

VARIETIES OF ELEMENTARY SUBALGEBRAS OF MAXIMAL DIMENSION FOR MODULAR LIE ALGEBRAS

JULIA PEVTSOVA AND JIM STARK

To Dave Benson on the occasion of his 60th birthday

ABSTRACT. Motivated by questions in modular representation theory, Carlson, Friedlander, and the first author introduced the varieties $\mathbb{E}(r, \mathfrak{g})$ of r -dimensional abelian p -nilpotent subalgebras of a p -restricted Lie algebra \mathfrak{g} in [CFP16]. In this paper, we identify the varieties $\mathbb{E}(r, \mathfrak{g})$ for a reductive restricted Lie algebra \mathfrak{g} and r the maximal dimension of an abelian p -nilpotent subalgebra of \mathfrak{g} .

CONTENTS

1. Introduction	2
2. Preliminaries	4
2.1. Notations and conventions	4
2.2. Maximal sets of commuting positive roots	6
2.3. Varieties $\mathbb{E}(r, \mathfrak{g})$	9
3. Unipotent case	10
3.1. Correspondence with sets of commuting roots	10
3.2. Types A_{2n+1} , B_2 , B_3 , C_n , and E_7	11
3.3. Type A_2	11
3.4. Type A_{2n} , $n \geq 2$	12
3.5. Type B_n , $n \geq 5$	12
3.6. Type B_4	14
3.7. Type D_n , $n \geq 5$	15
3.8. Type D_4	15
3.9. Type G_2	16
3.10. Types E_6 , E_8 , and F_4	18
3.11. Type G_2 when $p = 3$	19
4. Calculation of $\mathbb{E}(\mathfrak{g})$ for $\mathfrak{g} = \text{Lie } G$	20
4.1. Reduction to the unipotent case	20
4.2. Ideal orbits	21
4.3. Majority case	22
4.4. Type A_2	23
4.5. Type G_2	24
5. Applications to Chevalley groups	25
Appendix A. Maximal sets of commuting roots	26
References	30

Date: April 11, 2018.

2000 Mathematics Subject Classification. 17B50, 16G10.

1. INTRODUCTION

Let \mathfrak{g} be a restricted Lie algebra defined over a field of positive characteristic p . A Lie subalgebra $\mathcal{E} \subset \mathfrak{g}$ is *elementary* if it is abelian with trivial p -restriction. The study of the projective variety $\mathbb{E}(r, \mathfrak{g})$ of elementary subalgebras of \mathfrak{g} of a fixed dimension r was initiated by Carlson, Friedlander and the first author in [CFP15]. The interest in the geometry of $\mathbb{E}(r, \mathfrak{g})$ can be traced back to Quillen's foundational work on mod- p group cohomology which revealed the significance of elementary abelian subgroups of a finite group G to both the representations and cohomology of G . The theory of global nilpotent operators, which associates geometric and sheaf-theoretic invariants living on the space $\mathbb{E}(r, \mathfrak{g})$ to representations of \mathfrak{g} , was introduced by Friedlander and the first author in [FP11] and further developed in [CFP12, CFP16]. This theory serves as one of the motivations for our interest in $\mathbb{E}(r, \mathfrak{g})$. The variety $\mathbb{E}(r, \mathfrak{g})$ is also directly related to the much studied variety of r -tuples of nilpotent commuting elements in \mathfrak{g} and, consequently, to the cohomology of Frobenius kernels of algebraic groups (see Suslin, Friedlander, Bendel [SFB97]). Further geometric properties of $\mathbb{E}(r, \mathfrak{g})$ were recently investigated in [War15].

In this paper we give a description of the variety $\mathbb{E}(\mathfrak{g}) = \mathbb{E}(r_{\max}, \mathfrak{g})$ for $\mathfrak{g} = \text{Lie } G$ the Lie algebra of a reductive algebraic group and r_{\max} the maximal dimension of an elementary subalgebra of \mathfrak{g} . The maximal dimension of an abelian nilpotent subalgebra of a complex simple Lie algebra \mathfrak{g} is known thanks to the work of Malcev [Mal51] while the general linear case was considered by Schur at the turn of the previous century [Sch05]. Malcev has also classified such subalgebras up to automorphisms of the Lie algebra. It turns out that under a mild restriction on p the maximal dimension of an elementary subalgebra in the modular case agrees with Malcev's results. To compute the variety $\mathbb{E}(r_{\max}, \mathfrak{g})$ we need to consider elementary subalgebras of \mathfrak{g} up to conjugation by G so our calculations and the end result differ from Malcev's, who classified abelian subalgebras up to automorphisms of \mathfrak{g} . Nonetheless, we find his linear algebraic approach very useful for our purposes.

An analogous classification of elementary abelian p -subgroups in a Chevalley group has been considered by several authors, for example Barry [Bar79] and Milgram and Priddy [MP87], with the most complete account given by Gorenstein, Lyons and Solomon in [GLS98, §3.3]. Using a particularly nice Springer isomorphism from the nullcone of the Lie algebra $\mathfrak{g} = \text{Lie } G$ to the unipotent variety of G (constructed by P. Sobaĵe [Sob15]) we can quickly recover the known results on maximal dimensions of elementary abelian p -subgroups in (untwisted) Chevalley groups. We also obtain information on the conjugacy classes of elementary abelian p -subgroups of $G(\mathbb{F}_{p^r})$. Thanks to the celebrated Quillen stratification theorem this has an immediate application to mod- p group cohomology: The number of conjugacy classes of the elementary abelian p -subgroups gives the number of irreducible components of maximal dimension in $\text{Spec } H^*(G(\mathbb{F}_{p^\ell}), \overline{\mathbb{F}}_p)$.

The computation of $\mathbb{E}(\mathfrak{g}) = \mathbb{E}(r_{\max}, \mathfrak{g})$ proceeds in several steps. We first reduce to the case of a simple algebraic group G with root system Φ . For such G , we compute $\mathbb{E}(\mathfrak{g})$ under the assumption that p is *separably good* for G (see [definition 2.2](#)). We rely on the result of Levy, McNinch and Testerman [LMT09] to show that any elementary subalgebra of \mathfrak{g} can be conjugated into $\mathfrak{u} \subset \mathfrak{g}$, the Lie algebra of the unipotent radical U of the Borel subgroup $B \leq G$. Consequently, as the next step towards calculating the variety $\mathbb{E}(\mathfrak{g})$, we explicitly determine $\mathbb{E}(\mathfrak{u})$ for $\mathfrak{u} \subset \mathfrak{g}$ as a

set. We define a map $\text{Lie}: \text{Max}(\Phi) \rightarrow \mathbb{E}(\mathfrak{u})$ which sends a maximal set of commuting positive roots to an elementary subalgebra of maximal dimension in \mathfrak{u} and show that there is an inverse map $\text{LT}: \mathbb{E}(\mathfrak{u}) \rightarrow \text{Max}(\Phi)$ which splits Lie . The map Lie is not necessarily surjective but we show that for all irreducible root systems Φ except for G_2 and A_2 it is surjective up to conjugation by U . Hence, we effectively prove that the maximal elementary subalgebras in \mathfrak{u} up to conjugation are given by the combinatorics of the root system of G . This calculation largely relies on the linear algebraic approach of Malcev and is split into several cases dictated by the existence of certain orderings on the corresponding root systems:

- (1) $A_{2n+1}, B_2, B_3, C_n, E_7$ (in these cases, there is a unique maximal elementary subalgebra in \mathfrak{u} given by a maximal set of commuting positive roots),
- (2) A_{2n} (two maximal elementary abelian subalgebras, given by maximal sets of commuting roots),
- (3) D_n (three maximal elementary subalgebras for $n = 4$ and two for $n \geq 5$, all given by maximal sets of commuting positive roots),
- (4) B_n (three families of maximal elementary subalgebras for $n = 4$, two families for $n \geq 5$, only given by maximal sets of commuting positive roots up to conjugation by U),
- (5) G_2, A_2 (exceptional cases).

The cases of E_6, E_8 and F_4 are handled by a computer calculation for which we provide the code.

If we allow conjugation by G , then a stronger result holds: with the exception of G_2 and A_2 , every elementary subalgebra in $\mathbb{E}(\mathfrak{u})$ is G -conjugate to a subalgebra stabilized by the action of the Borel. This observation greatly simplifies the calculation of stabilizers of the conjugacy classes which is the next step in determination of $\mathbb{E}(\mathfrak{g})$. The stabilizers of such subalgebras are parabolic subgroups which implies that the G -orbits in $\mathbb{E}(\mathfrak{g})$ are partial flag varieties. To finish the calculation in all types, except for G_2 and A_2 , we prove in [theorem 4.9](#) that for \mathcal{E} with parabolic stabilizer the orbit map $G \rightarrow G \cdot \mathcal{E} \subset \mathbb{E}(\mathfrak{g})$ is separable.

When p is not separably good (as defined in [2.2](#)), such as $p = 2$ for $G = \text{PGL}_2$, the answer for $\mathbb{E}(\mathfrak{g})$ can be somewhat surprising. In [example 4.6](#) we utilize a construction from [\[LMT09\]](#) to illustrate that. We are grateful to Jared Warner for pointing out this example to us. We also illustrate in [example 4.10](#) that in general the orbit map $G \rightarrow G \cdot \epsilon \subset \mathbb{E}(\mathfrak{g})$ can fail to be separable.

The ultimate outcome is that the projective variety $\mathbb{E}(\mathfrak{g})$ is a product of $\mathbb{E}(\mathfrak{g}_i)$ where $\mathfrak{g}_i = \text{Lie}(G_i)$ and the G_i range over the simple algebraic subgroups of the derived group of G . When G is simple $\mathbb{E}(\mathfrak{g})$ is a finite disjoint union of partial flag varieties *unless* G is of type G_2 or A_2 . This is proved in [Theorem 4.11](#):

Theorem. *Let G be a simple algebraic group with root system Φ which is not of type A_2 or G_2 , and let $\mathfrak{g} = \text{Lie } G$. Assume that p is separably good for G . Then*

$$\mathbb{E}(\mathfrak{g}) = \coprod_{\substack{R \in \text{Max}(\Phi) \\ R \text{ an ideal}}} G/P_R,$$

where P_R is the parabolic stabilizer of the elementary Lie algebra associated with the root subset R .

Reinterpreted explicitly for each type, [theorem 4.11](#) implies that $\mathbb{E}(\mathfrak{g})$ is a disjoint union of at most three copies of generalized Grassmannians in types A_n ($n \neq 2$),

C, D, E , whereas in types B and F the “two step” partial flag varieties appear (see [Table 4](#)). For types A_2, G_2 we show that $\mathbb{E}(\mathfrak{g})$ is an irreducible variety and compute its dimension and G -orbits. We find that this non-homogeneous answer partially justifies the fact that we have to resort to case-by-case considerations in our calculations.

We note that calculation of $\mathbb{E}(\mathfrak{g})$ for $\mathfrak{g} = \text{Lie } G$ with G a special linear or symplectic group was done in [\[CFP15\]](#). The arguments in that paper are based on induction on the dimension and are qualitatively different from the arguments inspired by Malcev’s approach used in this paper.

The paper is organized as follows. In [section 2](#) we recall various conditions on p such as good, very good, and torsion and define what it means to be separably good. We also state the combinatorial classification of maximal sets of commuting roots for irreducible root systems in [Table 2](#). We refer to Malcev [\[Mal51\]](#) for this classification but also give a detailed explanation in the Appendix motivated largely by the fact that Malcev’s paper is very sketchy on details.

The classification of the maximal elementary subalgebras of the unipotent radical $\mathfrak{u} \subset \mathfrak{g}$ up to conjugation by G is settled in [section 3](#). [Section 4](#), where we calculate the variety $\mathbb{E}(\mathfrak{g})$, contains the main result of the paper. In [section 5](#) we apply our results to obtain information on conjugacy classes of maximal elementary abelian p -subgroups in Chevalley groups.

Throughout the paper, k will be an algebraically closed field of positive characteristic p . We follow the convention in Jantzen [\[Jan03\]](#) and assume that our reductive algebraic k -groups are defined and split over \mathbb{Z} . By Chevalley group we mean a group of the form $G(\mathbb{F}_{p^e})$ where G is a reductive k -group, defined and split over \mathbb{Z} .

Acknowledgments. The first author is indebted to Eric Friedlander for generously sharing his ideas and intuition about the variety $\mathbb{E}(r, \mathfrak{g})$. We are also grateful to George McNinch, Paul Sobaje, and Jared Warner for sharing their expertise and patiently answering our structural questions about reductive groups in positive characteristic and to Jim Humphreys for his helpful comments. We thank the referee for a very careful reading of the paper and many helpful suggestions. We wish to acknowledge the support provided by the NSF grants DMS-0953011 and DMS-0500946.

We would like to take this opportunity to express both our profound gratitude to our friend and colleague Steve Mitchell and our deep sadness about his passing away while this manuscript was under review. Steve had generously shared his insights and participated in stimulating discussions at the onset of this project, pointing out, in particular, Souter’s paper to us [\[Sut04\]](#). He was a caring mentor, a fantastic teacher, and an inspiring and prolific writer of both mathematical and non mathematical prose. He will be dearly, painfully missed by many including the authors.

2. PRELIMINARIES

2.1. Notations and conventions. Let $G_{\mathbb{Z}}$ be a split connected reductive algebraic \mathbb{Z} -group. Set $G_R = (G_{\mathbb{Z}})_R$ for any ring R and $G = G_k$. Let $T_{\mathbb{Z}} \subseteq G_{\mathbb{Z}}$ be a split maximal torus and define T_R and T as we did with G . Fix a Borel subgroup B containing T and let U be the unipotent radical of B . Let $X(T) = \text{Hom}(T, \mathbb{G}_m)$ be the character group and let $\Phi \subseteq X(T)$ be the root system associated to G with

respect to T . Let $\Lambda_r(\Phi) = \mathbb{Z}\Phi$ and $\Lambda(\Phi)$ be the root and integral weight lattices of Φ respectively. The quotient $\Lambda(\Phi)/\Lambda_r(\Phi)$ is called the fundamental group of the root system Φ . If Φ is irreducible then its fundamental group is cyclic, except for type D_n when n is even, in which case one gets the Klein 4-group.

If G is semisimple then the quotient $\Lambda(\Phi)/X(T)$ is called the fundamental group of G . We say that G is simply connected if its fundamental group is trivial. For reductive G we denote by G_{sc} the simply connected semisimple group of the same type as G . For any subgroup A of the fundamental group of Φ there is a semisimple group G with root system Φ , fundamental group A , and a central isogeny $G_{\text{sc}} \rightarrow G$ (see [KMRT98, 25] for more details). The following lemma clarifies the significance of the fundamental group of G for our calculations.

Lemma 2.1 ([Ste75, 2.4]). *Let G be a semisimple algebraic group with root system Φ . If p does not divide the order of the fundamental group of G then the isogeny $G_{\text{sc}} \rightarrow G$ is separable.*

Proof. Let H be the scheme-theoretic kernel of the isogeny $G_{\text{sc}} \rightarrow G$. Then H is a diagonalizable group scheme associated to the finite abelian group $\Lambda(\Phi)/X(T)$ (see, for example, Jantzen [Jan03, II.1.6]). In particular, the dimension of the coordinate ring $k[H]$ is equal to the order of $\Lambda(\Phi)/X(T)$. The assumption on p implies that $(p, \dim k[H]) = 1$, hence, H is an étale group scheme, $k[H]$ is a separable algebra, and the map $G_{\text{sc}} \rightarrow G$ is separable. \square

If $\beta = \sum_i m_i \alpha_i$ is the highest root written as a linear combination of simple roots then p is *bad* for Φ if $p = m_i$ for some i . Similarly we may write the dual of this root as a linear combination of dual simple roots $\beta^\vee = \sum_i m'_i \alpha_i^\vee$ and p is *torsion* for Φ if $p = m'_i$ for some i . A prime is *good* (respectively *non-torsion*) if it is not bad (respectively torsion). We say p is *very good* for Φ if p is good for Φ and p does not divide the order of the fundamental group of Φ . Some authors include this condition in the definition of non-torsion but we will not. We instead say that a prime is *very non-torsion* if it is non-torsion and p does not divide the order of the fundamental group of Φ . As one sees from Table 1 the only very non-torsion prime which is not very good is $p = 3$ for G_2 .

Definition 2.2. If G is a semisimple algebraic group we say that p is *separably good* for G if

- (1) p is good for G ,
- (2) the isogeny $G_{\text{sc}} \rightarrow G$ is separable.

If G is a connected reductive group we say that p is *separably good* for G if it is separably good for its derived group $[G, G]$.

Note that very good implies separably good by lemma 2.1 but in type A the separably good condition is less restrictive. The simple groups of type A_{n-1} are SL_n/μ_d for $d \mid n$ and by lemma 2.1 if $p \nmid d$ then p is separably good. In particular, p is always separably good if $d = 1$ so the Special Linear group SL_n is covered by our results for all p .

Let \mathfrak{g} and \mathfrak{u} be the Lie algebras of G and U respectively. For any ring R let $\mathfrak{g}_R = \text{Lie } G_R$ so that $\mathfrak{g} = \mathfrak{g}_k$. If S is an R -algebra with the structure map $R \rightarrow S$, then one has $\mathfrak{g}_S = \mathfrak{g}_R \otimes_R S$.

Definition 2.3. Let k be an R -algebra. An element $x \in \mathfrak{g}$ is *defined over R* if there exists an $x' \in \mathfrak{g}_R$ such that $x = x' \otimes 1$. A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *defined over*

R if there exists a Lie subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}_R$ such that $\mathfrak{h} = \mathfrak{h}' \otimes_R k$. We will call x' and \mathfrak{h}' R -forms of x and \mathfrak{h} respectively.

For a simply connected semisimple group G the Lie algebra \mathfrak{g} has a Chevalley basis $\{x_\alpha, h_i \mid \alpha \in \Phi, 1 \leq i \leq \text{rank } \Phi\}$ defined over \mathbb{Z} . In particular, this means $[x_\alpha, x_\beta] = N_{\alpha, \beta} x_{\alpha + \beta}$, where $N_{\alpha, \beta} = 0$ if $\alpha + \beta \notin \Phi$ and when $\alpha + \beta \in \Phi$ we have $N_{\alpha, \beta} = \pm(r + 1)$ where $-r\alpha + \beta, \dots, s\alpha + \beta$ is the α -string through β . The sign can be inductively determined depending on a choice of ordering for the roots and our choice that $N_{\alpha, \beta} = +(r + 1)$ when (α, β) is an extraspecial pair defined by this ordering (see Carter [Car72, 4.2] for details).

For a general reductive group there exists a central isogeny $G_{\text{sc}} \times D \rightarrow G$ where D is some torus. The preimage of the Borel $B \subseteq G$ is a Borel in the reductive group $G_{\text{sc}} \times D$ and the isogeny restricts to an isomorphism between the unipotent parts of these Borels. We may therefore define $x_\alpha \in \mathfrak{g}$ and $h_i = [x_{\alpha_i}, x_{-\alpha_i}]$ as the image under the isogeny of the corresponding elements of Lie G_{sc} . Even though h_i do not necessarily span $\text{Lie}(T)$, we have that $\{x_\alpha \mid \alpha \in \Phi^+\}$ is still a basis for \mathfrak{u} and satisfies the same relations as in the simply connected case. We say that the set $\{x_\alpha \mid \alpha \in \Phi\}$ is induced from a Chevalley basis.

Definition 2.4. A subalgebra $\mathcal{E} \subseteq \mathfrak{g}$ is called *Chevalley* if it is spanned by some subset of the $\{x_\alpha \mid \alpha \in \Phi\}$ induced from a Chevalley basis.

Note that the non-uniqueness of a Chevalley basis is up to scaling of the basis vectors, so the property of being a Chevalley subalgebra does not depend on the choice of Chevalley basis from which the x_α are induced. This also follows from an equivalent description of a Chevalley subalgebra as a T -stable subalgebra which intersects $\text{Lie}(T)$ trivially.

Finally, we note that good primes are greater than or equal to the length of the longest root string in the root system Φ . In particular, this implies that the structure constants $N_{\alpha, \beta} \in \mathbb{Z}$ for the Chevalley basis are not divisible by p . Thus in \mathfrak{g} one has $[x_\alpha, x_\beta] \neq 0$ if and only if $\alpha + \beta \in \Phi$.

TABLE 1. Bad and torsion primes, fundamental groups, and maximal root string lengths. [GM06, 2.13] [MT11, 9.2]

Type	A_n	B_n ($n \geq 2$)	C_n ($n \geq 3$)	D_n ($n \geq 4$)	E_6	E_7	E_8	F_4	G_2
Bad	none	2	2	2	2, 3	2, 3	2, 3, 5	2, 3	2, 3
Torsion	none	2	none	2	2, 3	2, 3	2, 3, 5	2, 3	2
$ \Lambda/\Lambda_r $	$n + 1$	2	2	4	3	2	1	1	1
Longest root string length	2	3	3	2	2	2	2	3	4

2.2. Maximal sets of commuting positive roots. Let Φ be an irreducible root system with positive roots Φ^+ , corresponding base $\Delta = \{\alpha_1, \dots, \alpha_n\}$, and the Weyl group W . Throughout the paper, we follow the labeling in Bourbaki [Bou02, 6, §4]. Let $S \subseteq \Delta$ be a set of simple roots and $\Delta \setminus S$ be its complement. Then we define

$$\Phi_S^{\text{rad}} = \Phi^+ \setminus \mathbb{N}(\Delta \setminus S)$$

to be the positive roots that cannot be written as a linear combination of the simple roots not in S . Note that $S \mapsto \Phi_S^{\text{rad}}$ commutes with unions and intersections. If $S = \{\alpha_i\}$ then we will write Φ_i^{rad} instead of $\Phi_{\{\alpha_i\}}^{\text{rad}}$. These are simply all positive roots whose expression in terms of simple roots includes α_i .

For any $I \subset \Delta$ define the parabolic subgroup W_I and its corresponding root system Φ_I as in Humphreys [Hum90, 1.10]. For $I = \Delta \setminus S$, we have $\Phi_S^{\text{rad}} = \Phi^+ \setminus \Phi_I^+$. If $P_I = L_I \ltimes U_I$ is the standard parabolic determined by the subset $I = \Delta \setminus S$ with the Levi factor L_I and the unipotent radical U_I , then the root subgroups U_α with $\alpha \in \Phi_S^{\text{rad}}$ are precisely the ones generating U_I .

Lemma 2.5. *Let $S \subseteq \Delta$. Then $\text{Stab}_W(\Phi_S^{\text{rad}}) = W_{\Delta \setminus S}$.*

Proof. Let $I = \Delta \setminus S$. As W_I stabilizes Φ and Φ_I , it also stabilizes $\Phi \setminus \Phi_I = \Phi_S^{\text{rad}} \cup -\Phi_S^{\text{rad}}$. Now for any element $w \in W_I$ its length as an element of W equals its length as an element of W_I . Length is characterized by the number of positive roots that are sent to negative roots so such roots are in Φ_I . In particular, $w\Phi_S^{\text{rad}} \subseteq \Phi_S^{\text{rad}} \cup -\Phi_S^{\text{rad}}$ are positive so $w\Phi_S^{\text{rad}} \subseteq \Phi_S^{\text{rad}}$. This proves that W_I stabilizes Φ_S^{rad} .

Conversely assume $w \in W \setminus W_I$ is of minimal length stabilizing Φ_S^{rad} . Then w stabilizes Φ_I and for any $\alpha \in I$ we have $\ell(ws_\alpha) = \ell(w) + 1$ so $w(\alpha)$ is positive. This means w stabilizes Φ_I^+ , so it stabilizes all of Φ^+ , but the identity is the only such element in W . \square

Lemma 2.6. *The set Φ_S^{rad} is not conjugate to any other set of positive roots.*

Proof. If w does not stabilize Φ_S^{rad} then it has a reduced expression of the form $w = us_{\alpha_i}v$ where $v \in W_{\Delta \setminus S}$ and $s_{\alpha_i} \notin W_{\Delta \setminus S}$, i.e., $\alpha_i \in S$. By Humphreys [Hum90, 1.7] we have $us_{\alpha_i}(\alpha_i) < 0$ so w sends $v^{-1}(\alpha_i) \in \Phi_S^{\text{rad}}$ to a negative root. \square

Two positive roots *commute* if their sum is not a root. A set of commuting positive roots is a set of positive roots which pairwise commute and $R \subseteq \Phi^+$ is a *maximal* set of commuting positive roots if it is maximal with respect to order, i.e., if R' is any other set of commuting positive roots then $|R'| \leq |R|$.

Notation 2.7. Let $\text{Max}(\Phi)$ be the set of all maximal sets of commuting positive roots in Φ . Let $m(\Phi)$ be the order of a maximal set of commuting roots in Φ^+ . If Φ is irreducible of type T then we may write $\text{Max}(T)$ and $m(T)$ instead.

To formulate the theorem on the maximal sets of commuting positive roots, we need to introduce additional notation for type B_n . We first recall notation from Bourbaki [Bou02, 6, §4.5]:

Notation 2.8. Type B_n .

$$\begin{aligned} \epsilon_i &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_n, & 1 \leq i \leq n \\ \epsilon_i + \epsilon_j &= (\alpha_i + \alpha_{i+1} + \cdots + \alpha_n) + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_n), & 1 \leq i < j \leq n \\ \epsilon_i - \epsilon_j &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}, & 1 \leq i < j \leq n \end{aligned}$$

Define the following subsets of positive roots, as in Malcev [Mal51].

$$\begin{aligned} S_t &= \{\epsilon_t, \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\}, & t = 1, 2, \dots, n \\ S_t^* &= \{\epsilon_t, \epsilon_i + \epsilon_j, \epsilon_{i'} - \epsilon_n \mid 1 \leq i < j < n, 1 \leq i' < n\}, & t = 1, 2, \dots, n-1 \end{aligned}$$

We present what is known about the maximal sets of commuting positive roots in Table 2. The proofs can be found in Malcev [Mal51], see also Appendix A.

TABLE 2. Maximal sets of commuting positive roots.

Type T	Restrictions on rank	$\text{Max}(T)$	$\#\text{Max}(T)$	$m(T)$
A_{2n}	$n \geq 1$	$\Phi_{n+1}^{\text{rad}}, \Phi_n^{\text{rad}}$	2	$n(n+1)$
A_{2n+1}	$n \geq 0$	Φ_{n+1}^{rad}	1	$(n+1)^2$
B_n	$n = 2, 3$	Φ_1^{rad}	1	$2n - 1$
	$n = 4$	$\Phi_1^{\text{rad}}, S_1, S_2, S_3, S_4, S_1^*, S_2^*, S_3^*$	8	7
	$n \geq 5$	$S_t, 1 \leq t \leq n$ $S_t^*, 1 \leq t < n$	$2n - 1$	$\binom{n}{2} + 1$
C_n	$n \geq 3$	Φ_n^{rad}	1	$\binom{n+1}{2}$
D_n	$n = 4$	$\Phi_1^{\text{rad}}, \Phi_3^{\text{rad}}, \Phi_4^{\text{rad}}$	3	6
	$n \geq 5$	$\Phi_{n-1}^{\text{rad}}, \Phi_n^{\text{rad}}$	2	$\binom{n}{2}$
E_6		$\Phi_1^{\text{rad}}, \Phi_6^{\text{rad}}$	2	16
E_7		Φ_7^{rad}	1	27
E_8		none of the form Φ_i^{rad}	134	36
F_4		none of the form Φ_i^{rad}	28	9
G_2		none of the form Φ_i^{rad}	5	3

Remark 2.9 (On cominusule roots). Recall that a simple root α_i is called *cominusule* if it occurs with coefficient 1 in the highest positive root (see Billey and Lakshmibai [BL00] or Richardson, Rohle, and Steinberg [RRS92] for more on cominusule roots and cominusule parabolics). Note that in types A, C, D, E_6, E_7 , the maximal sets of commuting positive roots are given by Φ_i^{rad} with α_i a simple cominusule root. One of the equivalent definitions of a simple cominusule root α_i is that the unipotent radical of the corresponding parabolic is abelian [RRS92, Lem 2.2] so the set Φ_i^{rad} is a natural candidate to be in $\text{Max}(T)$. As one sees from the table, the sets Φ_i^{rad} defined by a simple cominusule root do have maximal dimension in almost all cases when they exist *except* for the most mysterious case of B_n .

Definition 2.10. We say that $R \subseteq \Phi^+$ is an *ideal* if $\alpha + \beta \in R$ whenever $\alpha \in \Phi^+, \beta \in R$, and $\alpha + \beta \in \Phi^+$.

Note, for example, that the sets Φ_i^{rad} are always ideals. The computations below require knowing which maximal sets of commuting positive roots are ideals

and for such sets R what is the stabilizer $\text{Stab}_W(R) = \{w \in W \mid wR = R\}$. This information can be found in [Table 3](#) (see [Appendix A](#) for the calculation).

Note that $\text{Max}(\Phi)$ contains non-ideals only in types $B_n (n \geq 4)$, E_8 , F_4 , and G_2 . One can check in types B_n , E_8 , and F_4 that every set in $\text{Max}(\Phi)$ is W -conjugate to an ideal in $\text{Max}(\Phi)$. In any type there is at most one ideal that is not of the form Φ_i^{rad} for some i , therefore in all types except G_2 [lemma 2.6](#) gives that each set in $\text{Max}(\Phi)$ is conjugate to a *unique* ideal in $\text{Max}(\Phi)$. The exceptional case is G_2 , where one finds that there are two orbits under the partial action of W on $\text{Max}(\Phi)$ and only one contains an ideal.

TABLE 3. Maximal commuting ideals and their stabilizers

Type T	Ideal R	$\text{Stab}_W(R)$
Any	Φ_i^{rad}	$W_{\Delta \setminus \{\alpha_i\}}$
$B_n, n \geq 4$	S_1	$W_{\Delta \setminus \{\alpha_1, \alpha_n\}}$
E_8	R is unique	$W_{\Delta \setminus \{\alpha_2\}}$
F_4	R is unique	$W_{\{\alpha_1, \alpha_3\}}$
G_2	R is unique	$W_{\{\alpha_2\}}$

2.3. Varieties $\mathbb{E}(r, \mathfrak{g})$.

Definition 2.11 ([\[CFP15\]](#)). An elementary subalgebra $\mathcal{E} \subseteq \mathfrak{g}$ of dimension r is a p -restricted Lie subalgebra of dimension r which is commutative and has p -restriction equal to 0. We define

$$\mathbb{E}(r, \mathfrak{g}) = \{\mathcal{E} \subseteq \mathfrak{g} \mid \mathcal{E} \text{ elementary subalgebra of dimension } r\}$$

So defined, $\mathbb{E}(r, \mathfrak{g})$ is a closed subset of the Grassmannian of r -planes in the vector space \mathfrak{g} and hence has a natural structure of a projective algebraic variety. In this paper, we are concerned with the varieties of elementary subalgebras of maximal dimension. We set

$$\begin{aligned} r_{\max} &= \text{Max} \{r \mid \mathbb{E}(r, \mathfrak{g}) \text{ is nonempty}\}, \\ \mathbb{E}(\mathfrak{g}) &= \mathbb{E}(r_{\max}, \mathfrak{g}). \end{aligned}$$

Theorem 2.12. *Let \mathfrak{g} and \mathfrak{h} be restricted Lie algebras whose maximal elementary subalgebras have dimensions r and s respectively. Then $r + s$ is the dimension of a maximal elementary subalgebra in $\mathfrak{g} \oplus \mathfrak{h}$ and $\mathbb{E}(\mathfrak{g} \oplus \mathfrak{h}) \simeq \mathbb{E}(\mathfrak{g}) \times \mathbb{E}(\mathfrak{h})$.*

Proof. If $\mathcal{E} \subseteq \mathfrak{g} \oplus \mathfrak{h}$ is maximal and $\mathcal{E}_1 \subseteq \mathfrak{g}$ and $\mathcal{E}_2 \subseteq \mathfrak{h}$ are its images under the projections to \mathfrak{g} and \mathfrak{h} respectively then $\mathcal{E}_1 \oplus \mathcal{E}_2 \subseteq \mathfrak{g} \oplus \mathfrak{h}$ is an elementary subalgebra and contains \mathcal{E} , hence equals \mathcal{E} . This proves that every maximal elementary subalgebra is a sum of necessarily maximal elementary subalgebras from \mathfrak{g} and \mathfrak{h} . Thus $r + s$ is the maximal dimension of an elementary subalgebra in $\mathfrak{g} \oplus \mathfrak{h}$ and there is a bijection $\mathbb{E}(\mathfrak{g}) \times \mathbb{E}(\mathfrak{h}) \rightarrow \mathbb{E}(\mathfrak{g} \oplus \mathfrak{h})$. One then checks on the standard affine open sets of the Grassmannian that this is an isomorphism. \square

3. UNIPOTENT CASE

In this section we assume that G is a simple algebraic group and compute $\mathbb{E}(\mathfrak{u})$ as a set. We do this by defining the leading terms associated to a particular subalgebra as a subset of the root system and showing that such a subset must be a maximal set of commuting positive roots. We then perform a case by case analysis based on the information in [Table 2](#) which shows that in most cases $\mathbb{E}(\mathfrak{u}) = \text{Lie}(\text{Max}(\Phi))$, with $\text{Lie}(\text{Max}(\Phi))$ defined in [Equation 3.1.1](#). The exceptions are type A_2 where $\mathbb{E}(\mathfrak{u}) \simeq \mathbb{P}^1$, type B_n for $n \geq 4$ where $\mathbb{E}(\mathfrak{u})$ includes the lie subalgebras $B(a_1, \dots, a_n)$ and $\exp(\text{ad}(a_n x_{\alpha_n}))(C(a_1, \dots, a_{n-1}))$ defined below, and the exceptional types of which only G_2 is calculated explicitly. For types F and E we verify later that under the adjoint action one has $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \text{Lie}(\text{Max}(\Phi))$. We note that the condition $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \text{Lie}(\text{Max}(\Phi))$ holds in all types except G_2 .

Though we are interested in the case when p is separably good, the results in this section are valid so long as p is greater than or equal to the length of the longest root string in Φ .

3.1. Correspondence with sets of commuting roots. We must first choose a total ordering \preceq on Φ^+ which respects addition of positive roots, that is, if $\beta, \gamma, \lambda, \beta + \lambda, \gamma + \lambda \in \Phi^+$ and $\beta \preceq \gamma$ then $\beta + \lambda \preceq \gamma + \lambda$. In examples the choice of ordering will depend on the root system but we note here that such a total ordering always exists: The standard ordering \leq on Φ respects addition, as does a reverse lexicographical ordering with respect to any ordering of the simple roots. This ordering will define the extraspecial pairs in our root system and consequently the signs in the structure constants of the Chevalley basis which induces the x_α . We will often construct such orderings through refinement.

Definition 3.1. Let $\succeq_1, \succeq_2, \dots, \succeq_n$ be relations on a set X which define \succ_i and $=_i$ in the obvious way. The relation

$$\succeq = (\succeq_1, \succeq_2, \dots, \succeq_n)$$

is defined as follows: $x \succeq y$ if either

- (1) For some $1 \leq i \leq n$ we have $x \succ_i y$ and $x =_j y$ for all $j < i$, or
- (2) $x =_i y$ for all $1 \leq i \leq n$.

We say that \succeq is given by *refining* \succeq_1 , first by \succeq_2 , then by \succeq_3 , and so on.

In simple terms we compare by first trying \succeq_1 and inductively trying \succeq_{i+1} if \succeq_i gives equality. We will always choose the \succeq_i to be preorders which respect addition of positive roots and we will choose \succeq_n to be, moreover, a total order. Then \succeq is a total order which respects addition.

Now for each set $R \subseteq \Phi^+$ of commuting positive roots there is an abelian Chevalley subalgebra $\text{Lie}(R) = \text{span}_k \{x_\alpha \mid \alpha \in R\}$ associated to R with $\dim \text{Lie}(R) = |R|$. As the elements x_α induced from a Chevalley basis are always p -nilpotent this is in fact an elementary subalgebra. We wish to show that this induces a map

$$(3.1.1) \quad \text{Lie: Max}(\Phi) \longrightarrow \mathbb{E}(\mathfrak{u}),$$

that is, that the Lie subalgebra associated to a maximal commuting set of positive roots has maximal dimension among all elementary subalgebras. We do this using an argument of Malcev [[Mal51](#)] which shows that there exists a surjection

$$(3.1.2) \quad \text{LT: } \mathbb{E}(\mathfrak{u}) \longrightarrow \text{Max}(\Phi)$$

that splits Lie.

Let $\mathcal{E} \subseteq \mathfrak{u}$ be an elementary subalgebra. The ordering \succeq on Φ^+ gives an ordering on the basis elements x_β of \mathfrak{u} . Choose the unique basis of \mathcal{E} which is in reduced echelon form with respect to this ordering and let $\text{LT}(\mathcal{E})$ be the set of roots β such that the corresponding x_β are the leading terms in this reduced basis. Observe that if β and γ are the leading terms of $b_1 = x_\beta + \langle \text{lower terms} \rangle$ and $b_2 = x_\gamma + \langle \text{lower terms} \rangle$ respectively, and if $\beta + \gamma \in \Phi^+$ then $[x_\beta, x_\gamma] = N_{\beta, \gamma} x_{\beta+\gamma}$ is the leading term of $[b_1, b_2]$. Thus if $[b_1, b_2] = 0$ then β and γ commute. This proves that $\text{LT}(\mathcal{E})$ is a commuting set of roots. Clearly $\text{LT}(\text{Lie}(R)) = R$ so LT splits Lie and both maps preserve maximality.

3.2. Types A_{2n+1} , B_2 , B_3 , C_n , and E_7 . In these types there is a unique maximal set of commuting roots of the form Φ_i^{rad} for some i . Let \succeq be the reverse lexicographic ordering given by $\alpha_i < \alpha_1 < \alpha_2 < \dots$, so $\beta \succeq \gamma$ if when written as a linear combination of simple roots the coefficient of α_i in γ is larger than the coefficient of α_i in β , or if those coefficients are equal then coefficient of α_1 in γ is larger, and so on. With this ordering the roots in Φ_i^{rad} , which all have nonzero α_i coefficient, are strictly smaller than the roots in $\Phi^+ \setminus \Phi_i^{\text{rad}}$, which have 0 as the α_i coefficient. The following lemma then gives that $\text{Lie}(\Phi_i^{\text{rad}})$ is the only possible maximal elementary subalgebra.

Lemma 3.2. *If $\Phi^+ \setminus \text{LT}(\mathcal{E}) \succ \text{LT}(\mathcal{E})$ then $\mathcal{E} = \text{Lie}(\text{LT}(\mathcal{E}))$.*

Proof. Choose $\beta \in \text{LT}(\mathcal{E})$ and let b be the element in the reduced echelon form basis of \mathcal{E} with leading term x_β . The terms in $b - x_\beta$ are of the form cx_γ with $\beta \succeq \gamma$, equivalently, $\gamma \in \text{LT}(\mathcal{E})$. So $c = 0$ because x_γ is the leading term of some other basis element. Thus we have $b = x_\beta$ and our reduced basis is $\{x_\beta \mid \beta \in \text{LT}(\mathcal{E})\}$. \square

3.3. Type A_2 . Along with type G_2 this case will be exceptional in that it is not true that every elementary subalgebra is conjugate to a subalgebra in $\text{Lie}(\text{Max}(\Phi))$ and it is not true that every Weyl group orbit in $\text{Max}(\Phi)$ contains an ideal. Consequently we will find below that the variety $\mathbb{E}(\mathfrak{g})$ for a simple algebraic group of type A_2 is not a disjoint union of flag varieties.

The highest root $\alpha_1 + \alpha_2$ commutes with all positive roots so $x_{\alpha_1 + \alpha_2}$ is contained in any maximal elementary \mathcal{E} . As the dimension of such \mathcal{E} is 2 this means \mathcal{E} may be generated by $x_{\alpha_1 + \alpha_2}$ and an element of the form $ax_{\alpha_1} + bx_{\alpha_2}$ where $a, b \in k$ are not both 0. One can check that no other conditions on a and b are needed to get an elementary subalgebra therefore we have a bijection

$$\begin{aligned} \mathbb{P}^1 &\simeq \mathbb{E}(\mathfrak{u}) \\ [a : b] &\mapsto \langle ax_{\alpha_1} + bx_{\alpha_2}, x_{\alpha_1 + \alpha_2} \rangle. \end{aligned}$$

We note that each of these subalgebras is fixed under conjugation by $U < G$, conjugation by the torus allows us to scale the constants a and b , and no two of these subalgebras are conjugate via a representative of a Weyl group element. Thus up to conjugation there are three subalgebras:

$$\begin{aligned} &\langle x_{\alpha_1}, x_{\alpha_1 + \alpha_2} \rangle, \\ &\langle x_{\alpha_2}, x_{\alpha_1 + \alpha_2} \rangle, \\ &\langle x_{\alpha_1} + x_{\alpha_2}, x_{\alpha_1 + \alpha_2} \rangle. \end{aligned}$$

3.4. **Type A_{2n} , $n \geq 2$.** Consider positive roots β and γ written as a linear combination of the simple roots α_i . Define $\beta \succeq_1 \gamma$ if the sum of the coefficients of α_n and α_{n+1} in the expression for γ is greater or equal the sum for β . Define \succeq_2 to be the reverse lexicographic ordering given by $\alpha_{n+1} \prec \alpha_n \prec \alpha_1 \prec \alpha_2 \prec \dots$. Finally let \succeq be the refinement of \succeq_1 by \succeq_2 . One can now check that

$$\Phi^+ \setminus (\Phi_n^{\text{rad}} \cup \Phi_{n+1}^{\text{rad}}) \succ \Phi_n^{\text{rad}} \setminus \Phi_{n+1}^{\text{rad}} \succ \Phi_{n+1}^{\text{rad}} \setminus \Phi_n^{\text{rad}} \succ \Phi_n^{\text{rad}} \cap \Phi_{n+1}^{\text{rad}}.$$

Let $\mathcal{E} \subseteq \mathfrak{u}$ be an elementary subalgebra. As seen in [Table 2](#), we have either $\text{LT}(\mathcal{E}) = \Phi_{n+1}^{\text{rad}}$ or $\text{LT}(\mathcal{E}) = \Phi_n^{\text{rad}}$. If $\text{LT}(\mathcal{E}) = \Phi_{n+1}^{\text{rad}}$ then [lemma 3.2](#) gives $\mathcal{E} = \text{Lie}(\Phi_{n+1}^{\text{rad}})$. Assume $\text{LT}(\mathcal{E}) = \Phi_n^{\text{rad}}$. Then there is a basis whose leading terms are contained in either $\Phi_n^{\text{rad}} \setminus \Phi_{n+1}^{\text{rad}}$ or $\Phi_{n+1}^{\text{rad}} \cap \Phi_n^{\text{rad}}$. For the latter the argument in the proof of [lemma 3.2](#) applies and we get that those basis elements are just the x_β for $\beta \in \Phi_{n+1}^{\text{rad}} \cap \Phi_n^{\text{rad}}$. We must show that the same is true for basis elements whose leading term is in $\Phi_n^{\text{rad}} \setminus \Phi_{n+1}^{\text{rad}}$.

We have

$$\begin{aligned} \Phi_n^{\text{rad}} \setminus \Phi_{n+1}^{\text{rad}} &= \{\epsilon_i - \epsilon_{n+1} \mid 1 \leq i < n+1\} \\ \Phi_{n+1}^{\text{rad}} \setminus \Phi_n^{\text{rad}} &= \{\epsilon_{n+1} - \epsilon_j \mid n+1 < j \leq 2n+1\} \end{aligned}$$

(where the notation follows Bourbaki [[Bou02](#), 4.7]) so the remaining basis elements are of the form

$$b_i = x_{\epsilon_i - \epsilon_{n+1}} + \sum_{n+1 < j \leq 2n+1} a_{ij} x_{\epsilon_{n+1} - \epsilon_j}$$

for $1 \leq i < n+1$. Now we compute

$$[b_i, b_{i'}] = \sum_{n+1 < j \leq 2n+1} a_{i'j} N_{\epsilon_i - \epsilon_{n+1}, \epsilon_{n+1} - \epsilon_j} x_{\epsilon_i - \epsilon_j} + a_{ij} N_{\epsilon_{n+1} - \epsilon_j, \epsilon_i - \epsilon_{n+1}} x_{\epsilon_i - \epsilon_j}.$$

As $n \geq 2$ we may choose $i \neq i'$. That this expression must equal 0 consequently gives $a_{ij} = 0$ for all i and j . Thus $b_i = x_{\epsilon_i - \epsilon_{n+1}}$ and $\mathcal{E} = \text{Lie}(\Phi_n^{\text{rad}})$ as desired.

3.5. **Type B_n , $n \geq 5$.** Recall that we define

$$\epsilon_i = \alpha_i + \alpha_{i+1} + \dots + \alpha_n$$

where the α_i are the simple roots numbered as in Bourbaki [[Bou02](#), 6, §4]. Let \succeq be the reverse lexicographical ordering given by

$$\alpha_1 \succ \dots \succ \alpha_{n-1} \succ \alpha_n.$$

Specifically, we have

$$\epsilon_r - \epsilon_s \succeq \epsilon_t \succeq \epsilon_i + \epsilon_j$$

for all i, j, t, r, s with $r < s$. If we also have that $i < j$ then

$$\begin{aligned} \epsilon_r &\succeq \epsilon_i && \text{if } i < r, \\ \epsilon_r + \epsilon_s &\succeq \epsilon_i + \epsilon_j && \text{if } j < s \text{ or if } j = s \text{ and } i < r, \\ \epsilon_r - \epsilon_s &\succeq \epsilon_i - \epsilon_j && \text{if } s < j \text{ or if } j = s \text{ and } i < r. \end{aligned}$$

One can also compute that if $i < j < n$ then $N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} = -N_{\epsilon_j + \epsilon_n, \epsilon_i - \epsilon_n} = 1$

Now if we define

$$\begin{aligned} R_1 &= \{\epsilon_i + \epsilon_j \mid 1 \leq i < j < n\} \\ R_2 &= \{\epsilon_i + \epsilon_n \mid 1 \leq i < n\} \\ R_3 &= \{\epsilon_i - \epsilon_n \mid 1 \leq i < n\} \end{aligned}$$

then the set of positive roots, Φ^+ , of B_n is the union of the sets

$$\{\epsilon_i - \epsilon_j \mid 1 \leq i < j < n\} \succ R_3 \succ \{\epsilon_i \mid 1 \leq i \leq n\} \succ R_2 \succ R_1.$$

and

$$\begin{aligned} S_t &= R_1 \cup R_2 \cup \{\epsilon_t\} \\ S_t^* &= R_1 \cup R_3 \cup \{\epsilon_t\} \end{aligned}$$

(see [notation 2.8](#) for the definition of S_t, S_t^*).

One can check that the following subalgebras are elementary and are maximal if not all a_i are zero.

$$\begin{aligned} B(a_1, \dots, a_n) &= \text{span}_k \left\{ x_\beta, \sum_{i=1}^n a_i x_{\epsilon_i} \mid \beta \in R_1 \cup R_2 \right\} \\ C(a_1, \dots, a_{n-1}) &= \text{span}_k \left\{ x_\beta, \sum_{i=1}^{n-1} a_i x_{\epsilon_i} \mid \beta \in R_1 \cup R_3 \right\}. \end{aligned}$$

Theorem 3.3. *If $n \geq 4$ and $\mathcal{E} \in \text{Max}(B_n)$ satisfies $\text{LT}(\mathcal{E}) = S_t$ or S_t^* then there exist a_1, \dots, a_n such that $\mathcal{E} = B(a_1, \dots, a_n)$ or $C(a_1, \dots, a_{n-1})^{\exp(\text{ad}(a_n x_{\alpha_n}))}$ respectively.*

Proof. If $\text{LT}(\mathcal{E}) = S_t$ for some t then the argument of [lemma 3.2](#) immediately gives $\mathcal{E} = B(0, \dots, 0, 1, a_{t+1}, \dots, a_n)$ for some a_{t+1}, \dots, a_n . Now assume that $\text{LT}(\mathcal{E}) = S_t^*$ for some $1 \leq t < n$. The reduced echelon form basis of \mathcal{E} then consists of the elements $x_{\epsilon_i + \epsilon_j}$ where $1 \leq i < j < n$ and for $1 \leq i < n$ the elements

$$x = x_{\epsilon_t} + \sum_{s=1}^{t-1} a_s x_{\epsilon_s} + \sum_{s=1}^{n-1} b_s x_{\epsilon_s + \epsilon_n} \quad \text{and} \quad y_i = x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^n c_{is} x_{\epsilon_s} + \sum_{s=1}^{n-1} d_{is} x_{\epsilon_s + \epsilon_n}$$

for some a_s, b_s, c_{sk} , and d_{sk} . That it's reduced means $c_{it} = 0$ for all i . Notice that $\exp(\text{ad}(\lambda x_{\alpha_n}))$ is upper triangular with respect to \preceq so $\text{LT}(\exp(\text{ad}(\lambda x_{\alpha_n}))(\mathcal{E})) = \text{LT}(\mathcal{E})$ and the element in the reduced basis of $\exp(\text{ad}(\lambda x_{\alpha_n}))(\mathcal{E})$ with leading term x_{ϵ_t} is $\exp(\text{ad}(\lambda x_{\alpha_n}))(x)$. When $\lambda = -b_t N_{\epsilon_n, \epsilon_t}^{-1}$ we have

$$\exp(\text{ad}(\lambda x_{\alpha_n}))(x) = x_{\epsilon_t} + \sum_{s=1}^{t-1} a_s x_{\epsilon_s} + \sum_{s=1}^{t-1} (b_s - a_s b_t N_{\epsilon_n, \epsilon_t}^{-1} N_{\epsilon_n, \epsilon_s}) x_{\epsilon_s + \epsilon_n} + \sum_{s=t+1}^{n-1} b_s x_{\epsilon_s + \epsilon_n}.$$

As $x_{\epsilon_t + \epsilon_n}$ is not a term in this basis element it suffices to show that \mathcal{E} is the subalgebra $C(0, \dots, 0, 1, a_{t+1}, \dots, a_{n-1})$ when $b_t = 0$. We do this by showing that the b_s, c_{ij} , and d_{ij} must all be zero.

One can check that the coefficient of x_{ϵ_i} in $[y_i, y_j]$ is $N_{\epsilon_i - \epsilon_n, \epsilon_n} c_{jn}$ so $c_{jn} = 0$ for all j . Also if $j \neq t$ then the coefficient of $x_{\epsilon_j + \epsilon_t}$ in $[x, y_i]$ is $N_{\epsilon_t, \epsilon_j} c_{ij}$ so $c_{ij} = 0$ for all i, j . For $i \neq t$ the coefficient of $x_{\epsilon_i + \epsilon_t}$ in $[x, y_t]$ is $N_{\epsilon_i + \epsilon_n, \epsilon_t - \epsilon_n} b_i$ so $b_i = 0$ for all i .

If $i, j, t < n$ are distinct then the coefficients of $x_{\epsilon_j + \epsilon_t}$ and $x_{\epsilon_i + \epsilon_j}$ in $[y_i, y_t]$ are $N_{\epsilon_j + \epsilon_n, \epsilon_t - \epsilon_n} d_{ij}$ and $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} d_{tj}$ respectively. As n is at least 4 this gives $d_{ij} = 0$ for all $i \neq j \neq t$.

To show that $d_{it} = 0$, we note that the coefficient of $x_{\epsilon_j + \epsilon_t}$ in $[y_i, y_j]$ is $N_{\epsilon_t + \epsilon_n, \epsilon_j - \epsilon_n} d_{it}$.

Finally, if $i, j < n$ are distinct then $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} d_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} d_{ii}$ is the coefficient of $x_{\epsilon_i + \epsilon_j}$ in $[y_i, y_j]$. For $i < j < t < n$ we thus get a system of equations

$$\begin{aligned} N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} d_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} d_{ii} &= d_{jj} + d_{ii} = 0 \\ N_{\epsilon_i - \epsilon_n, \epsilon_t + \epsilon_n} d_{tt} + N_{\epsilon_i + \epsilon_n, \epsilon_t - \epsilon_n} d_{ii} &= d_{tt} + d_{ii} = 0 \\ N_{\epsilon_j - \epsilon_n, \epsilon_t + \epsilon_n} d_{tt} + N_{\epsilon_j + \epsilon_n, \epsilon_t - \epsilon_n} d_{jj} &= d_{tt} + d_{jj} = 0 \end{aligned}$$

whose unique solution is $d_{ii} = d_{jj} = d_{tt} = 0$. This gives $d_{ii} = 0$ for all i and completes the proof of the theorem. \square

We note that $\mathbb{E}(\mathfrak{u}) = \text{Lie}(\text{Max}(\Phi))$ does not hold in type B_n , $n \geq 5$. In the next proposition we show that any elementary subalgebra in $\mathbb{E}(\mathfrak{u})$ is G -conjugate to a subalgebra in $\text{Lie}(\text{Max}(\Phi))$.

Proposition 3.4. *Let F/\mathbb{F}_p be a field extension. Any F -point of $\mathbb{E}(\mathfrak{u})$ is $G(F)$ -conjugate to an elementary subalgebra in $\text{Lie}(\text{Max}(\Phi))$.*

Proof. We show that any elementary subalgebra in $\mathbb{E}(\mathfrak{u})$ defined over F is $G(F)$ -conjugate to $\text{Lie}(S_1)$ where S_1 is as defined in [notation 2.8](#).

The simple reflection s_n acts by negating ϵ_n and fixing the remaining ϵ_i therefore any representative $s'_n \in N_G(T)$ conjugates $C(a_1, \dots, a_{n-1})$ to $B(a_1, \dots, a_{n-1}, 0)$. Similarly s_i , where $i < n$, swaps ϵ_i with ϵ_{i+1} and fixes the remaining ϵ_j so by conjugation we may assume our elementary subalgebra is of the form $B(a_1, \dots, a_{n-1}, 1)$. Finally conjugation by $\exp(\text{ad}(a_i N_{\epsilon_i - \epsilon_n, \epsilon_n}^{-1} x_{\epsilon_i - \epsilon_n}))$ lets us assume $a_i = 0$ and does not alter the remaining a_j , thus we have conjugated our subalgebra to $B(0, \dots, 0, 1)$. Using simple reflections we conjugate to $B(1, 0, \dots, 0) = \text{Lie}(S_1)$ and are done. Note that if the variables $(a_1, \dots, a_{n-1}, a_n)$ belong to the field F then all conjugations we have to perform to reduce $C(a_1, \dots, a_{n-1})$ or $B(a_1, \dots, a_{n-1}, a_n)$ to $B(1, 0, \dots, 0)$ are by elements in $G(F)$. \square

3.6. Type B_4 . We keep the ordering and choice of basis from the last section. As [theorem 3.3](#) applies here as well, all that is left is to prove the following.

Theorem 3.5. *If $\text{LT}(\mathcal{E}) = \Phi_1^{\text{rad}}$ then $\mathcal{E} = \text{Lie}(\Phi_1^{\text{rad}})$.*

Proof. The reduced echelon form basis of \mathcal{E} is of the form

$$\begin{aligned} x_i &= x_{\epsilon_1 + \epsilon_i} + \sum_{2 \leq s < r < i} c_{isr} x_{\epsilon_s + \epsilon_r}, \quad \text{and} \\ y_1 &= x_{\epsilon_1} + \sum_{2 \leq s < r \leq 4} d_{1sr} x_{\epsilon_s + \epsilon_r} \\ y_i &= x_{\epsilon_1 - \epsilon_i} + \sum_{\substack{i < r \leq 4 \\ 2 \leq s < r}} e_{isr} x_{\epsilon_s - \epsilon_r} + \sum_{s=2,3,4} f_{is} x_{\epsilon_s} + \sum_{2 \leq s < r \leq 4} g_{isr} x_{\epsilon_s + \epsilon_r} \end{aligned}$$

for $i = 2, 3, 4$. These 7 basis elements can be formed into 21 possible commutators which must equal zero. Setting their coefficients equal to zero gives a system of equations which can be solved by hand and whose unique solution is $c_{isr} = d_{1sr} = e_{isr} = f_{is} = g_{isr} = 0$ as desired. \square

As above we do not get $\mathbb{E}(\mathfrak{u}) = \text{Lie}(\text{Max}(\Phi))$. Instead we get $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \text{Lie}(S_1) \cup \{\text{Lie}(\Phi_1^{\text{rad}})\}$.

3.7. **Type D_n , $n \geq 5$.** Define

$$\epsilon_i = \begin{cases} \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2} + \frac{1}{2}(\alpha_{n-1} + \alpha_n) & \text{if } i \leq n-2 \\ \frac{1}{2}(\alpha_{n-1} + \alpha_n) & \text{if } i = n-1 \\ \frac{1}{2}(\alpha_n - \alpha_{n-1}) & \text{if } i = n. \end{cases}$$

where the α_i are the simple roots ordered as in Bourbaki [Bou02, 6, §4] and let \succ be the reverse lexicographical ordering given by

$$\alpha_{n-2} \succ \cdots \succ \alpha_2 \succ \alpha_1 \succ \alpha_{n-1} \succ \alpha_n.$$

One can check that for $i < j < n$ this gives $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} = -N_{\epsilon_j - \epsilon_n, \epsilon_i + \epsilon_n} = 1$.

If we define

$$R = \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n-1\}$$

then

$$\Phi_n^{\text{rad}} = R \cup \{\epsilon_i + \epsilon_n \mid 1 \leq i \leq n-1\}$$

$$\Phi_{n-1}^{\text{rad}} = R \cup \{\epsilon_i - \epsilon_n \mid 1 \leq i \leq n-1\}$$

and Φ^+ is the union of the sets

$$\Phi^+ \setminus \Phi_{\{\alpha_1, \alpha_{n-1}, \alpha_n\}}^{\text{rad}} \succ \Phi_1^{\text{rad}} \setminus \Phi_{\{\alpha_{n-1}, \alpha_n\}}^{\text{rad}} \succ \Phi_{n-1}^{\text{rad}} \setminus R \succ \Phi_n^{\text{rad}} \setminus R \succ R.$$

Theorem 3.6. *If $n \geq 4$ and $\mathcal{E} \in \text{Max}(D_n)$ satisfies $\text{LT}(\mathcal{E}) = \Phi_n^{\text{rad}}$ or Φ_{n-1}^{rad} then $\mathcal{E} = \text{Lie}(\Phi_n^{\text{rad}})$ or $\text{Lie}(\Phi_{n-1}^{\text{rad}})$ respectively.*

Proof. If $\text{LT}(\mathcal{E}) = \Phi_n^{\text{rad}}$ then lemma 3.2 immediately gives $\mathcal{E} = \text{Lie}(\Phi_n^{\text{rad}})$, so assume $\text{LT}(\mathcal{E}) = \Phi_{n-1}^{\text{rad}}$. The reduced basis of \mathcal{E} then consists of the x_α corresponding to $\alpha \in R$ and the elements

$$y_i = x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^{n-1} a_{is} x_{\epsilon_s + \epsilon_n}$$

for $1 \leq i \leq n-1$ and we want to show that $a_{ij} = 0$ for all i, j .

If $i, j, t < n$ are distinct then the coefficient of $x_{\epsilon_i + \epsilon_t}$ in $[y_i, y_j]$ is $N_{\epsilon_i - \epsilon_n, \epsilon_t + \epsilon_n} a_{jt}$. As $n \geq 4$ this gives $a_{jt} = 0$ for all $j \neq t$. Now for $i \neq j$ the coefficient of $x_{\epsilon_i + \epsilon_j}$ in $[y_i, y_j]$ is $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} a_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} a_{ii}$. Thus if $i < j < t < n$ we get a system of equations

$$\begin{aligned} N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} a_{jj} + N_{\epsilon_i + \epsilon_n, \epsilon_j - \epsilon_n} a_{ii} &= a_{jj} + a_{ii} = 0 \\ N_{\epsilon_i - \epsilon_n, \epsilon_t + \epsilon_n} a_{tt} + N_{\epsilon_i + \epsilon_n, \epsilon_t - \epsilon_n} a_{ii} &= a_{tt} + a_{ii} = 0 \\ N_{\epsilon_j - \epsilon_n, \epsilon_t + \epsilon_n} a_{tt} + N_{\epsilon_j + \epsilon_n, \epsilon_t - \epsilon_n} a_{jj} &= a_{tt} + a_{jj} = 0 \end{aligned}$$

whose unique solution is $a_{ii} = a_{jj} = a_{tt} = 0$. This gives $a_{ii} = 0$ for all i and completes the proof of the theorem. \square

3.8. **Type D_4 .** We keep the ordering from the last section. As theorem 3.6 applies here as well, all that is left is to prove that if $\text{LT}(\mathcal{E}) = \Phi_1^{\text{rad}}$ then $\mathcal{E} = \text{Lie}(\Phi_1^{\text{rad}})$.

The reduced basis of \mathcal{E} is of the form

$$\begin{aligned}
& x_{\epsilon_1 - \epsilon_2} + a_{11}x_{\epsilon_2 - \epsilon_3} + a_{12}x_{\epsilon_3 - \epsilon_4} + a_{13}x_{\epsilon_2 - \epsilon_4} + a_{14}x_{\epsilon_3 + \epsilon_4} + a_{15}x_{\epsilon_2 + \epsilon_4} + a_{16}x_{\epsilon_2 + \epsilon_3} \\
& x_{\epsilon_1 - \epsilon_3} + a_{21}x_{\epsilon_3 - \epsilon_4} + a_{22}x_{\epsilon_2 - \epsilon_4} + a_{23}x_{\epsilon_3 + \epsilon_4} + a_{24}x_{\epsilon_2 + \epsilon_4} + a_{25}x_{\epsilon_2 + \epsilon_3} \\
& x_{\epsilon_1 - \epsilon_4} + a_{31}x_{\epsilon_3 + \epsilon_4} + a_{32}x_{\epsilon_2 + \epsilon_4} + a_{33}x_{\epsilon_2 + \epsilon_3} \\
& x_{\epsilon_1 + \epsilon_4} + a_{41}x_{\epsilon_2 + \epsilon_3} \\
& x_{\epsilon_1 + \epsilon_3} \\
& x_{\epsilon_1 + \epsilon_2}
\end{aligned}$$

and we wish to show that the a_{ij} are zero. As in the B_4 case, setting commutators equal to zero yields a system of equations for the a_{ij} which can be solved by hand and whose unique solution is $a_{ij} = 0$ for all i, j .

3.9. Type G_2 . As with type A_2 this case is exceptional in that it is not true that every elementary subalgebra is conjugate to a subalgebra in $\text{Lie}(\text{Max}(\Phi))$ and it is not true that every Weyl group orbit in $\text{Max}(\Phi)$ contains an ideal.

Recall that we assume the characteristic p is good for Φ , so explicitly we assume $p \neq 2, 3$. Let α_1 be the short root and α_2 the long root in the basis, with $s_1, s_2 \in W$ the corresponding simple reflections. We choose the reverse graded lexicographic ordering on Φ with $\alpha_1 \succ \alpha_2$ so that the positive roots are ordered as follows:

$$\alpha_1 \succ \alpha_2 \succ \alpha_1 + \alpha_2 \succ 2\alpha_2 + \alpha_2 \succ 3\alpha_1 + \alpha_2 \succ 3\alpha_1 + 2\alpha_2.$$

The s_i act via

$$\begin{aligned}
s_i(\alpha_i) &= -\alpha_i, \\
s_1(\alpha_2) &= 3\alpha_1 + \alpha_2, \\
s_2(\alpha_1) &= \alpha_1 + \alpha_2,
\end{aligned}$$

and W is the dihedral group of order 12 generated by the reflection s_1 and rotation s_1s_2 . It is simple to check that the maximal sets of commuting positive roots fall into two orbits:

$$\begin{aligned}
C_1 &= \{\alpha_1, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\
C_2 &= \{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\
C_3 &= \{\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},
\end{aligned}$$

and

$$\begin{aligned}
C_4 &= \{\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\
C_5 &= \{2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.
\end{aligned}$$

Only C_5 is an ideal.

Theorem 3.7. *Every $\mathcal{E} \in \mathbb{E}(\mathfrak{u})$ is G -conjugate to one of $\text{Lie}(C_3)$, $\text{Lie}(C_5)$, or $L = \langle x_{\alpha_2} + x_{3\alpha_1 + \alpha_2}, x_{2\alpha_1 + \alpha_2}, x_{3\alpha_1 + 2\alpha_2} \rangle$. Moreover, these subalgebras are pairwise non-conjugate.*

Proof. As C_3 and C_5 are representatives of the two orbits we must show that every $\mathcal{E} \in \mathbb{E}(\mathfrak{g})$ is conjugate to either L or $\text{Lie}(C_i)$ for any i . We handle each of the five possible leading terms separately.

An abelian subalgebra with leading terms C_1 is generated by elements of the form

$$\begin{aligned} & x_{\alpha_1} + a_1 x_{\alpha_1 + \alpha_2} + a_2 x_{2\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + 2\alpha_2}. \end{aligned}$$

for some $a_1, a_2 \in k$. Conjugating by $\exp(\text{ad}(\frac{1}{2}a_2 x_{\alpha_1 + \alpha_2})) \exp(\text{ad}(a_1 x_{\alpha_2}))$ gives $\text{Lie}(C_1)$.

An abelian subalgebra with leading terms C_2 is generated by elements of the form

$$\begin{aligned} & x_{\alpha_1 + \alpha_2} + a_1 x_{2\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + 2\alpha_2}, \end{aligned}$$

for some $a_1 \in k$. Conjugating by $\exp(\text{ad}(-\frac{1}{2}a_1 x_{\alpha_1}))$ gives $\text{Lie}(C_2)$.

An abelian subalgebra with leading terms C_3 is generated by elements of the form

$$\begin{aligned} & x_{\alpha_2} + a_1 x_{\alpha_1 + \alpha_2} + a_2 x_{3\alpha_1 + \alpha_2}, \\ & x_{2\alpha_1 + \alpha_2} + 3a_1 x_{3\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + 2\alpha_2}. \end{aligned}$$

for some $a_1, a_2 \in k$. Conjugation by $\exp(\text{ad}(-a_1 x_{\alpha_1}))$ allows us to assume $a_1 = 0$, giving

$$\begin{aligned} & x_{\alpha_2} + a_2 x_{3\alpha_1 + \alpha_2}, \\ & x_{2\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + 2\alpha_2}. \end{aligned}$$

If $a_2 = 0$ this is $\text{Lie}(C_3)$. If $a_2 \neq 0$ then conjugation by $\alpha_2^\vee(\sqrt[3]{a_2})$ sends the subalgebra above to L .

The only subalgebra with leading terms C_5 is $\text{Lie}(C_5)$ so all that is left is C_4 . An abelian subalgebra with leading terms C_4 is generated by elements of the form

$$\begin{aligned} & x_{\alpha_2} + a_1 x_{2\alpha_1 + \alpha_2} + a_2 x_{3\alpha_1 + \alpha_2}, \\ & x_{\alpha_1 + \alpha_2} + a_3 x_{2\alpha_1 + \alpha_2} - 3a_1 x_{3\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + 2\alpha_2}. \end{aligned}$$

for some $a_1, a_2, a_3 \in k$. Conjugation by $\exp(\text{ad}(-\frac{1}{2}a_3 x_{\alpha_1}))$ allows us to assume $a_3 = 0$. If $a_1 = 0$ then conjugation by s_1 yields a subalgebra generated by elements of the form

$$\begin{aligned} & a x_{\alpha_2} + x_{3\alpha_1 + \alpha_2}, \\ & x_{2\alpha_1 + \alpha_2}, \\ & x_{3\alpha_1 + 2\alpha_2}, \end{aligned}$$

and the leading terms are either C_3 or C_5 , so assume $a_1 \neq 0$. Then conjugation by $\alpha_2^\vee(\sqrt[3]{a_1})$ allows us to assume $a_1 = 1$. Our generators are now of the form

$$\begin{aligned} x_{\alpha_2} + x_{2\alpha_1+\alpha_2} + ax_{3\alpha_1+\alpha_2}, \\ x_{\alpha_1+\alpha_2} - 3x_{3\alpha_1+\alpha_2}, \\ x_{3\alpha_1+2\alpha_2}. \end{aligned}$$

For any choice of $u, v \in k$ we may conjugate by $\exp(\text{ad}(ux_{\alpha_1}))s_1 \exp(\text{ad}(vx_{\alpha_1}))$ to get a subalgebra generated by terms of the form

$$\begin{aligned} \lambda_1 x_{\alpha_2} + ((u^3 + 3u + a)v + u^2 + 1)x_{\alpha_1+\alpha_2} + \text{lower terms} \dots, \\ \lambda_2 x_{\alpha_2} + (3(u^2 - 1)v + 2u)x_{\alpha_1+\alpha_2} + \text{lower terms} \dots, \\ x_{3\alpha_1+2\alpha_2}. \end{aligned}$$

From the constant term one sees that for any a the polynomial $u^4 - 6u^2 - 2ua - 3$ must always have a root distinct from ± 1 and necessarily nonzero. Choose u to be such a root. Then

$$\frac{4u + a}{u^2 - 1} = \frac{u^2 + 3}{2u}$$

holds and, moreover, implies that the equations

$$\begin{aligned} 3(u^2 - 1)v + 2u &= 0, \\ 3(4u + a)v + u^2 + 3 &= 0 \end{aligned}$$

determine a unique v . Finally

$$u[3(u^2 - 1)v + 2u] + [3(4u + a)v + u^2 + 3] = 3[(u^3 + 3u + a)v + u^2 + 1]$$

so $(u^3 + 3u + a)v + u^2 + 1 = 0$. Thus with this choice of u, v we have conjugated the subgroup to one for whom $\alpha_1 + \alpha_2$ is not a leading term. This means we are no longer in the case of leading terms C_4 and our previous arguments apply. Hence we have shown that every $\mathcal{E} \in \mathbb{E}(\mathfrak{u})$ is G -conjugate to one of $\text{Lie}(C_3)$, $\text{Lie}(C_5)$, or L .

All that is left is to show that these three subalgebras are not conjugate. For this we simply observe that their normalizers

$$\begin{aligned} N_{\mathfrak{g}}(\text{Lie}(C_3)) &= \langle h_1, h_2, x_\beta \mid \beta \in \Phi^+ \setminus \{\alpha_1\} \rangle, \\ N_{\mathfrak{g}}(\text{Lie}(C_5)) &= \langle h_1, h_2, x_\beta \mid \beta \in \Phi^+ \cup \{-\alpha_2\} \rangle, \\ N_{\mathfrak{g}}(L) &= \langle h_1 + 2h_2, x_\beta \mid \beta \in \Phi^+ \setminus \{\alpha_1\} \rangle, \end{aligned}$$

have dimensions 7, 9, and 6 respectively. Conjugate subalgebras have conjugate, hence equidimensional, normalizers therefore the subalgebras $\text{Lie}(C_3)$, $\text{Lie}(C_5)$, and L are non-conjugate. \square

3.10. Types E_6 , E_8 , and F_4 . For these types there are too many cases for us to reasonably tackle them by hand. Instead we have written Magma code that attempts to confirm that $\mathbb{E}(\mathfrak{u}) \subseteq U \cdot \text{Lie}(\text{Max}(\Phi))$ holds for Lie algebras of a given type. This code is available online [Sta14b], is successful for types E_6 , E_8 , and F_4 , and moreover confirms that if $\mathcal{E} \in \mathbb{E}(\mathfrak{u})$ is defined over a subfield $F \subseteq k$ then the conjugating element may be taken from the F -points of U . We warn that while the computations for E_6 and F_4 are very quick the computation for E_8 takes several hours and it is helpful to import the list of maximal sets of commuting roots from code written in Sage [Sta14a], where that aspect of the computation is much faster.

Summarizing the above discussion, we now have the following result.

Theorem 3.8. *Let G be simple algebraic group with root system Φ , not of type A_2 or G_2 . Then*

$$\mathbb{E}(\mathbf{u}) = \bigcup_{I \in \text{Max}(\Phi)} U \cdot \text{Lie}(I).$$

Moreover, for any subfield $F \subseteq k$ the F -points of $\mathbb{E}(\mathbf{u})$ are contained in $U(F) \cdot \text{Lie}(I)$ for some $I \in \text{Max}(\Phi)$.

If $\dot{w} \in N_G(T)$ is a representative of the Weyl group element $w \in W$ then $\dot{w} \cdot x_\alpha$ is proportional to $x_{w \cdot \alpha}$. Thus if both I and wI are contained in $\text{Max}(\Phi)$ then $\dot{w} \cdot \text{Lie}(I) = \text{Lie}(wI)$. As Φ is not of type G_2 we know that every set in $\text{Max}(\Phi)$ is W -conjugate to an ideal. This gives the following corollary.

Corollary 3.9. *Let G be simple algebraic group with root system Φ , not of type A_2 or G_2 . Then*

$$\mathbb{E}(\mathbf{u}) \subseteq \bigcup_{\substack{I \in \text{Max}(\Phi) \\ I \text{ an ideal}}} G \cdot \text{Lie}(I).$$

Moreover, for any subfield $F \subseteq k$ the F -points of $\mathbb{E}(\mathbf{u})$ are contained in $G(F) \cdot \text{Lie}(I)$ for some ideal $I \in \text{Max}(\Phi)$.

3.11. Type G_2 when $p = 3$. We now break with the assumption, made at the start of the current section, that p is equal or greater than the length of the longest root string in Φ . In type G_2 there are root strings of length 4 and we now consider the case $p = 3$.

The results of this section are based on the property that if $\alpha \neq \beta$ are positive roots then $[x_\alpha, x_\beta] = 0$ if and only if α and β commute. The techniques that follow from that assumption are still valid in the current case if we replace ‘‘commuting’’ with a different combinatorial property that holds if and only if the corresponding x_α and x_β commute.

Definition 3.10. Let $\alpha \neq \beta$ be positive roots and p a prime. We say that α and β *p -commute* if they commute or if the α -string through β begins at $(-p+1)\alpha + \beta$.

The above notion is a direct translation of the condition that the structure constant $N_{\alpha,\beta}$ is either 0 or p (it cannot be a larger multiple of p) therefore the product $[x_\alpha, x_\beta] = 0$ if and only if α and β p -commute. There are 3 sets of maximal p -commuting roots in Φ^+ , they are

$$\begin{aligned} R_1 &= \{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ R_2 &= \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ R_3 &= \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}. \end{aligned}$$

Geometrically each set consists of 4 rotationally consecutive roots. As long and short roots alternate and the Weyl group is the Dihedral group on the long roots we see that these three sets are conjugate.

The three cases of elementary abelian Lie subalgebras that come from these roots are as follows. If $\text{LT}(\mathcal{E}) = R_1$ then $\mathcal{E} = \text{Lie}(R_1)$ because R_1 is minimal in the ordering. If $\text{LT}(\mathcal{E}) = R_2$ then

$$\mathcal{E} = \langle x_{\alpha_1} + cx_{\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle.$$

Conjugating by $\exp(\text{ad}(-cx_{\alpha_2}))$ yields $\text{Lie}(R_2)$. Lastly if $\text{LT}(\mathcal{E}) = R_3$ then

$$\mathcal{E} = \langle x_{\alpha_2} + cx_{3\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle.$$

Conjugating by $\exp(\text{ad}(\sqrt[3]{c}x_{\alpha_2}))$ yields $\text{Lie}(R_3)$. Thus we have $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \text{Lie}(R_1)$.

4. CALCULATION OF $\mathbb{E}(\mathfrak{g})$ FOR $\mathfrak{g} = \text{Lie } G$

We now calculate, when p is separably good, the variety $\mathbb{E}(\mathfrak{g})$ when \mathfrak{g} is the Lie algebra of a reductive group G . We start by reducing to the case of a simple algebraic group.

Theorem 4.1. *Let G_1, \dots, G_n be the simple algebraic subgroups of the derived group $[G, G]$ and let $\mathfrak{g}_i = \text{Lie}(G_i)$. Then*

$$\mathbb{E}(\mathfrak{g}) = \prod_{i=1}^n \mathbb{E}(\mathfrak{g}_i).$$

Proof. Follows from [theorem 2.12](#). □

Henceforth we assume that G is a simple algebraic group. In all but the A_2 and G_2 cases we find that $\mathbb{E}(\mathfrak{g})$ is a disjoint union of flag varieties corresponding to ideals of maximal commuting roots. In the A_2 and G_2 cases we find that the $\mathbb{E}(\mathfrak{g})$ are irreducible varieties of dimensions 5 and 8 respectively, each given as the union of three G -orbits.

4.1. Reduction to the unipotent case. Our first step is to show that every maximal elementary subalgebra may be conjugated into \mathfrak{u} , the Lie algebra of the unipotent radical of the Borel. This is a result of Levy, McNinch and Testerman based on the following theorem.

Theorem 4.2 ([\[LMT09, 2.2\]](#)). *Let G be a semisimple group and p a very non-torsion prime for G . Let U be a unipotent subgroup scheme of G . Then U is contained in a Borel subgroup of G .*

Remark 4.3. Reading the proof in [\[LMT09\]](#), one finds that the theorem above holds for any prime when G is SL_n or Sp_n , and consequently it holds for any group which is separably isogeneous to SL_n or Sp_n .

Corollary 4.4. *Let G be semisimple and p either separably good or very non-torsion for G . Then any elementary subalgebra \mathcal{E} of \mathfrak{g} may be conjugated into \mathfrak{u} .*

Proof. Let $G_{(1)} \leq G$ be the first Frobenius kernel of G which is a closed normal subgroup scheme. An elementary subalgebra \mathcal{E} of \mathfrak{g} corresponds to an infinitesimal subgroup scheme $E \leq G_{(1)}$, and this correspondence is compatible with the conjugation action of G on $G_{(1)}$ and the adjoint action on \mathfrak{g} [\[Jan03, I.9.6\]](#). As \mathcal{E} consists entirely of nilpotent elements of \mathfrak{g} we have, by Engel's theorem, that E is a unipotent subgroup scheme of G . [Theorem 4.2](#) then gives that E may be conjugated into our choice of Borel B and therefore \mathcal{E} may be conjugated into the Lie algebra of B . As \mathcal{E} is nilpotent its conjugate must then lie in the nilpotent radical \mathfrak{u} of this Lie algebra. □

Remark 4.5. Explicitly, the condition on p in [corollary 4.4](#) amounts to p separably good or G of type G_2 and $p = 3$. The latter produces an interesting counter-example to the separability [lemma 4.8](#) which we explain in [example 4.10](#).

Example 4.6. This conjugation property is not true for a general prime. One can check that when $p = 2$ the nilpotent cone of \mathfrak{pgl}_2 is a linear subspace of dimension 2. This is necessarily maximal among elementary subalgebras and is not contained in the Lie algebra of any Borel. Examples of maximal elementary subalgebras which are not contained in a Borel exist for \mathfrak{pgl}_3 when $p = 3$ (see [LMT09]) and \mathfrak{pgl}_4 when $p = 2$. While it is true that for all n there exists an elementary subalgebra of \mathfrak{pgl}_n which is not contained in a Borel, it is not known for larger n whether such an elementary subalgebra can be maximal.

Corollary 4.7. *Let G be semisimple and p separably good for G . Then*

$$\mathbb{E}(\mathfrak{g}) = \bigcup_{\substack{R \in \text{Max}(\Phi) \\ R \text{ an ideal}}} G \cdot \text{Lie}(R).$$

Proof. Follows immediately from [corollary 3.9](#) and the corollary above. \square

4.2. Ideal orbits. Now we geometrically identify orbits of the form $G \cdot \text{Lie}(R)$ where R is an ideal in $\text{Max}(\Phi)$. Observe that such $\mathcal{E} = \text{Lie}(R)$ are fixed by B under the adjoint action (see Malle and Testerman [MT11, 15.4]). The stabilizer $P = \text{Stab}_G(\mathcal{E})$ is then a standard parabolic and is generated by B and representatives $\dot{s}_i \in N_G(T)$ of some collection of simple roots s_i in the Weyl group $W = N_G(T)/C_G(T)$. As $\dot{s}_i \text{Lie}(R) = \text{Lie}(s_i R)$ we have $\dot{s}_i \in P$ if and only if $s_i \in \text{Stab}_W(R)$. Thus P is identified by the information in [Table 3](#) above.

The orbit map $\pi: G \rightarrow G \cdot \mathcal{E}$ factors to a bijective morphism $G/P \rightarrow G \cdot \mathcal{E}$ and we show below that this morphism is in fact an isomorphism. Note that the fact that the orbit map π induces an isomorphism $G/P \cong G \cdot \mathcal{E}$ is equivalent to π being separable. This, in turn, is equivalent to the condition that the kernel of the tangent map at the identity $\ker d\pi_1$ is contained in $\text{Lie}(P)$ (see, for example, [Bor91, 6.7]). We show in [theorem 4.9](#) that this condition does, indeed, hold.

Lemma 4.8. *If $\pi: G \rightarrow \mathbb{E}(r, \mathfrak{g})$ is the orbit map $g \mapsto g \cdot \mathcal{E}$ for some elementary subalgebra $\mathcal{E} \in \mathbb{E}(r, \mathfrak{g})$ then $\ker d\pi_1 = N_{\mathfrak{g}}(\mathcal{E})$.*

Proof. Let $\underline{e} = (e_1, \dots, e_r)$ be a basis of \mathcal{E} . As $\mathbb{E}(r, \mathfrak{g})$ is a closed subvariety in $\text{Grass}(r, \mathfrak{g})$ we may consider the latter to be the codomain of π . This gives the following diagram

$$(4.8.1) \quad \begin{array}{ccc} & & (\mathfrak{g}^{\times r})^\circ \\ & \nearrow \tilde{\pi} & \downarrow \phi \\ G & \xrightarrow[\substack{g \mapsto g \cdot \mathcal{E}}]{\pi} & \text{Grass}(r, \mathfrak{g}) \end{array}$$

where $(\mathfrak{g}^{\times r})^\circ$ is the open subset of $\mathfrak{g}^{\times r}$ consisting of linearly independent r -tuples and $\phi: (\mathfrak{g}^{\times r})^\circ \rightarrow \text{Grass}(r, \mathfrak{g})$ is the canonical projection. The map ϕ is, in particular, a GL_r -torsor and hence is locally trivial. Its tangent map at \underline{e} can be identified as the linear map

$$d\phi_{\underline{e}}: \text{Hom}(\mathcal{E}, \mathfrak{g}) \rightarrow \text{Hom}(\mathcal{E}, \mathfrak{g}/\mathcal{E})$$

induced by the projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathcal{E}$. Indeed, locally on the affine neighborhood defined by the non-vanishing of the Plucker coordinate associated with \mathcal{E} , the torsor

trivializes as follows:

$$\begin{array}{ccc} \mathbb{A}^{r(n-r)} \times \mathrm{GL}_r & \hookrightarrow & \mathbb{A}^{rn} = \mathbb{M}_{r \times n} \simeq \mathrm{Hom}_k(\mathcal{E}, \mathfrak{g}) \\ \downarrow \phi & & \downarrow \\ \mathbb{A}^{r(n-r)} & \xlongequal{\quad} & \mathrm{Hom}(\mathcal{E}, \mathfrak{g}/\mathcal{E}). \end{array}$$

Replace [diagram 4.8.1](#) with the corresponding diagram for tangent spaces:

$$\begin{array}{ccc} & \mathfrak{g}^{\times r} \cong \mathrm{Hom}(\mathcal{E}, \mathfrak{g}) & \\ & \nearrow^{g \mapsto ([g, \underline{e}])} & \downarrow \mathrm{d}\phi_{\underline{e}} \\ \mathfrak{g} & \xrightarrow{\mathrm{d}\tilde{\pi}_1} & \mathrm{Hom}(\mathcal{E}, \mathfrak{g}/\mathcal{E}). \\ & \xrightarrow{\mathrm{d}\pi_1} & \end{array}$$

We are interested in $\ker \mathrm{d}\pi_1 = \ker(\mathrm{d}\phi_1 \circ \mathrm{d}\tilde{\pi}_1)$. Note that in [diagram 4.8.1](#) the identification $\mathfrak{g}^{\times r} \cong \mathrm{Hom}(\mathcal{E}, \mathfrak{g})$ is given by sending an r -tuple (g_1, \dots, g_r) to the map $\mathcal{E} \rightarrow \mathfrak{g}$ defined by $e_i \mapsto g_i$. Hence, the kernel of the vertical map is $\mathcal{E}^{\times r} \subset \mathfrak{g}^{\times r}$. The map $\mathrm{d}\tilde{\pi}_1$ is given by $g \mapsto ([g, e_1], \dots, [g, e_r])$. To land in $\ker \mathrm{d}\phi_1 = \mathcal{E}^{\times r}$ we must have $[g, e_i] \in \mathcal{E}$ for any $1 \leq i \leq r$. Hence, $\mathrm{d}\phi_1 \circ \mathrm{d}\tilde{\pi}_1(g) = 0$ if and only if $g \in N_{\mathfrak{g}}(\mathcal{E})$. \square

Theorem 4.9. *If $P = \mathrm{Stab}_G(\mathcal{E})$ is parabolic then the orbit $G \cdot \mathcal{E} \subseteq \mathbb{E}(\mathfrak{g})$ is isomorphic to the flag variety G/P .*

Proof. By Borel [[Bor91](#), 6.7] and the above lemma this is equivalent to the statement that $N_{\mathfrak{g}}(\mathcal{E}) \subseteq \mathrm{Lie}(P)$. Without loss of generality we assume P is a standard parabolic, then \mathcal{E} is fixed by the torus and therefore is a Chevalley subalgebra. The normalizer $N_{\mathfrak{g}}(\mathcal{E})$ is then fixed by the torus as well so it suffices to choose $\alpha \in \Phi$ and show that $x_{\alpha} \in N_{\mathfrak{g}}(\mathcal{E})$ implies $x_{\alpha} \in \mathrm{Lie}(P)$.

We assume that p is separably good for G so, in particular, p is greater or equal to the length of the longest root string in Φ . This implies that any structure constants which are zero in k are zero in \mathbb{Z} . We get then that the \mathbb{Z} -form of x_{α} normalizes the \mathbb{Z} -form of \mathcal{E} . The action of the root space U_{α} on \mathfrak{g} is given by exponentiating the adjoint action of the \mathbb{Z} -form of x_{α} and then base changing to k , thus U_{α} stabilizes \mathcal{E} . As x_{α} spans the Lie algebra of U_{α} we have $x_{\alpha} \in \mathrm{Lie}(P)$ as desired. \square

Example 4.10. To see how the above argument can fail when p is less than the maximal length of a root string in Φ consider G of type G_2 and $p = 3$. We saw in [subsection 3.11](#) that $\mathcal{E} = \mathrm{Lie}(R_1)$ is a maximal elementary subalgebra. The stabilizer of \mathcal{E} is the Borel B but one can check that in $\mathfrak{g}_{\mathbb{Z}}$ we have $[x_{-\alpha_1}, \mathcal{E}] = 3\mathbb{Z}x_{\alpha_2} + \mathcal{E}$, thus $x_{-\alpha_1}$ normalizes \mathcal{E} in $\mathfrak{g}_{\mathbb{F}_3}$. If $M_0 \in \mathbb{M}_{14}(\mathbb{Z})$ is the matrix of $\mathrm{ad}(x_{-\alpha_1})$ in $\mathfrak{g}_{\mathbb{Z}}$ and $M_3 \in \mathbb{M}_{14}(\mathbb{F}_3)$ is its mod 3 image, i.e., the matrix of $\mathrm{ad}(x_{-\alpha_1})$ in $\mathfrak{g}_{\mathbb{F}_3}$, then $M_0^3 \neq 0$ but $M_3^3 = 0$. Thus even though $\exp(M_0)$ and $\exp(M_3)$ are both well defined and $\exp(M_3)$ stabilizes \mathcal{E} , the action of $U_{-\alpha_1}$ is given by the mod 3 image of $\exp(M_0)$ and this does not equal $\exp(M_3)$.

4.3. Majority case. Retaining the notation from the previous section we now consider G which is *not* of type A_2 or G_2 .

Theorem 4.11. *Let G be a simple algebraic group, not of type A_2 or G_2 . Assume that p is separably good for G . Then*

$$\mathbb{E}(\mathfrak{g}) = \coprod_{\substack{R \in \text{Max}(\Phi) \\ R \text{ an ideal}}} G/P_R,$$

where $P_R = \text{Stab}_G(\text{Lie}(R))$.

Proof. From [corollary 4.7](#) the variety $\mathbb{E}(\mathfrak{g})$ is a union of orbits $G \cdot \text{Lie}(R)$ where R ranges over the ideals in $\text{Max}(\Phi)$ and [theorem 4.9](#) gives that each orbit $G \cdot \text{Lie}(R)$ is isomorphic to the flag variety G/P_R . These orbits are therefore closed and we need only prove that they are distinct.

By the Bruhat decomposition two such $\text{Lie}(R)$ are conjugate if and only if the corresponding R are conjugate via the Weyl group. At most one ideal in $\text{Max}(\Phi)$ is not of the form Φ_i^{rad} for some i so [lemma 2.6](#) gives that distinct maximal commuting ideals are non-conjugate as desired. \square

Combining [theorem 4.11](#) with [Table 2](#) we get, except for types A_2 and G_2 , the explicit type-by-type calculation of $\mathbb{E}(\mathfrak{g})$ found in [Table 4](#).

TABLE 4. $\mathbb{E}(\mathfrak{g})$ for p separably good.

Type	Restrictions on rank	$\mathbb{E}(\mathfrak{g})$
A_{2n}	$n = 1$	Irreducible, 5-dimensional
	$n \geq 2$	$G/P_{\Delta \setminus \{\alpha_n\}} \coprod G/P_{\Delta \setminus \{\alpha_{n+1}\}}$
A_{2n+1}	$n \geq 0$	$G/P_{\Delta \setminus \{\alpha_{n+1}\}}$
B_n	$n = 2, 3$	$G/P_{\Delta \setminus \{\alpha_1\}}$
	$n = 4$	$G/P_{\Delta \setminus \{\alpha_1\}} \coprod G/P_{\{\alpha_2, \alpha_3\}}$
	$n \geq 5$	$G/P_{\Delta \setminus \{\alpha_1, \alpha_n\}}$
C_n	$n \geq 3$	$G/P_{\Delta \setminus \{\alpha_n\}}$
D_n	$n = 4$	$G/P_{\Delta \setminus \{\alpha_1\}} \coprod G/P_{\Delta \setminus \{\alpha_3\}} \coprod G/P_{\Delta \setminus \{\alpha_4\}}$
	$n \geq 5$	$G/P_{\Delta \setminus \{\alpha_{n-1}\}} \coprod G/P_{\Delta \setminus \{\alpha_n\}}$
E_6		$G/P_{\Delta \setminus \{\alpha_1\}} \coprod G/P_{\Delta \setminus \{\alpha_6\}}$
E_7		$G/P_{\Delta \setminus \{\alpha_7\}}$
E_8		$G/P_{\Delta \setminus \{\alpha_2\}}$
F_4		$G/P_{\{\alpha_1, \alpha_3\}}$
G_2		Irreducible, 8-dimensional

4.4. Type A_2 . In type A_2 there are 3 disjoint orbits, two of which contain Chevalley subalgebras corresponding to ideals and one which does not contain a Chevalley

subalgebra. The following are representatives of the three orbits

$$\begin{aligned} L_1 &= \langle x_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle, \\ L_2 &= \langle x_{\alpha_1}, x_{\alpha_1+\alpha_2} \rangle, \\ L_3 &= \langle x_{\alpha_1} + x_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle. \end{aligned}$$

The stabilizers of these subalgebras are

$$\begin{aligned} \text{Stab}_G(L_1) &= P_1, \\ \text{Stab}_G(L_2) &= P_2, \\ \text{Stab}_G(L_3) &= \langle \alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_2), U \mid \lambda_1^3 = \lambda_2^3 \rangle, \end{aligned}$$

and so the orbits have dimensions

$$\begin{aligned} \dim(G \cdot L_1) &= 2, \\ \dim(G \cdot L_2) &= 2, \\ \dim(G \cdot L_3) &= 5. \end{aligned}$$

As $G \cdot L_1 \simeq G/P_1$ and $G \cdot L_2 \simeq G/P_2$ they are closed orbits and $G \cdot L_3$ is open. The map $\mathbb{P}^1 \rightarrow \mathbb{E}(\mathfrak{g})$ given by $[a : b] \mapsto \langle ax_{\alpha_1} + bx_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle$ then yields that the closure of $G \cdot L_3$ contains the other two orbits. So $G \cdot L_3$ is dense, hence $\mathbb{E}(\mathfrak{g})$ is irreducible of dimension 5.

4.5. Type G_2 . In type G_2 there are again 3 disjoint orbits, two of which contain Chevalley subalgebras and one which does not, but now only one orbit contains a Chevalley subalgebra corresponding to an ideal. The following are representatives of the three orbits

$$\begin{aligned} L &= \langle x_{\alpha_2} + x_{3\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle \\ \text{Lie}(C_3) &= \langle x_{\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle, \\ \text{Lie}(C_5) &= \langle x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle, \end{aligned}$$

and C_5 is the ideal. The stabilizers of these subalgebras are

$$\begin{aligned} \text{Stab}_G(L) &= \langle \alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_2), U_\alpha \mid \lambda_1^2 = \lambda_2^3, \alpha \in \Phi^+ \setminus \{\alpha_1\} \rangle, \\ \text{Stab}_G(\text{Lie}(C_3)) &= \langle T, U_\alpha \mid \alpha \in \Phi^+ \setminus \{\alpha_1\} \rangle, \\ \text{Stab}_G(\text{Lie}(C_5)) &= P_2, \end{aligned}$$

so the orbits have dimensions

$$\begin{aligned} \dim(G \cdot L) &= 8, \\ \dim(G \cdot \text{Lie}(C_3)) &= 7, \\ \dim(G \cdot \text{Lie}(C_5)) &= 5. \end{aligned}$$

As $\mathbb{E}(\mathfrak{g})$ is a closed subvariety of the Grassmannian it is complete. The Borel fixed point theorem then gives that $G \cdot \text{Lie}(C_5)$ is the only closed orbit. Boundaries of orbits are unions of orbits of smaller dimension therefore the closure of $G \cdot \text{Lie}(C_3)$ is its union with $G \cdot \text{Lie}(C_5)$ and the orbit $G \cdot L$ is open. To see that $G \cdot \text{Lie}(C_3)$ is not open we need that the closure of $G \cdot L$ contains $G \cdot \text{Lie}(C_3)$; one sees this from the map $\mathbb{A}^1 \rightarrow \mathbb{E}(\mathfrak{g})$ defined by $a \mapsto \langle x_{\alpha_2} + ax_{3\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle$ which sends $\mathbb{A}^1 \setminus \{0\}$ into $G \cdot L$ and 0 to $\text{Lie}(C_3)$. In particular, we have now shown that $G \cdot L$ is dense and so $\mathbb{E}(\mathfrak{g})$ is irreducible of dimension 8.

5. APPLICATIONS TO CHEVALLEY GROUPS

Let $k = \overline{\mathbb{F}_p}$ be the algebraic closure of \mathbb{F}_p and let G be a reductive k -group, defined and split over \mathbb{Z} as above. Our calculation of $\mathbb{E}(\mathfrak{g})$ yields information about the maximal elementary abelian p -subgroups of the Chevalley groups $G(\mathbb{F}_q)$ for $q = p^\ell$ and their conjugacy classes. Since any such subgroup can be conjugated into the Sylow p -subgroup $U(\mathbb{F}_q)$, we can assume that G is semi-simple simply connected. Assume p is good for G , and let $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$ be the varieties of p -unipotent and p -nilpotent elements in G and \mathfrak{g} respectively.

To translate between the Lie algebra and Chevalley group we use the theorem below.

Theorem 5.1 ([Sob15, 4.3]). *There exists a unique isomorphism $\phi: \mathcal{N}_1(\mathfrak{g}) \rightarrow \mathcal{U}_1(G)$ satisfying*

- (1) ϕ is G -equivariant,
- (2) $x, y \in \mathfrak{g}$ commute if and only if $\phi(x), \phi(y) \in G$ commute,
- (3) $x \in \mathfrak{g}_{\mathbb{F}_q}$ if and only if $\phi(x) \in G(\mathbb{F}_q)$.

Such a ϕ gives an inclusion preserving bijection between p -nilpotent commutative subsets of $\mathfrak{g}_{\mathbb{F}_q}$ and p -unipotent commutative subsets of $G(\mathbb{F}_q)$. A maximal commuting set of p -unipotent elements in $G(\mathbb{F}_q)$ is necessarily a maximal elementary abelian subgroup. Similarly a maximal set of commuting p -nilpotent elements in $\mathfrak{g}_{\mathbb{F}_q}$ is necessarily a maximal elementary subalgebra, and therefore corresponds to a maximal elementary subalgebra of \mathfrak{g} that is defined over \mathbb{F}_q . As the \mathbb{F}_q -rational points of the Grassmannian are exactly the subspaces defined over \mathbb{F}_q we now have the following.

Theorem 5.2. *The map ϕ induces a $G(\mathbb{F}_q)$ -equivariant bijection between the \mathbb{F}_q -rational points of $\mathbb{E}(\mathfrak{g})$ and the maximal elementary abelian subgroups of $G(\mathbb{F}_q)$.*

We note that when p is greater than the Coxeter number of the root system this theorem is a special case of a more general theorem of Warner [War15, 3.9] which gives a bijection between *all* elementary subalgebras of \mathfrak{g} defined over \mathbb{F}_q and what he calls the \mathbb{F}_q -linear elementary abelian subgroups of $G(\mathbb{F}_q)$. Our version has a weaker restriction on the prime p , but consequently only applies in the case of maximal elementary subalgebras and maximal elementary abelian subgroups.

Even though we don't need this observation for our application to Chevalley groups, the following enhancement of [corollary 4.4](#) is worthy of pointing out:

Corollary 5.3. *Let \mathcal{E} be an \mathbb{F}_q -rational point of $\mathbb{E}(\mathfrak{g})$. Then \mathcal{E} is $G(\mathbb{F}_q)$ -conjugate to a subalgebra of \mathfrak{u} .*

Proof. The subalgebra \mathcal{E} corresponds, using ϕ , to a maximal elementary abelian subgroup of $G(\mathbb{F}_q)$. As $U(\mathbb{F}_q)$ is a p -Sylow subgroup of $G(\mathbb{F}_q)$ an element $g \in G(\mathbb{F}_q)$ conjugates this elementary abelian into $U(\mathbb{F}_q)$. The equivariance of ϕ then gives that g conjugates \mathcal{E} into \mathfrak{u} . □

[Theorem 3.9](#) together with the calculation for A_2 in [section 3](#) show that in all types except for G_2 if two elementary subalgebras in $\mathbb{E}(\mathfrak{u})$ are defined over \mathbb{F}_q and conjugate by an element in $G(k)$ then they are already conjugate by an element in $G(\mathbb{F}_q)$. This observation, combined with [theorem 5.2](#), allows us to translate the results for $\mathbb{E}(\mathfrak{u})$ from [section 3](#) to the classification of conjugacy classes of maximal elementary abelian p -subgroups of $G(\mathbb{F}_q)$. Consequently, we recover the results

of Barry [Bar79] on maximal elementary abelian p -subgroups of Chevalley groups of classical type, and supplement Barry's results with similar information for the exceptional types.

Theorem 5.4. *Let G be a simple algebraic group defined and split over \mathbb{Z} . Assume p is good for G . Then the conjugacy classes and ranks of the elementary abelian p -subgroups of $G(\mathbb{F}_q)$ of maximal rank are given in Table 5, where in types E_8 and F_4 we take R to be the unique ideal in $\text{Max}(\Phi)$.*

Note that for G reductive, the representatives of conjugacy classes of maximal elementary abelian p -subgroups of $G(\mathbb{F}_q)$ are given by products of representatives for each simple entry in the direct product decomposition of the derived group $[G, G]$.

Example 5.5. In type G_2 we used the fact that k is algebraically closed in a nontrivial way in subsection 3.9 to conclude that $\mathbb{E}(\mathfrak{g})$ had three G -orbits. Thus we can only conclude that there are *at least* three conjugacy classes of maximal elementary abelian subgroups in $G(\mathbb{F}_q)$. In fact the number of conjugacy classes depends on q . For example, if $q = 5^\ell$ then $G(\mathbb{F}_q)$ has three conjugacy classes of elementary abelian subgroups when ℓ is odd but has six such conjugacy classes when ℓ is even.

As implied by the Quillen stratification theorem [Qui71] the above calculation gives the number of irreducible components of maximal dimension for $\text{Spec } H^*(G(\mathbb{F}_q), k)$.

Corollary 5.6. *Let G be a simple algebraic group defined and split over \mathbb{Z} . Assume p is good for G . Then the dimension of $\text{Spec } H^*(G(\mathbb{F}_q), k)$ and the number of irreducible components of maximal dimension are given in Table 6.*

Proof. By Quillen [Qui71] the dimension of $\text{Spec } H^*(G(\mathbb{F}_q), k)$ equals the maximal rank of an elementary abelian p -subgroup of $G(\mathbb{F}_q)$ which can be read from Table 5. The Quillen stratification theorem also implies that the number of irreducible components equals the number of conjugacy classes of maximal elementary abelian p -subgroups which once again can be read from Table 5. \square

APPENDIX A. MAXIMAL SETS OF COMMUTING ROOTS

In the Appendix we give some details for the description of the maximal subsets of commuting positive roots found in Table 2 and the stabilizers of certain ideals in Table 3. Note that the maximal subsets of commuting positive roots are computed in Malcev [Mal51] (except that he skips the proof for E_8) so we give details here only for completeness of our arguments.

We begin with the computation of maximal subsets of commuting positive roots. For type E , D_n when $n < 7$, and B_n when $n < 5$ we use a computer program which we have made available online [Sta14a]. For the remaining types we provide the following arguments. In several cases we start by looking at commutative subsets of the entire root system Φ , not just the positive roots. In this case, $\Gamma \subset \Phi$ is a commutative subset if for any $\alpha, \beta \in \Gamma$, $\alpha + \beta$ is not a root and $\beta \neq -\alpha$.

A.1. Type A_n . One can check, as in Grantcharov and Serganova [GS13], that sending $J \subseteq \{1, 2, \dots, n+1\}$ to the set of roots $\{\epsilon_i - \epsilon_j \mid i \in J, j \notin J\}$ yields a bijection between proper nontrivial subsets of $\{1, 2, \dots, n+1\}$ and sets of inclusion

TABLE 5. Maximal elementary abelian subgroups of $G(\mathbb{F}_q)$, $q = p^\ell$.

Type		# conjugacy classes	rank	Representatives
A_{2n}	$n = 1$	3	2ℓ	$\langle U_{\alpha_1}, U_{\alpha_1+\alpha_2} \rangle$ $\langle U_{\alpha_2}, U_{\alpha_1+\alpha_2} \rangle$ $\langle \phi(\mathbb{F}_q(x_{\alpha_1} + x_{\alpha_2})), U_{\alpha_1+\alpha_2} \rangle$
	$n \geq 2$	2	$n(n+1)\ell$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_{n+1}^{\text{rad}} \rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_n^{\text{rad}} \rangle$
A_{2n+1}	$n \geq 0$	1	$(n+1)^2\ell$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_{n+1}^{\text{rad}} \rangle$
B_n	$n = 2, 3$	1	$(2n-1)\ell$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_1^{\text{rad}} \rangle$
	$n = 4$	2	7ℓ	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_1^{\text{rad}} \rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in S_1 \rangle$
	$n \geq 5$	1	$\binom{n}{2}\ell + \ell$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in S_1 \rangle$
C_n	$n \geq 2$	1	$\binom{n+1}{2}\ell$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_n^{\text{rad}} \rangle$
D_n	$n = 4$	3	6ℓ	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_1^{\text{rad}} \rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_3^{\text{rad}} \rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_4^{\text{rad}} \rangle$
	$n \geq 5$	2	$\binom{n}{2}\ell$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_{n-1}^{\text{rad}} \rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_n^{\text{rad}} \rangle$
E_6		2	16ℓ	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_1^{\text{rad}} \rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_6^{\text{rad}} \rangle$
E_7		1	27ℓ	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi_7^{\text{rad}} \rangle$
E_8		1	36ℓ	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in R \rangle$
F_4		1	9ℓ	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in R \rangle$
G_2		≥ 3	3ℓ	

maximal commutative subsets of Φ . As J gets sent to a set of size $|J|(n+1-|J|)$ we see that this set is of maximal order when $n = 2m$ and $|J| = m, m+1$ or when $n = 2m+1$ and $|J| = m+1$. It is a set of positive roots if and only if $J < \{1, \dots, n+1\} \setminus J$, thus in type A_{2m} we have $J = \{1, \dots, m\}$ or $\{1, \dots, m+1\}$ yielding Φ_m^{rad} and Φ_{m+1}^{rad} , respectively, and in type A_{2m+1} we have $\{1, \dots, m+1\}$ yielding Φ_{m+1}^{rad} .

A.2. Type B_n . We assume $n \geq 5$. The set $R = \{\epsilon_i \mid 1 \leq i \leq n\}$ is an inclusion maximal set of non-commuting roots so any maximal set of commuting roots in Φ^+ consists of a maximal set of commuting roots in $\Phi^+ \setminus R$ together with at most one

TABLE 6. $\text{Spec } H^*(G(\mathbb{F}_p^\ell), k)$.

Type		# of irreducible components of max dimension	dimension
A_{2n}	$n = 1$	3	2ℓ
	$n \geq 2$	2	$n(n+1)\ell$
A_{2n+1}	$n \geq 0$	1	$(n+1)^2\ell$
B_n	$n = 2, 3$	1	$\ell(2n-1)$
	$n = 4$	1	7ℓ
	$n \geq 4$	1	$\binom{n}{2}\ell + \ell$
C_n	$n \geq 2$	1	$\binom{n+1}{2}\ell$
D_n	$n = 4$	3	6ℓ
	$n \geq 5$	2	$\binom{n}{2}\ell$
E_6		2	16ℓ
E_7		1	27ℓ
E_8		1	36ℓ
F_4		1	9ℓ
G_2		≥ 3	3ℓ

element from R . Observe that $\Psi = \Phi \setminus \pm R$ is a root system of type D_n with simple roots $\{\alpha_1, \dots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\}$. The maximal set Ψ_n^{rad} can commute with any ϵ_i and yields S_i . The maximal set Ψ_{n-1}^{rad} commutes with ϵ_i when $i < n$ and yields S_i^* .

A.3. Type C_n . Sending $J \subseteq \{1, \dots, n\}$ to the set

$$\phi(J) = \{\epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j, -\epsilon_j - \epsilon_{j'} \mid i, i' \in J \text{ and } j, j' \notin J\}$$

gives a bijection ϕ between the power set of $\{1, \dots, n\}$ and inclusion maximal unipotent commuting subsets of Φ . Among those subsets J satisfying $|J| = m$, the number of positive roots in $\phi(J)$ attains a maximum of $\frac{1}{2}m(m+1) + m(n-m)$ when $J < \{1, \dots, n\} \setminus J$ and this maximum value for a given m attains a maximum of $\frac{1}{2}n(n+1)$ when $m = n$. Thus we take the positive roots of $\phi(\{1, \dots, n\})$ and get Φ_n^{rad} .

A.4. Type D_n . We assume $n \geq 7$.

Lemma A.1. *Let Φ be type D_n . If $R \subseteq \Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ is an inclusion maximal set of commuting roots which contains $\epsilon_1 - \epsilon_2$ then $R = \Phi_1^{\text{rad}}$ has order $2n - 2$. If it does not contain $\epsilon_1 - \epsilon_2$ then it consists of the root $\epsilon_1 + \epsilon_2$ together with one choice of root from each of the sets $\{\epsilon_1 + \epsilon_r, \epsilon_2 - \epsilon_r\}_{2 < r \leq n}$ and $\{\epsilon_1 - \epsilon_r, \epsilon_2 + \epsilon_r\}_{2 < r \leq n}$, and hence has order $2n - 3$.*

Proof. We have $\Phi_{\alpha_1, \alpha_2}^{\text{rad}} = \{\epsilon_1 \pm \epsilon_i, \epsilon_2 \pm \epsilon_j \mid 2 \leq i \leq n \text{ and } 3 \leq j \leq n\}$. If $\epsilon_1 - \epsilon_2$ is contained in our maximal set then the roots $\epsilon_2 \pm \epsilon_j$ are not, so the set contains at most the roots $\epsilon_1 \pm \epsilon_i$, i.e., the roots of Φ_1^{rad} . These indeed commute and there are $2n - 2$ of them. If $\epsilon_1 - \epsilon_2$ is not contained in our maximal set then note that $\epsilon_1 + \epsilon_2$ is the longest root and therefore is contained in any inclusion maximal set of commuting roots. The remaining roots form the sets of non-commuting pairs given in the statement. One sees that roots from distinct pairs commute and there are $2n - 4$ such pairs. \square

Observe that $m(\Phi) \geq |\Phi_n^{\text{rad}}| = \frac{1}{2}n(n-1)$. Also Φ_1^{rad} is inclusion maximal and of smaller order so no element of $\text{Max}(\Phi)$ contains Φ_1^{rad} . Now $\Psi = \Phi \setminus \pm\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ is a root system of type D_{n-2} with simple roots $\{\alpha_3, \dots, \alpha_n\}$. Every set of commuting roots in Φ^+ is the union of sets of commuting roots from Ψ^+ and $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$. By the lemma above the set of commuting roots from $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ can have at most $2n - 3$ elements and by induction the set from Ψ^+ can have at most $\frac{1}{2}(n-2)(n-3)$. These sum to the order of Φ_n^{rad} so a maximal set of commuting roots must be the union of a maximal set from Ψ^+ and a set of order $2n - 3$ from $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$.

Now it suffices to take $R \in \text{Max}(\Psi)$ and check which roots in $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ it commutes with. If $R = \Psi_n^{\text{rad}}$ then $\epsilon_i + \epsilon_n \in R$ for all $2 < i < n$ so $\epsilon_1 - \epsilon_j, \epsilon_2 - \epsilon_j \notin R$ for all $2 < j \leq n$. This identifies a unique inclusion maximal set of commuting roots in $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ and its union with Ψ_n^{rad} is Φ_n^{rad} . If $R = \Psi_{n-1}^{\text{rad}}$ then $\epsilon_i - \epsilon_n \in R$ for all $2 < i < n$ so $\epsilon_1 - \epsilon_i, \epsilon_2 - \epsilon_i, \epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_n \notin R$. Again this identifies the inclusion maximal set in $\Phi_{\alpha_1, \alpha_2}^{\text{rad}}$ and its union with Ψ_{n-1}^{rad} is Φ_{n-1}^{rad} .

A.5. Type F_4 . Recall that the roots of F_4 are

$$\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j, \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4).$$

We will denote positive roots of the last type by $\epsilon_{ijk} = \frac{1}{2}(\epsilon_1 + i\epsilon_2 + j\epsilon_3 + k\epsilon_4)$ where $i, j, k \in \{\pm 1\}$ and will write, for example, ϵ_{+-+} instead of $\epsilon_{1,-1,1}$.

Lemma A.2. *If $\epsilon_{ijk} \neq \epsilon_{i'j'k'}$ then ϵ_{ijk} and $\epsilon_{i'j'k'}$ commute if and only if there is exactly one sign change between (i, j, k) and (i', j', k') . In particular, a commuting set of roots can have at most 2 roots of the form ϵ_{ijk} .*

Proof. Observe that the terms in $\epsilon_{ijk} + \epsilon_{i'j'k'}$ are exactly those e_t for which the sign did not change (including ϵ_1). If there are 1 or 2 such terms then $\epsilon_{ijk} + \epsilon_{i'j'k'}$ is a root. As the roots are distinct there cannot be 4 such terms, therefore for ϵ_{ijk} and $\epsilon_{i'j'k'}$ to commute there must be 3 such terms, hence exactly one sign change.

If (i, j, k) , (i', j', k') , and (i'', j'', k'') are mutually distinct and there is exactly one sign change from (i, j, k) to (i', j', k') and (i'', j'', k'') then there are 2 sign changes from (i', j', k') to (i'', j'', k'') . This proves that 3 roots of the form (i, j, k) cannot pairwise commute. \square

Now observe that the roots $\pm\epsilon_i$ and $\pm\epsilon_i \pm \epsilon_j$ give $B_4 \subseteq F_4$. As the commuting property is preserved when intersecting with a subroot system the lemma above gives that a maximal set of commuting roots in F_4 can have at most 9 roots: 7 from a maximal set in B_4 plus 2 additional roots of the form ϵ_{ijk} . This maximum is indeed attained so every maximal set of commuting positive roots in F_4 is identified by a triple $(C, ijk, i'j'k')$, where $C \subseteq B_4$ is a maximal set of commuting roots and ϵ_{ijk} and $\epsilon_{i'j'k'}$ are the two additional roots. We now compute all the possibilities.

A.5.1. $C = \Phi_1^{\text{rad}}$. Every ϵ_{ijk} commutes with $C = \{\epsilon_1, \epsilon_1 \pm \epsilon_i \mid i = 2, 3, 4\}$. Once (i, j, k) are chosen there are three tripels (i', j', k') which differ by a single sign change. This gives 24 ordered pairs of roots $(\epsilon_{ijk}, \epsilon_{i'j'k'})$ that can be added, thus 12 possible sets of roots for this case.

A.5.2. $C = S_t$. We have $C = \{\epsilon_t, \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq 4\}$ and for ϵ_{ijk} to commute with ϵ_t the sign on ϵ_t must be positive. We cannot have more than one negative sign in (i, j, k) otherwise ϵ_{ijk} would not commute with some root of the form $\epsilon_i + \epsilon_j$. Thus the two roots of the form ϵ_{ijk} must be ϵ_{+++} and ϵ_{ijk} where there is exactly one negative sign in (i, j, k) and this negative sign is not on the ϵ_t term.

Thus for $C = S_1$ we get three maximal sets corresponding to the three choices for a negative sign and for $C = S_2, S_3, S_4$ we get two maximal sets each. This gives 9 possible sets of roots for this case.

A.5.3. $C = S_t^*$. We have $C = \{\epsilon_t, \epsilon_i + \epsilon_j, \epsilon_{i'} - \epsilon_4 \mid 1 \leq i < j < 4, 1 \leq i' < 4\}$. Because of the $\epsilon_{i'} - \epsilon_4$ terms the only ϵ_{ijk} with a single negative that commutes with C is ϵ_{++-} . As ϵ_{--+} and ϵ_{---} don't commute with $\epsilon_2 + \epsilon_3$ we find that the two additional elements must be ϵ_{++-} and ϵ_{+++} or ϵ_{++-} and ϵ_{ij-} where exactly one of i, j is negative and the negative is not on the ϵ_t term.

The first choice is valid for any t . For the second when $C = S_1$ there are two choices for the additional negative and when $C = S_2, S_3$ there is one choice for the additional negative. This gives 7 possible sets of roots for this case.

A.6. **Type G_2 .** There are 3 short and 3 long positive roots. No pair of short positive roots commute so there can be at most 1 short root in a maximal commuting set. The pair of long roots $(\alpha_2, 3\alpha_1 + \alpha_2)$ does not commute so a maximal set contains the highest root $3\alpha_1 + 2\alpha_2$ together with at most 1 other long and 1 short root. With this one can check that the maximal sets are

$$\begin{aligned} & \{\alpha_1, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ & \{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ & \{\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ & \{\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ & \{2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}. \end{aligned}$$

REFERENCES

- [Bar79] Michael J. J. Barry. Large abelian subgroups of Chevalley groups. *J. Austral. Math. Soc. Ser. A*, 27(1):59–87, 1979.
- [BL00] Sara Billey and V. Lakshmibai. *Singular loci of Schubert varieties*, volume 182 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2000.
- [Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [Bou02] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.
- [Car72] Roger W. Carter. *Simple groups of Lie type*. John Wiley & Sons, London-New York-Sydney, 1972. Pure and Applied Mathematics, Vol. 28.
- [CFP12] Jon F. Carlson, Eric Friedlander, and Julia Pevtsova. Representations of elementary abelian p -groups and bundles on grassmannians. *Adv. Math.*, 229:2985–3051, 2012.

- [CFP15] Jon F. Carlson, Eric Friedlander, and Julia Pevtsova. Elementary subalgebras of modular lie algebras. *J. Algebra*, 442:155–189, 2015.
- [CFP16] Jon F. Carlson, Eric Friedlander, and Julia Pevtsova. Vector bundles associated to lie algebras. *Journal für die Reine und Ang Math. [Crelle's Journal]*, 716:147–178, 2016.
- [FP11] Eric M. Friedlander and Julia Pevtsova. Constructions for infinitesimal group schemes. *Trans. Amer. Math. Soc.*, 363(11):6007–6061, 2011.
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998. Almost simple K -groups.
- [GM06] Meinolf Geck and Gunter Malle. Reflection groups. In *Handbook of algebra. Vol. 4*, volume 4 of *Handb. Algebr.*, pages 337–383. Elsevier/North-Holland, Amsterdam, 2006.
- [GS13] D. Grantcharov and V. Serganova. On weight modules of algebras of twisted differential operators on the projective space. *ArXiv e-prints*, June 2013.
- [Hum90] James E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [Jan03] Jens Carsten Jantzen. *Representations of Algebraic Groups*. American Mathematical Society, second edition, 2003.
- [KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
- [LMT09] Paul Levy, George McNinch, and Donna M. Testerman. Nilpotent subalgebras of semisimple Lie algebras. *C. R. Math. Acad. Sci. Paris*, 347(9-10):477–482, 2009.
- [Mal51] A. I. Malcev. Commutative subalgebras of semi-simple Lie algebras. *Amer. Math. Soc. Translation*, 1951(40):15, 1951. Translated from *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]*, 9:291-300, 1945.
- [MP87] R. James Milgram and Stewart B. Priddy. Invariant theory and $H^*(\mathrm{GL}_n(\mathbb{F}_p); \mathbb{F}_p)$. *J. Pure Appl. Algebra*, 44(1-3):291–302, 1987.
- [MT11] Gunter Malle and Donna Testerman. *Linear algebraic groups and finite groups of Lie type*, volume 133 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011.
- [Qui71] Daniel Quillen. The spectrum of an equivariant cohomology ring. I, II. *Ann. of Math. (2)*, 94:549–572; *ibid. (2)* 94 (1971), 573–602, 1971.
- [RRS92] Roger Richardson, Gerhard Röhrle, and Robert Steinberg. Parabolic subgroups with abelian unipotent radical. *Invent. Math.*, 110(3):649–671, 1992.
- [Sch05] Issai Schur. Zur theorie der vertauschbaren matrizen. *J. Reine Angew. Math.*, 130:66–76, 1905.
- [SFB97] Andrei Suslin, Eric M. Friedlander, and Christopher P. Bendel. Support varieties for infinitesimal group schemes. *J. Amer. Math. Soc.*, 10(3):729–759, 1997.
- [Sob15] Paul Sobaje. Springer isomorphisms in characteristic p . *Transformation Groups*, 20(4):1141–1153, 2015.
- [Sta14a] Jim Stark. Commuting root subgroups. <http://sites.math.washington.edu/~julia/Preprints/varieties/CommutingRootSubgroups.sws>, September 2014.
- [Sta14b] Jim Stark. Conjugacy of abelian lie subalgebras. <http://sites.math.washington.edu/~julia/Preprints/varieties/roots.mgm>, October 2014.
- [Ste75] Robert Steinberg. Torsion in reductive groups. *Advances in Math.*, 15:63–92, 1975.
- [Sut04] Ruedi Suter. Abelian ideals in a Borel subalgebra of a complex simple Lie algebra. *Invent. Math.*, 156(1):175–221, 2004.
- [War15] Jared Warner. \mathbb{F}_p -expressible subalgebras and orbits of $\mathbb{E}(r, \mathfrak{g})$. *JPAA*, 219(8):3355–3371, August 2015.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA
E-mail address: julia@math.washington.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA
E-mail address: jstarx@gmail.com