma, 78

zero angle, 84