VARIETIES FOR MODULES OF QUANTUM ELEMENTARY ABELIAN GROUPS

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Abstract. We define a rank variety for a module of a noncocommutative Hopf algebra $A = \Lambda \rtimes G$ where $\Lambda = k[X_1, \ldots, X_m]/(X_1^\ell, \ldots, X_m^\ell)$, $G = (\mathbb{Z}/\ell\mathbb{Z})^m$, and char k does not divide ℓ, in terms of certain subalgebras of A playing the role of "cyclic shifted subgroups". We show that the rank variety of a finitely generated module M is homeomorphic to the support variety of M defined in terms of the action of the cohomology algebra of A. As an application we derive a theory of rank varieties for the algebra Λ. When $\ell = 2$, rank varieties for Λ-modules were constructed by Erdmann and Holloway using the representation theory of the Clifford algebra. We show that the rank varieties we obtain for Λ-modules coincide with those of Erdmann and Holloway.

1. Introduction

The theory of varieties for modules of a finite group G began with the groundbreaking work of Quillen [27], a stratification of the maximal ideal spectrum of the cohomology ring of G into pieces indexed by elementary abelian subgroups. This idea was taken further by Avrunin and Scott [3], to a stratification of an affine variety associated to any finitely generated module. These results depended on earlier work of Venkov [33] and Evens [17], showing that the cohomology of G, a graded commutative ring, is finitely generated.

The theory took a different twist with the introduction by Carlson [10] of the *rank variety* for a module of an elementary abelian group E. The rank variety is yet another geometric invariant of a module, and is defined in terms of *cyclic shifted subgroups* of E. Carlson conjectured that the variety arising from the action of cohomology, and the rank variety defined purely in terms of representation-theoretic properties of a module, coincide. The conjecture was proven by Avrunin and Scott [3].

This theory was adapted to restricted Lie algebras by Friedlander and Parshall [18]. It was then further generalized to other finite group schemes (see [19, 31, 32]) based upon the fundamental theorem of Friedlander and Suslin stating that the

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cohomology of any finite group scheme, or equivalently finite dimensional cocommutative Hopf algebra, is finitely generated [21]. In particular, the notion of rank variety was recently generalized to all finite group schemes by Friedlander and the first author [19]. One important aspect of the rank variety in the context of finite group schemes is that it satisfies the ultimate generalization of the Avrunin-Scott Theorem: The rank variety of a module defined in a purely representation-theoretic way is homeomorphic to the support variety defined cohomologically. The interplay between the two seemingly very different descriptions of the variety of a module allows for applications both in cohomology and in representation theory.

Much less is known in the context of finite dimensional noncocommutative Hopf algebras. Ginzburg and Kumar computed the cohomology rings of quantum groups at roots of unity, and these happen to be finitely generated [22]. This fact allowed mathematicians to start development of the theory of support varieties for modules of these small quantum groups (see [26], [28]). However it appears difficult to give an equivalent representation-theoretic definition of variety for these quantum groups in general. Even less has been done for other types of finite dimensional noncocommutative Hopf algebras, and in particular it is an open question as to whether their cohomology is finitely generated.

In this paper, we have modest goals. We only consider Hopf algebras that are quantum analogues of elementary abelian groups, namely tensor products of Taft algebras (which are also Borel subalgebras of $u_q(sl_2^{\times m})$). We define the rank variety of a module for such a Hopf algebra (Definition 3.2), giving the first definition of rank varieties for modules of a noncocommutative Hopf algebra. Our definition employs the analogue of a "shifted cyclic subgroup" (2.1.1) in this context. These "subgroups" are subalgebras generated by certain nilpotent elements. As in the case of elementary abelian p-groups, they are parametrized by the affine space $\mathbb{A}^m$.

The cohomology of a tensor product of Taft algebras is finitely generated, so we may also associate a support variety, defined cohomologically, to any module (4.2.1). We show that the rank variety of any finitely generated module is homeomorphic to the support variety (Theorem 5.6), thus providing an analogue of the Avrunin-Scott Theorem in our context. We use "Carlson's modules" L_ζ as our main tool and apply the techniques developed in [15] and [16] in the study of support varieties defined via Hochschild cohomology. We expect that our results will shed light on the problem of constructing a rank variety for a broader class of finite dimensional Hopf algebras, including the small quantum groups.

One of the most important applications of the identification of the rank and support varieties in the setting of finite group schemes is the proof of the "tensor product property" which expresses the variety of a tensor product as the intersection of varieties (see [3], [18], [19], [32]). Another common application is a classification of thick tensor ideal subcategories in the stable module category (see [8], [20]). Both of these applications will be addressed in a sequel to this paper.

Our results have consequences beyond Hopf algebras. A tensor product of Taft algebras is isomorphic to a skew group algebra $A = \Lambda \rtimes G$ where the group $G \cong (\mathbb{Z}/\ell\mathbb{Z})^m$ is elementary abelian (in nondefining characteristic) and $\Lambda = k[X_1, \ldots, X_m]/(X_1^\ell, \ldots, X_m^\ell)$. When $\ell = 2$, that is, the generators of Λ have square 0, Erdmann and Holloway have used Hochschild cohomology to define support varieties for Λ-modules [15], applying a theory of varieties for modules of algebras initiated by Snashall and Solberg [30]. The support variety of a Λ-module in this case is equivalent to a rank variety defined representation-theoretically by Erdmann and Holloway. Their approach is quite different from ours: They use a "stable map description" of the rank variety and representation theory of the Clifford algebra. In this paper we use the extension of Λ to A to give definitions of support and rank varieties for Λ-modules more generally (see (6.0.1) and (6.5.1)), that is for *any* ℓ not divisible by the characteristic of the field k, and to show that the varieties we obtain are homeomorphic (Corollary 6.7). In case the generators of Λ have square 0, our varieties coincide with those of Erdmann and Holloway, giving an alternative approach to their theory. In order to make this connection, we found it necessary to record some basic facts relating cohomology and Hochschild cohomology of finite dimensional Hopf algebras in an appendix.

When this article was nearly complete, the authors learned that Benson, Erdmann, and Holloway had found a different way to define rank varieties for Λ-modules for arbitrary ℓ, involving an algebra extension of Λ that is a tensor product of Λ with a twisted group algebra of G [9]. Their algebra extension has some features in common with ours, leading to a parallel theory. We thank Benson, Erdmann, and Holloway for some very helpful conversations.

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Throughout this paper, k will denote an algebraically closed field containing a primitive ℓth root of unity q; in particular ℓ is not divisible by the characteristic of k. All tensor products and dimensions will be over k unless otherwise indicated. We shall use the notation $V^\#$ for the k-linear dual of a finite dimensional vector space V.

2. Quantum analogues of cyclic shifted subgroups

Let m be a positive integer and let G denote the group $(\mathbb{Z}/\ell\mathbb{Z})^m$ with generators $g_1, \ldots, g_m$. Define an action of G by automorphisms on the polynomial ring $R = k[X_1, \ldots, X_m]$ by setting

$$g_i \cdot X_j = q^{\delta_{ij}} X_j$$

for all i, j, where δ_{ij} is the Kronecker delta. Let $\widetilde{A} = R \rtimes G$, the skew group algebra, that is $\widetilde{A}$ is a free left R-module having R-basis G, with the semidirect (or smash) product multiplication

$$(rg)(sh) = r(g\cdot s)gh$$

for all $r, s \in R$ and $g, h \in G$. Then $\widetilde{A}$ is a Hopf algebra with

$$\Delta(X_i) = X_i \otimes 1 + g_i \otimes X_i, \qquad \Delta(g_i) = g_i \otimes g_i,$$

$\varepsilon(X_i) = 0$, $\varepsilon(g_i) = 1$, $S(X_i) = -g_i^{-1}X_i$, and $S(g_i) = g_i^{-1}$, for all i. Letting $h_1 = 1$ and $h_j = \prod_{i=1}^{j-1} g_i$ $(2 \le j \le m)$, we have

$$X_jh_j \cdot X_ih_i = qX_ih_i \cdot X_jh_j \quad \text{for all } j > i. \tag{2.0.1}$$

The following consequence of this q-commutativity of the elements X_ih_i will be most essential in what follows.

Lemma 2.1. *For any* $\lambda_1, \ldots, \lambda_m \in k$, $\left(\sum_{j=1}^{m} \lambda_j X_j h_j\right)^{\ell} = \sum_{j=1}^{m} \lambda_j^{\ell} X_j^{\ell}$.

Proof. This is a consequence of the q-binomial formula which in this context gives, for all $n \le \ell$ and $j > i$,

$$(\lambda_iX_ih_i + \lambda_jX_jh_j)^n = \sum_{s=0}^{n} \frac{(n)_q!}{(s)_q!(n-s)_q!}(\lambda_iX_ih_i)^s(\lambda_jX_jh_j)^{n-s},$$

where $(s)_q = 1 + q + q^2 + \cdots + q^{s-1}$, $(s)_q! = (s)_q(s-1)_q\cdots(1)_q$, and $(0)_q! = 1$ by definition. If $n = \ell$, the coefficients of $\lambda_i^{\ell}X_i^{\ell}$ and $\lambda_j^{\ell}X_j^{\ell}$ should be interpreted to be 1. As q is a primitive ℓth root of 1, induction on m yields the desired result. □

Another application of the q-binomial formula, to $\Delta(X_i^{\ell}) = (X_i \otimes 1 + g_i \otimes X_i)^{\ell}$, shows that the ideal $(X_1^{\ell}, \ldots, X_m^{\ell})$ is a Hopf ideal. Thus

$$A = \widetilde{A}/(X_1^{\ell}, \ldots, X_m^{\ell})$$

is a Hopf algebra of dimension ℓ^{2m}, a tensor product of m copies of a Taft algebra, a quantum analogue of an elementary abelian group. We may identify A with the skew group algebra $\Lambda \rtimes G$ where

$$\Lambda = k[X_1, \ldots, X_m]/(X_1^{\ell}, \ldots, X_m^{\ell}),$$

a truncated polynomial algebra. We will primarily be interested in the finite dimensional Hopf algebra A in this paper, but will need to use $\widetilde{A}$ as well in some of the proofs. Note that since A is a finite dimensional Hopf algebra, it is a Frobenius algebra [25, Thm. 2.1.3], and in particular is self-injective.

We now introduce algebra maps $\tau_{\underline{\lambda}}$ which will play the role of "cyclic shifted subgroups" (see [5, II]) or p-points ([19]) for the algebra A. By Lemma 2.1, for each point $\underline{\lambda} = [\lambda_1 : \ldots : \lambda_m]$ in k-projective space $\mathbb{P}^{m-1}$, there is an embedding of algebras

$$\tau_{\underline{\lambda}} : k[t]/(t^{\ell}) \to A \tag{2.1.1}$$

defined by $\tau_{\underline{\lambda}}(t) = \sum_{i=1}^{m} \lambda_i X_i h_i$. Denote the image of $\tau_{\underline{\lambda}}$ by $k\langle\tau_{\underline{\lambda}}(t)\rangle$.

Lemma 2.2. *Let $\underline{\lambda} = [\lambda_1 : \ldots : \lambda_m] \in \mathbb{P}^{m-1}$. Then A is free as a left (respectively, right) $k\langle\tau_{\underline{\lambda}}(t)\rangle$-module, with $k\langle\tau_{\underline{\lambda}}(t)\rangle$-basis*

$$\mathcal{B} = \{X_2^{a_2} \cdots X_m^{a_m} g_1^{b_1} \cdots g_m^{b_m} \mid 0 \leq a_i, b_i \leq \ell - 1\}$$

in case $\lambda_1 \neq 0$. Analogous statements hold if $\lambda_i \neq 0$ for other values of i.

Proof. We shall prove that $\mathcal{B}$ is a free $k\langle\tau_{\underline{\lambda}}(t)\rangle$-basis of A as a left $k\langle\tau_{\underline{\lambda}}(t)\rangle$-module. That it is also a basis of A as a right module is proved similarly.

We may assume that $\lambda_1 = 1$. Since the number of elements in $\mathcal{B}$ is $\ell^{2m-1} = \dim_k A / \dim_k k\langle\tau_{\underline{\lambda}}(t)\rangle$, it suffices to show that $A = k\langle\tau_{\underline{\lambda}}(t)\rangle\mathcal{B}$.

We use induction on a_1 to show that $X_1^{a_1} \cdots X_m^{a_m} g_1^{b_1} \cdots g_m^{b_m} \in k\langle\tau_{\underline{\lambda}}(t)\rangle\mathcal{B}$ for any choice of exponents $0 \leq a_i, b_i \leq \ell - 1$. The statement is trivial for $a_1 = 0$. Assume it is proved for all monomials with $a_1 < n \leq \ell - 1$. It remains to show that $X_1^n X_2^{a_2} \cdots X_m^{a_m} g_1^{b_1} \cdots g_m^{b_m} \in k\langle\tau_{\underline{\lambda}}(t)\rangle\mathcal{B}$. The defining relations on X_i and g_j, together with the definition of $\tau_{\underline{\lambda}}(t)$ given above (2.1.1), immediately imply that

$$X_1^n X_2^{a_2} \cdots X_m^{a_m} g_1^{b_1} \cdots g_m^{b_m} - \tau_{\underline{\lambda}}(t) X_1^{n-1} X_2^{a_2} \cdots X_m^{a_m} g_1^{b_1} \cdots g_m^{b_m}$$

is a sum of monomials $X_1^{a_1'} \cdots X_m^{a_m'} g_1^{b_1'} \cdots g_m^{b_m'}$ for some exponents a_i', b_i' with $a_1' < n$. The statement follows by induction. $\square$

To every point $\underline{\lambda}$ in k-projective space $\mathbb{P}^{m-1}$ we associate two special *left* A-modules: $V(\underline{\lambda})$ and $V'(\underline{\lambda})$, which will be used extensively throughout the paper. We point out that our modules are different from those used in [14], [15] even though we choose to use similar names for them. As will be shown in Corollary 5.7, they share one of the main properties with the modules introduced in [14]: The rank variety of each of $V(\underline{\lambda})$ and $V'(\underline{\lambda})$ will be the point $\underline{\lambda} \in \mathbb{P}^{m-1}$.

For each $\underline{\lambda} \in \mathbb{P}^{m-1}$, let

(2.2.1) $$V(\underline{\lambda}) = A \cdot \tau_{\underline{\lambda}}(t)^{\ell-1} \quad \text{and} \quad V'(\underline{\lambda}) = A \cdot \tau_{\underline{\lambda}}(t),$$

that is $V(\underline{\lambda})$ (respectively, $V'(\underline{\lambda})$) is the left ideal generated by $\tau_{\underline{\lambda}}(t)^{\ell-1}$ (respectively, $\tau_{\underline{\lambda}}(t)$).

Recall that for an A-module M, the *Heller shift* of M, denoted $\Omega(M)$, is the kernel of the projection $P(M) \to M$ where $P(M)$ is the projective cover of M. Similarly, $\Omega^{-1}(M)$ is the cokernel of the embedding of M into its injective hull.

Lemma 2.3. *For each $\underline{\lambda} \in \mathbb{P}^{m-1}$ we have:*

(i) $V(\underline{\lambda}) \cong k \uparrow_{k\langle\tau_{\underline{\lambda}}(t)\rangle}^{A} = A \otimes_{k\langle\tau_{\underline{\lambda}}(t)\rangle} k$.

(ii) *The restriction $V(\underline{\lambda}) \downarrow_{k\langle\tau_{\underline{\lambda}}(t)\rangle}$ contains the trivial module as a direct summand. In particular, $V(\underline{\lambda})$ is not projective as a $k\langle\tau_{\underline{\lambda}}(t)\rangle$-module.*

(iii) $\dim_k V(\underline{\lambda}) = \ell^{2m-1}$, $\dim_k V'(\underline{\lambda}) = (\ell-1)\ell^{2m-1}$, *and there is a short exact sequence of* A*-modules* $0 \to V'(\underline{\lambda}) \xrightarrow{\iota} A \xrightarrow{\pi} V(\underline{\lambda}) \to 0$.

(iv) $\cdots \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)} A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}} A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)} A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}} V(\underline{\lambda}) \to 0$
is a minimal projective resolution of $V(\underline{\lambda})$, *and*

$$0 \to V(\underline{\lambda}) \to A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)} A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}} A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)} A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}} \cdots$$

is a minimal injective resolution of $V(\underline{\lambda})$. *This remains true if* $V(\underline{\lambda})$ *is replaced by* $V'(\underline{\lambda})$, *with appropriate changes in the powers of* $\tau_{\underline{\lambda}}(t)$.

(v) $\Omega(V(\underline{\lambda})) \cong V'(\underline{\lambda})$ *and* $\Omega(V'(\underline{\lambda})) \cong V(\underline{\lambda})$.

Proof. (i) Define a map $\phi : A \times k \to V(\underline{\lambda}) = A\tau_{\underline{\lambda}}(t)^{\ell-1}$ by $\phi(a,c) = ca\tau_{\underline{\lambda}}(t)^{\ell-1}$ for all $a \in A, c \in k$, a k-bilinear map that commutes with left multiplication by elements of A. Note that $\phi(a\tau_{\underline{\lambda}}(t), c) = 0 = \phi(a, \tau_{\underline{\lambda}}(t)\cdot c)$, the latter equality due to the trivial action of $k\langle\tau_{\underline{\lambda}}(t)\rangle$ on k. Thus ϕ induces an A-map from the tensor product $A \otimes_{k\langle\tau_{\underline{\lambda}}(t)\rangle} k$ to $V(\underline{\lambda})$. One readily checks that this map gives a bijection between the k-bases $\mathcal{B}\otimes 1$ of $A\otimes_{k\langle\tau_{\underline{\lambda}}(t)\rangle} k$ and $\mathcal{B}\tau_{\underline{\lambda}}(t)^{\ell-1}$ of $V(\underline{\lambda})$ where $\mathcal{B}$ is defined in Lemma 2.2.

(ii) The trivial $k\langle\tau_{\underline{\lambda}}(t)\rangle$-submodule $1\otimes k$ of $A\otimes_{k\langle\tau_{\underline{\lambda}}(t)\rangle} k$ is complemented by the k-linear span of $(\mathcal{B}-\{1\})\otimes_{k\langle\tau_{\underline{\lambda}}(t)\rangle} k$.

(iii) By the proof of (i) above, $\mathcal{B}$ is in bijection with a k-basis of $V(\underline{\lambda})$, so $\dim_k V(\underline{\lambda}) = \ell^{2m-1}$. Similarly, a k-basis of $V'(\underline{\lambda})$ is $\cup_{i=1}^{\ell-1}\mathcal{B}\tau_{\underline{\lambda}}(t)^i$, of cardinality $(\ell-1)\ell^{2m-1}$. The map $\iota : V'(\underline{\lambda}) \to A$ in the statement of the lemma is inclusion, and the map $\pi : A \to V(\underline{\lambda})$ is given by $\pi(a) = a\otimes 1 \in A\otimes_{k\langle\tau_{\underline{\lambda}}(t)\rangle} k \cong V(\underline{\lambda})$. Again by considering bases of each of these modules, the sequence given in the lemma is seen to be exact.

(iv) Note that the Jacobson radical of A is $\operatorname{rad}(A) = A\cdot\operatorname{rad}(\Lambda)$, the ideal generated by $X_1,\dots,X_m$. The first resolution is minimal as $A/\operatorname{rad}(A) \cong kG \cong V(\underline{\lambda})/\operatorname{rad}(V(\underline{\lambda}))$ as A-modules. For minimality of the second resolution, note that the socle of A, $\operatorname{soc}(A)$, is the k-linear span of all $X_1^{\ell-1}\cdots X_m^{\ell-1}g_1^{b_1}\cdots g_m^{b_m}$, where $0\le b_i\le \ell-1$. We claim that in the notation of Lemma 2.2, the socle of $V(\underline{\lambda})$ has a basis in one-to-one correspondence with the subset

$$\{X_2^{\ell-1}\cdots X_m^{\ell-1}g_1^{b_1}\cdots g_m^{b_m} \mid 0\le b_i\le \ell-1\}$$

of $\mathcal{B}$. Clearly $X_2,\dots,X_m$ act trivially on all elements $X_2^{\ell-1}\cdots X_m^{\ell-1}g_1^{b_1}\cdots g_m^{b_m}\otimes 1$ (in the notation of part (i) of this lemma). We will check that X_1 also acts trivially:

$$\begin{aligned} X_1X_2^{\ell-1}\cdots X_m^{\ell-1}g_1^{b_1}\cdots g_m^{b_m}\otimes 1 &= q^{-b_1}X_2^{\ell-1}\cdots X_m^{\ell-1}g_1^{b_1}\cdots g_m^{b_m}(\tau_{\underline{\lambda}}(t)-\sum_{i=2}^m \lambda_i X_i h_i)\otimes 1 \\ &= 0. \end{aligned}$$

It follows that $\operatorname{soc}(V(\underline{\lambda})) \cong kG \cong \operatorname{soc}(A)$ as A-modules.

(v) This follows immediately from (iv). $\square$

We are also interested in simple A-modules. The quotient $A/\operatorname{rad}(A) \cong kG$ is a commutative semisimple algebra. Thus the simple A-modules are all one-dimensional, and correspond to the irreducible characters of G, with Λ acting trivially.

We shall use the notation $\underline{\mathrm{Hom}}$ to denote morphisms in the stable module category. In other words,

$$\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/\operatorname{PHom}_A(M,N)$$

where $\operatorname{PHom}_A(M,N)$ is the set of all A-homomorphisms $f : M \to N$ which factor through a projective A-module. Recall that for $n > 0$,

$$\mathrm{Ext}^n_A(M,N) \cong \underline{\mathrm{Hom}}_A(\Omega^n(M),N) \cong \underline{\mathrm{Hom}}_A(M,\Omega^{-n}(N)),$$

where Ω^n (respectively, Ω^{-n}) is the composition of n copies of Ω (respectively, Ω^{-1}). The isomorphism also holds for $n = 0$ if M is a simple A-module. The following lemma will be needed in Section 5.

Lemma 2.4. *Let S be a simple A-module. Then $\mathrm{Ext}^n_A(S,V(\underline{\lambda})) \neq 0$ for each n, $\underline{\lambda}$, and the restriction map $\tau^*_{\underline{\lambda}} : \mathrm{Ext}^n_A(S,V(\underline{\lambda})) \to \mathrm{Ext}^n_{k[t]/(t^\ell)}(S,V(\underline{\lambda}))$ is injective.*

Proof. Since A is free as a $\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle$-module by Lemma 2.2, an A-injective resolution of $V(\underline{\lambda})$ restricts to a $\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle$-injective resolution. It follows that $\Omega^{-n}_{\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle}(V(\underline{\lambda}))$ is isomorphic to $\Omega^{-n}_A(V(\underline{\lambda}))$ in the stable module category, that is up to projective direct summands. Thus

$$\mathrm{Ext}^n_{\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle}(S,V(\underline{\lambda})) \cong \underline{\mathrm{Hom}}_{\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle}(S,\Omega^{-n}_{\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle}(V(\underline{\lambda})) \cong \underline{\mathrm{Hom}}_{\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle}(S,\Omega^{-n}_A(V(\underline{\lambda})).$$

As A is self-injective, Ω and Ω^{-1} are inverse operators up to projective direct summands, so by Lemma 2.3(v), $\Omega^{-n}_A(V(\underline{\lambda})) = V'(\underline{\lambda})$ if n is odd, and $\Omega^{-n}_A(V(\underline{\lambda})) = V(\underline{\lambda})$ if n is even. Assume without loss of generality that $\lambda_1 = 1$. Since $\operatorname{soc}(V(\underline{\lambda})) = kGX_2^{\ell-1}\cdots X_m^{\ell-1}\otimes 1$ as a submodule of $V(\underline{\lambda}) \cong k\uparrow^A_{k\langle\tau_{\underline{\lambda}}(t)\rangle}$ (see the proof of Lemma 2.3(iv)), there is a unique (up to scalar) nonzero A-homomorphism f from S to $V(\underline{\lambda})$, sending S to $ke_SX_2^{\ell-1}\cdots X_m^{\ell-1}\otimes 1 \subset \operatorname{soc}(V(\underline{\lambda}))$ where e_S is the primitive central idempotent of kG corresponding to S. This does not factor through a projective A-module: If it did, it would factor through $A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}} V(\underline{\lambda})$ since A surjects onto $V(\underline{\lambda})$. The image of S in A must be contained in the socle of A, however the map $\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}$ sends $\operatorname{soc}(A)$ to 0. Therefore this map represents an A-homomorphism from S to $V(\underline{\lambda})$ that is nonzero in $\underline{\mathrm{Hom}}_A(S,V(\underline{\lambda}))$. A similar argument applies to $V'(\underline{\lambda})$, proving that $\mathrm{Ext}^n_A(S,V(\underline{\lambda})) \neq 0$ for each n.

Next we show that the image of the map f above, under restriction $\tau^*_{\underline{\lambda}}$, remains nonzero in $\underline{\mathrm{Hom}}_{\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle}(S,V(\underline{\lambda}))$. Again, if it does not, then $f : S \to V(\underline{\lambda})$ factors as a $\mathrm{k}\langle\tau_{\underline{\lambda}}(\mathrm{t})\rangle$-map through $A \xrightarrow{\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}} V(\underline{\lambda})$. The image of S in A must be a one-dimensional $k\langle\tau_{\underline{\lambda}}(t)\rangle$-submodule, spanned by an element $a \in A$ for which $\tau_{\underline{\lambda}}(t)a =$

0. Since f sends a generator of S to a non-zero element in $ke_SX_2^{\ell-1}\cdots X_m^{\ell-1}\otimes 1\subset \operatorname{soc} V(\underline{\lambda})$, we get that $a\tau_{\underline{\lambda}}(t)^{\ell-1}\in k^\times e_SX_2^{\ell-1}\cdots X_m^{\ell-1}\tau_{\underline{\lambda}}(t)^{\ell-1}=k^\times e_SX_1^{\ell-1}\cdots X_m^{\ell-1}$ under the identification of $V(\underline{\lambda})$ with $\operatorname{Span}_k(\mathcal{B}\tau_{\underline{\lambda}}(t)^{\ell-1})\subset A$ in the notation of Lemma 2.2. By Lemma 7.4 of the appendix, this cannot happen. Hence, f does not factor through $A\xrightarrow{\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}}V(\underline{\lambda})$. A similar argument applies in odd degrees, involving $V'(\underline{\lambda})$. □

Define an action of G on projective space $\mathbb{P}^{m-1}$ by

$$g_1^{a_1}\dots g_m^{a_m}\cdot[\lambda_1:\cdots:\lambda_m]=[q^{a_1}\lambda_1:\cdots:q^{a_m}\lambda_m]. \tag{2.4.1}$$

Lemma 2.5. *Let M be a finitely generated A-module.*

(i) $\underline{\operatorname{Hom}}_A(V(\underline{\lambda}),M)=0$ *if, and only if, the restriction* $M\downarrow_{k\langle\tau_{\underline{\lambda}}(t)\rangle}$ *is projective as a* $k\langle\tau_{\underline{\lambda}}(t)\rangle$*-module.*

(ii) *For each* $g\in G$, $M\downarrow_{k\langle\tau_{\underline{\lambda}}(t)\rangle}$ *is projective if, and only if,* $M\downarrow_{k\langle\tau_{g\cdot\underline{\lambda}}(t)\rangle}$ *is projective.*

Proof. Lemma 2.3(i) together with the Eckmann-Shapiro Lemma implies the isomorphism

$$\underline{\operatorname{Hom}}_A(V(\underline{\lambda}),M)\cong\underline{\operatorname{Hom}}_A(k\uparrow^A_{k\langle\tau_{\underline{\lambda}}(t)\rangle},M)\cong\underline{\operatorname{Hom}}_{k\langle\tau_{\underline{\lambda}}(t)\rangle}(k,M).$$

This proves (i) since $\underline{\operatorname{Hom}}_{k\langle\tau_{\underline{\lambda}}(t)\rangle}(k,M)=0$ if, and only if, $M\downarrow_{k\langle\tau_{\underline{\lambda}}(t)\rangle}$ is projective. For (ii), note that $\tau_{g\cdot\underline{\lambda}}(t)=g\cdot\tau_{\underline{\lambda}}(t)$. Since g defines an inner automorphism of A, we now have $V(g\cdot\underline{\lambda})\cong g\cdot V(\underline{\lambda})\cong V(\underline{\lambda})$. Thus the statement follows from (i). □

3. Rank varieties

In this section we define rank varieties for A-modules in the spirit of [10]. The subalgebras $k\langle\tau_{\underline{\lambda}}(t)\rangle$, defined in the text following (2.1.1), will play the role of cyclic shifted subgroups of A.

Lemma 3.1. *Let M be a finitely generated A-module. The subset of projective space $\mathbb{P}^{m-1}$, consisting of all points $\underline{\lambda}$ such that $M\downarrow_{k\langle\tau_{\underline{\lambda}}(t)\rangle}$ is not projective, is closed in the Zariski topology.*

Proof. Let $n=\dim M$ and $S(\underline{\lambda})\in M_n(k)$ a matrix representing the action of $\tau_{\underline{\lambda}}(t)$ on M. Then M is projective (equivalently, free) as a $k\langle\tau_{\underline{\lambda}}(t)\rangle$-module if and only if the Jordan form of $S(\underline{\lambda})$ has n/ℓ blocks of size ℓ. That is $S(\underline{\lambda})$ has the maximal possible rank for an ℓ-nilpotent matrix, $n-n/\ell$. The subset of $\mathbb{P}^{m-1}$,

$$\{\underline{\lambda}\in\mathbb{P}^{m-1}\mid\tau_{\underline{\lambda}}(t)\text{ does not have rank }n-n/\ell\},$$

is described by the equations produced by the minors of $S(\underline{\lambda})$ of size $(n-n/\ell)\times(n-n/\ell)$. All these minors must be 0, and they give homogeneous polynomial equations in the coefficients λ_i of X_ih_i. Thus this subset is defined by a set of homogeneous polynomials and is therefore closed. □

The action of G by automorphisms on the polynomial algebra $k[X_1, \ldots, X_m]$, defined by $g_i \cdot X_j = q^{\delta_{ij}} X_j$, gives $k[X_1, \ldots, X_m]$ the structure of a free kG-module. It is easily seen to have the invariants

$$k[X_1, \ldots, X_m]^G = k[X_1^\ell, \ldots, X_m^\ell] \cong k[X_1, \ldots, X_m].$$

Thus, $\mathbb{A}^m/G = \operatorname{Spec} k[X_1, \ldots, X_m]^G \cong \mathbb{A}^m$, where $\mathbb{A}^m$ is the affine space k^m (see for example [23, I.5.5(6)] for the first equality). Since the action of G commutes with the standard action of k^* on $\mathbb{A}^m$, and the induced action on $\mathbb{P}^{m-1} = \mathbb{A}^m/k^*$ is the action defined as in (2.4.1), we have $\mathbb{P}^{m-1}/G \cong \mathbb{P}^{m-1}$. Furthermore, Lemma 2.5(ii) implies that the set $\{\underline{\lambda} \in \mathbb{P}^{m-1} \mid M \downarrow_{k\langle \tau_{\underline{\lambda}}(t)\rangle} \text{ is not projective}\}$ is stable under the action of G. Thus, we can make the following definition.

Definition 3.2. The *rank variety* of an A-module M is

$$V_A^r(M) = \{\underline{\lambda} \in \mathbb{P}^{m-1} \mid M \downarrow_{k\langle \tau_{\underline{\lambda}}(t)\rangle} \text{ is not projective}\}/G.$$

We will sometimes abuse notation and write $\underline{\lambda} \in V_A^r(M)$ when we mean that $\underline{\lambda}$ is a representative of a G-orbit in $V_A^r(M)$. Note that Lemma 3.1 ensures $V_A^r(M)$ is a projective variety for any finitely generated A-module M. The following properties of these varieties are immediate.

Proposition 3.3. *Let M, N, M_1, M_2, M_3 be A-modules.*

(i) $V_A^r(k) = \mathbb{P}^{m-1}/G \cong \mathbb{P}^{m-1}$.

(ii) $V_A^r(M \oplus N) = V_A^r(M) \cup V_A^r(N)$.

(iii) $V_A^r(\Omega^i(M)) = V_A^r(M)$ *for all i.*

(iv) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of A-modules, then $V_A^r(M_i) \subset V_A^r(M_j) \cup V_A^r(M_k)$ for any $\{i, j, k\} = \{1, 2, 3\}$.*

We will denote $V_A^r(k)$ by V_A^r.

The rank variety characterizes projectivity of modules by the following lemma, a version of Dade's Lemma for finite group representations [13]. We thank K. Erdmann and D. Benson for suggesting to us that the proof of a generalization of Dade's Lemma in [7] should apply almost verbatim in our setting. For completeness, we give our adaptation of the proof in [7] here (cf. [9, Thm. 2.6]).

Theorem 3.4. *Let M be a finitely generated A-module. Then $V_A^r(M) = \emptyset$ if, and only if, M is a projective A-module.*

Proof. If M is projective, then $M \downarrow_{k\langle \tau_{\underline{\lambda}}(t)\rangle}$ is projective for all $\underline{\lambda}$ by Lemma 2.2, so $V_A^r(M) = \emptyset$.

For the converse, we argue by induction on m. Let $Y_i = X_i h_i$ $(i = 1, \ldots, m)$, where h_i is defined in the text preceding (2.0.1). Let $\Lambda'_m = k\langle Y_1, \ldots, Y_m\rangle$ and note that M is projective if and only if $M \downarrow_{\Lambda'_m}$ is projective: We may write $A \cong \Lambda'_m \rtimes G$. If $M \downarrow_{\Lambda'_m}$ is projective, any surjective A-map from another A-module N onto M splits on restriction to Λ'_m. The splitting map may be averaged by applying

$\frac{1}{|G|}\sum_{g\in G} g$ to obtain an A-map, as the characteristic of k does not divide $|G|$. If $m = 1$, this immediately implies that M is projective if, and only if, $V_A^r(M) = \emptyset$.

Let $m = 2$ and assume $V_A^r(M) = \emptyset$ but that M is not projective. We will show that M is forced to be 0. The Jacobson radical of A is $J = (Y_1, Y_2)$, the ideal generated by Y_1, Y_2. Let

$$N = \{u \in M | Ju \subset J^{\ell-1}M\}.$$

Let $Y = \lambda_1 Y_1 + \lambda_2 Y_2 = \tau_{\underline{\lambda}}(t)$. We will first show that the map induced by Y:

$$N/J^{\ell-1}M \xrightarrow{\cdot Y} J^{\ell-1}M/J^{\ell}M$$

is an isomorphism for any pair (λ_1, λ_2) where $\lambda_2 \neq 0$. We will need the observation that $YJ^{\ell-1} = J^{\ell}$, which follows from $Y_1^{\ell-i}Y_2^i \in YJ^{\ell-1}$ for $i \in \{1, \dots, \ell-1\}$ as may be proven by induction on i: If $i = 1$, then $YY_1^{\ell-1} = \lambda_2 q^{\ell-1} Y_1^{\ell-1} Y_2$, so $Y_1^{\ell-1}Y_2 \in YJ^{\ell-1}$. If $i \geq 2$, then $YY_1^{\ell-i}Y_2^{i-1} = \lambda_1 Y_1^{\ell-i+1}Y_2^{i-1} + \lambda_2 q^{\ell-i} Y_1^{\ell-i} Y_2^i$, so $Y_1^{\ell-i}Y_2^i \in YJ^{\ell-1}$ by induction.

Injectivity of $\cdot Y$: Let $v \in N$. Suppose $Yv \in J^{\ell}M = YJ^{\ell-1}M$. Then there exists $u \in J^{\ell-1}M$ such that $Yv = Yu$. Therefore, $Y(v-u) = 0$. Since $M\downarrow_{k\langle Y\rangle}$ is projective, we have $v - u = Y^{\ell-1}u'$ for some $u' \in M$. Hence, $v = u + Y^{\ell-1}u' \in J^{\ell-1}M$. In other words, $\bar{v} = 0 \in N/J^{\ell-1}M$. We conclude that $\cdot Y$ is injective.

Surjectivity of $\cdot Y$: We may assume that M does not have projective summands. This implies that $\mathrm{soc}(k[Y_1, Y_2]/(Y_1^{\ell}, Y_2^{\ell}))M = Y_1^{\ell-1}Y_2^{\ell-1}M = 0$. Now the relations on Y_1, Y_2 easily imply that

$$Y^{\ell-1}Y_1^{\ell-1}M = Y_1^{\ell-1}Y_2^{\ell-1}M = 0. \tag{3.4.1}$$

To show surjectivity we need to show that $YN = J^{\ell-1}M$. Observe that $J^{\ell-1} = kY_1^{\ell-1} + YJ^{\ell-2}$. Thus, $J^{\ell-1}M = Y_1^{\ell-1}M + YJ^{\ell-2}M$. The definition of N immediately implies that $J^{\ell-2}M \subset N$. Therefore $YJ^{\ell-2}M \subset YN$. Hence, to show the inclusion $Y_1^{\ell-1}M + YJ^{\ell-2}M \subset YN$, it suffices to show that $Y_1^{\ell-1}M \subset YN$.

Take any element of the form $Y_1^{\ell-1}u$, $u \in M$. By (3.4.1), $Y^{\ell-1}Y_1^{\ell-1}u = 0$. Since $M\downarrow_{k\langle Y\rangle}$ is projective, there is an element $u' \in M$ such that

$$Y_1^{\ell-1}u = Yu'. \tag{3.4.2}$$

Multiplying both sides by Y_1, we get $Y_1Yu' = 0$. Thus, $Y_1(\lambda_1Y_1 + \lambda_2Y_2)u' = 0$. Using the relation $Y_2Y_1 = qY_1Y_2$, we get $(\lambda_1Y_1 + q^{-1}\lambda_2Y_2)Y_1u' = 0$. Applying our projectivity hypothesis to the restriction of M to $k\langle\lambda_1Y_1 + q^{-1}\lambda_2Y_2\rangle$, there is a $u'' \in M$ for which

$$Y_1u' = (\lambda_1Y_1 + q\lambda_2Y_2)^{\ell-1}u''. \tag{3.4.3}$$

Combining (3.4.2) and (3.4.3), we get

$$Yu' \in J^{\ell-1}M, \quad Y_1u' \in J^{\ell-1}M.$$

Since $\lambda_2 \neq 0$, Y and Y_1 generate J. Thus, $Ju' \subset J^{\ell-1}M$, so that $u' \in N$ by definition. By (3.4.2) it follows that $Y_1^{\ell-1}M \subset YN$, as was needed.

Thus, we obtain that for any non-zero pair (λ_1, λ_2), the map

$$\cdot(\lambda_1 Y_1 + \lambda_2 Y_2) : N/J^{\ell-1}M \to J^{\ell-1}M/J^{\ell}M$$

is an isomorphism. Assume $J^{\ell-1}M/J^{\ell}M \neq 0$. Use the isomorphism $\cdot Y_2$ to identify $N/J^{\ell-1}M$ with $J^{\ell-1}M/J^{\ell}M$, so that we may consider the maps $\cdot(\lambda_1 Y_1 + \lambda_2 Y_2)$ to be endomorphisms of $J^{\ell-1}M/J^{\ell}M$. Taking $\lambda_2 = 1$, the determinant of the operator $\cdot(\lambda_1 Y_1 + Y_2)$ is a polynomial in λ_1 and thus there is a value of λ_1 for which the operator is not invertible, a contradiction. Thus $J^{\ell-1}M/J^{\ell}M = 0$, and Nakayama's Lemma implies that $J^{\ell-1}M = 0$. In particular, $Y_2^{\ell-1}M = 0$. At the same time, M restricted to $k\langle Y_2\rangle$ is projective. Therefore $M = 0$, completing the proof in the case $m = 2$.

Now suppose $m \geq 3$ and $V_A^r(M) = \emptyset$. We will show that $M\downarrow_{\Lambda'_m}$ is projective, where $\Lambda'_m = k\langle Y_1, \dots, Y_m\rangle$. As noted at the beginning of the proof, this will imply M is projective.

We have a short exact sequence of algebras (in the sense of [12, XVI §6]):

$$k\langle Y_m\rangle \to \Lambda'_m \to \Lambda'_{m-1}.$$

Therefore there is a spectral sequence

$$\mathrm{H}_p(\Lambda'_{m-1}, \mathrm{H}_q(k\langle Y_m\rangle, M)) \Rightarrow \mathrm{H}_{p+q}(\Lambda'_m, M).$$

By our assumption, $M\downarrow_{k\langle Y_m\rangle}$ is projective. Thus the spectral sequence collapses at E^2 and we get an isomorphism

$$\mathrm{H}_p(\Lambda'_{m-1}, M/Y_mM) \cong \mathrm{H}_p(\Lambda'_m, M).$$

Therefore, to finish the proof it suffices to check that M/Y_mM is projective as a Λ'_{m-1}-module.

Write $\underline{\lambda}' = [\lambda_1 : \cdots : \lambda_{m-1}] \in \mathbb{P}^{m-2}$, and let $\tau'(\underline{\lambda}') = \lambda_1 Y_1 + \cdots + \lambda_{m-1}Y_{m-1}$. Consider the subalgebra $B \subset \Lambda'_m$ generated by $\tau'(\underline{\lambda}')$ and Y_m. Any element of B of the form $\mu_1\tau'(\underline{\lambda}') + \mu_2 Y_m$ is of the form $\tau(\underline{\lambda})$ for the algebra Λ'_m. Thus, $M\downarrow_{k\langle\mu_1\tau'(\underline{\lambda}')+\mu_2Y_m\rangle}$ is projective for any pair (μ_1, μ_2). Since $\tau'(\underline{\lambda}')$ and Y_m q-commute, the argument for $m = 2$ applies to B. We conclude that M is projective as a B-module. Therefore, M/Y_mM is projective as B/Y_mB-module. In other words, $M/Y_mM\downarrow_{k\langle\tau'(\underline{\lambda}')\rangle}$ is projective. By the induction hypothesis, M/Y_mM is projective as a Λ'_{m-1}-module. Thus, $\mathrm{H}_p(\Lambda'_m, M) = 0$ for all $p > 0$. Since the trivial module is the only simple module for the local algebra Λ'_m, we conclude that M is projective as a Λ'_m-module. Therefore, M is projective as an A-module. $\square$

4. Support varieties

In this section we introduce cohomological support varieties for A-modules. The proofs of their properties are standard and will be omitted when they are identical to those existing in the literature.

The cohomology of the quantum elementary abelian group A is

$$\mathrm{H}^*(A,k) = \mathrm{Ext}^*_A(k,k) \cong k[y_1,\dots,y_m] \tag{4.0.4}$$

as a graded algebra, where $\deg(y_i)=2$. Indeed, let $A_1 = (k[t]/(t^\ell)) \rtimes \mathbb{Z}/\ell\mathbb{Z}$, the algebra A in the case $m=1$. The periodic $k[t]/(t^\ell)$-free resolution of k,

$$\cdots \xrightarrow{\cdot t^{\ell-1}} k[t]/(t^\ell) \xrightarrow{\cdot t} k[t]/(t^\ell) \xrightarrow{\cdot t^{\ell-1}} k[t]/(t^\ell) \xrightarrow{\cdot t} k[t]/(t^\ell) \xrightarrow{\varepsilon} k \to 0, \tag{4.0.5}$$

becomes an A_1-projective resolution of k by giving $k[t]/(t^\ell)$ the standard $\mathbb{Z}/\ell\mathbb{Z}$-action $g\cdot t^i = q^i t^i$ in even degrees and the shifted $\mathbb{Z}/\ell\mathbb{Z}$-action $g\cdot t^i = q^{i+1}t^i$ in odd degrees, where g is a generator of $\mathbb{Z}/\ell\mathbb{Z}$. The resolution yields $\mathrm{H}^*(A_1,k)\cong k[y]$. The general case is obtained by applying the Künneth formula.

Recall that $\mathrm{H}^*(A,M) = \mathrm{Ext}^*_A(k,M)$ is an $\mathrm{H}^*(A,k)$-module under Yoneda composition, for any A-module M. To proceed with our geometric constructions, we will need to establish finite generation of $\mathrm{H}^*(A,M)$ over $\mathrm{H}^*(A,k)$ whenever M is finitely generated.

Lemma 4.1. *Let M be a finitely generated A-module. Then $\mathrm{H}^*(A,M)$ is finitely generated as an $\mathrm{H}^*(A,k)$-module.*

Proof. Since M is finitely generated, we can argue by induction on the length of its composition series. Hence, it suffices to prove the lemma for simple A-modules. Let S be such a module. The spectral sequence in cohomology arising from the sequence of augmented algebras (see, for example, [12, XVI §6]) $\Lambda \to A \to kG$ yields the isomorphism $\mathrm{H}^*(A,S) = \mathrm{H}^*(\Lambda,S)^G$. Let $R = \mathrm{H}^*(\Lambda,k)$, a finitely generated G-algebra. Since any simple A-module becomes trivial when restricted to Λ, we conclude that $\mathrm{H}^*(\Lambda,S)$ is a rank 1 free R-module with a compatible action of G. Since $\mathrm{H}^*(A,k) = \mathrm{H}^*(\Lambda,k)^G = R^G$, it remains to see that $\mathrm{H}^*(\Lambda,S)^G$ is finitely generated over R^G. As the characteristic of k does not divide the order of G, this is a consequence of the Noether Theorem stating that the Noetherian k-algebra R is finitely generated over R^G (see, for example, [6, 1.3.1]). □

Remark 4.2. We can compute $\mathrm{H}^*(A,S)$ explicitly when S is simple, yielding more insight in our special case. First let $m=1$. Let S_i be the (one-dimensional) simple A_1-module on which t acts as multiplication by 0 and the generator g of $\mathbb{Z}/\ell\mathbb{Z}$ acts as multiplication by q^i. Using the resolution (4.0.5), we get

$$\mathrm{H}^n(A_1,S_i) = \begin{cases} 0, & \text{for all } n \text{ if } i \notin \{0,1\} \\ 0, & \text{if } n \text{ is even and } i=1, \text{ or if } n \text{ is odd and } i=0 \\ k, & \text{if } n \text{ is odd and } i=1, \text{ or if } n \text{ is even and } i=0. \end{cases}$$

Since the action of the generator y of $k[y] \cong \mathrm{H}^*(A_1, k)$ induces a periodicity isomorphism, we get that $\mathrm{H}^*(A_1, S_i)$ is a rank 1 free $\mathrm{H}^*(A_1, k)$-module in case $i \in \{0, 1\}$.

Now if $m \geq 2$, a (one-dimensional) simple A-module may be written $S = S_\chi$ for some $\chi : G \to k^\times$, where any $g \in G$ acts as multiplication by $\chi(g)$. It may be factored as $S_\chi \cong S_{\chi_1} \otimes \cdots \otimes S_{\chi_m}$ where g_j acts trivially on S_{χ_i} if $i \neq j$. We may similarly factor $A \cong A_1 \otimes \cdots \otimes A_m$ where $A_i = (k[X_i]/(X_i^\ell)) \rtimes \langle g_i \rangle$. Apply the Künneth Theorem to obtain

$$\mathrm{H}^*(A, S_\chi) \cong \mathrm{H}^*(A_1, S_{\chi_1}) \otimes \cdots \otimes \mathrm{H}^*(A_m, S_{\chi_m}),$$

and apply the case $m = 1$ to each factor. If $k = \mathbb{C}$, this result and Lemma 4.1 follow from [22, Thm. 2.5], since in this case A is isomorphic to the quantized restricted Borel subalgebra of $u_q(sl_2^{\times m})$.

Let M, N be left A-modules. Let $I(M, N)$ be the annihilator of $\mathrm{Ext}^*_A(M, N)$ under the action of $\mathrm{H}^*(A, k)$ by cup product, equivalent to $- \otimes_k N$ followed by Yoneda composition (see [5, I. Prop. 3.2.1]). Since $I(M, N)$ is a homogeneous ideal, it defines a projective subvariety of Proj $\mathrm{H}^*(A, k) \cong \mathbb{P}^{m-1}$. We define

$$V_A^c(M, N) = \mathrm{Proj}\ \mathrm{H}^*(A, k)/I(M, N), \tag{4.2.1}$$

the set of all homogeneous prime ideals of dimension 1 (that is, those ideals not contained in any homogeneous prime ideal other than $\mathrm{H}^{*>0}(A, k)$) which contain $I(M, N)$. If $M = N$, we write simply $I(M) = I(M, M)$ and $V_A^c(M) = V_A^c(M, M)$. Note that

$$V_A^c(k) = \mathrm{Proj}\ \mathrm{H}^*(A, k) \cong \mathbb{P}^{m-1}.$$

We will denote $V_A^c(k)$ by V_A^c.

Proposition 4.3. *The following properties hold for all finitely generated A-modules M, N.*

(i) $V_A^c(M \oplus N) = V_A^c(M) \cup V_A^c(N)$.
(ii) $V_A^c(M) = V_A^c(\Omega(M))$.
(iii) $V_A^c(M, N) \subset V_A^c(M) \cap V_A^c(N)$.
(iv) $V_A^c(M) = \cup_S V_A^c(S, M) = \cup_S V_A^c(M, S)$, *where S runs over all simple A-modules.*
(v) $V_A^c(M) = \emptyset$ *if, and only if, M is projective.*
(vi) $V_A^c(M \otimes N) \subset V_A^c(M) \cap V_A^c(N)$.

Proof. Arguments from [5, II. §5.7] apply verbatim to prove (i)–(iv).

(v) If M is projective, then $\mathrm{Ext}^n_A(M, M) = 0$ for all $n > 0$. Thus, $V_A^c(M) = \emptyset$.

Assume $V_A^c(M) = \emptyset$. By (iv), we get $V_A^c(S, M) = \emptyset$ for every simple A-module S. Note that $\mathrm{Ext}^*_A(S, M) \cong \mathrm{H}^*(A, S^\# \otimes M)$ (see [5, I. §3.1] or the appendix), and Lemma 4.1 thus implies that this cohomology is finitely generated as an $\mathrm{H}^*(A, k)$-module. So there exists n_0 such that $\mathrm{Ext}^n_A(S, M) = 0$ for any S and any $n > n_0$. Hence, the minimal injective resolution of M is finite. Since injective A-modules are projective, any finite resolution splits. Hence, M is injective, and therefore projective.

(vi) The action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(M\otimes N, M\otimes N)$ factors through its action on $\mathrm{Ext}^*_A(M,M)$: We may first apply $-\otimes M$ to an n-extension of k by k, and then apply $-\otimes N$. Thus $I(M,M)\subset I(M\otimes N, M\otimes N)$, implying that $V^c_A(M\otimes N)\subset V^c_A(M)$. On the other hand, since M is finitely generated, we have the adjunction isomorphism $\mathrm{Ext}^*_A(M\otimes N, M\otimes N)\cong \mathrm{Ext}^*_A(M^\#\otimes M\otimes N, N)$ (see the appendix). Thus, $V^c_A(M\otimes N) = V^c_A(M^\#\otimes M\otimes N, N)$. The latter is contained in $V^c_A(N)$ by (iii). □

Following the original construction by Carlson [11] for finite groups, we introduce modules L_ζ. Let $\zeta\in \mathrm{H}^n(A,k)\cong \underline{\mathrm{Hom}}(\Omega^n(k),k)$, $\zeta\neq 0$. Then L_ζ is defined to be the kernel of the corresponding map $\zeta:\Omega^n(k)\to k$. In other words, we have a short exact sequence

$$0\to L_\zeta\to \Omega^n(k)\to k\to 0. \tag{4.3.1}$$

Let $\langle\zeta\rangle\subset \mathbb{P}^{m-1}$ be the projective hypersurface defined by ζ, that is the set of all homogeneous prime ideals in $\mathrm{H}^*(A,k)$ of dimension 1 which contain ζ.

The following result is an adaptation to our situation of [5, II.5.9.6] (see also [16, 3.3]).

Proposition 4.4. *$V^c_A(M\otimes L_\zeta) = \langle\zeta\rangle\cap V^c_A(M)$.*

Proof. Observe that for any M,N, a homogeneous prime ideal $\wp$ of dimension 1 belongs to $V^c_A(M,N)$ if and only if $I(M,N)\subset\wp$ if and only if $\mathrm{Ext}^*_A(M,N)_\wp\neq 0$.

We first show $\langle\zeta\rangle\cap V^c_A(M)\subset V^c_A(M\otimes L_\zeta)$. By Proposition 4.3(iv), $V^c_A(M\otimes L_\zeta) = \bigcup V^c_A(M\otimes L_\zeta, S)$ and $V^c_A(M)=\bigcup V^c_A(M,S)$, where S runs through all simple A-modules, so it suffices to show that

$$\langle\zeta\rangle\cap V^c_A(M,S)\subset V^c_A(M\otimes L_\zeta, S)$$

for any S. Let $\wp$ be a homogeneous prime ideal in $\langle\zeta\rangle\cap V^c_A(M,S)$, that is $\wp$ contains the ideal generated by $I(M,S)$ and ζ. We want to show that $\wp\in V^c_A(M\otimes L_\zeta, S)$, that is $I(M\otimes L_\zeta, S)\subset\wp$.

Suppose $I(M\otimes L_\zeta,S)\not\subset\wp$. This implies that $\mathrm{Ext}^*_A(M\otimes L_\zeta,S)_\wp=0$. Tensoring the short exact sequence

$$0\longrightarrow L_\zeta\longrightarrow \Omega^n(k)\xrightarrow{\ \zeta\ } k\longrightarrow 0$$

with M and applying $\mathrm{Ext}^*_A(-,S)$, we get a long exact sequence

$$\longrightarrow \mathrm{Ext}^i_A(M,S)\xrightarrow{\ \zeta\ }\mathrm{Ext}^{i+n}_A(M,S)\xrightarrow{\ \eta\ }\mathrm{Ext}^i_A(M\otimes L_\zeta,S)\xrightarrow{\ \delta\ }\mathrm{Ext}^{i+1}_A(M,S)\longrightarrow$$

Let $z\in \mathrm{Ext}^{i+n}_A(M,S)$. Then $\eta(z)\in\mathrm{Ext}^i_A(M\otimes L_\zeta,S)$. Since $\mathrm{Ext}^*_A(M\otimes L_\zeta,S)_\wp=0$, there exists a homogeneous element $a\notin\wp$ such that $\eta(az)=a\eta(z)=0$. The long exact sequence implies that $az = \zeta(y)$ for $y \in \operatorname{Ext}_A^{\deg(a)+i}(M,S)$. This implies that

$$\operatorname{Ext}_A^*(M,S)_\wp = \zeta \operatorname{Ext}_A^*(M,S)_\wp.$$

Indeed, if $z \in \operatorname{Ext}_A^j(M,S)$ for $j > n$, then $z = \zeta(a^{-1}y) \in \zeta \operatorname{Ext}_A^*(M,S)_\wp$ as we just showed. Assume $z \in \operatorname{Ext}_A^j(M,S)$ for $j \leq n$. Let $b \notin \wp$. We multiply z by an appropriate power of b so that $\deg(b^m z) > n$. Then $b^m z \in \operatorname{Ext}_A^*(M,S)_\wp$ and, hence, $z \in \operatorname{Ext}_A^*(M,S)_\wp$ since b is invertible in $\operatorname{Ext}_A^*(M,S)_\wp$.

Since $\zeta \in \wp$, and $\operatorname{Ext}_A^*(M,S) = \operatorname{Ext}_A^*(k, M^\# \otimes S)$ is finitely generated over $\mathrm{H}^*(A,k)$ by Lemma 4.1, Nakayama's Lemma implies that $\operatorname{Ext}_A^*(M,S)_\wp = 0$. This contradicts the assumption $I(M,S) \subset \wp$. We conclude that $I(M \otimes L_\zeta, S) \subset \wp$, and hence $\langle\zeta\rangle \cap V_A^c(M,S) \subset V_A^c(M \otimes L_\zeta, S)$.

To prove the opposite inclusion $V_A^c(M \otimes L_\zeta) \subset \langle\zeta\rangle \cap V_A^c(M)$, it suffices, by Proposition 4.3(vi), to show that $V_A^c(L_\zeta) \subset \langle\zeta\rangle$. Using 4.3(iv) again, we reduce to showing the inclusion $V_A^c(S, L_\zeta) \subset \langle\zeta\rangle$ for any simple A-module S. Thus we need to show that $\operatorname{Ext}_A^*(S, L_\zeta)_\wp \neq 0$ for a prime ideal $\wp \subset \mathrm{H}^*(A,k)$ implies $\zeta \in \wp$. We will prove the converse. Suppose $\zeta \notin \wp$. Then multiplication by ζ induces an isomorphism on $\operatorname{Ext}_A^*(S,k)_\wp$ since it is invertible in $\mathrm{H}^*(A,k)_\wp$. Since localization is exact, the short exact sequence (4.3.1) implies that $\operatorname{Ext}_A^*(S, L_\zeta)_\wp$ is the kernel of the isomorphism $\zeta : \operatorname{Ext}_A^*(S,k)_\wp \to \operatorname{Ext}_A^{*+n}(S,k)_\wp$. Thus, $\operatorname{Ext}_A^*(S, L_\zeta)_\wp = 0$. □

Induction yields the following corollary.

Corollary 4.5. $V_A^c(M \otimes L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_i}) = \langle\zeta_1, \ldots, \zeta_i\rangle \cap V_A^c(M)$.

5. Identification of varieties

In this section we will establish an analogue of the Avrunin-Scott Theorem, identifying the cohomological variety with the rank variety of a module. For $\underline{\lambda} \in \mathbb{P}^{m-1}$ we denote by $\tau_{\underline{\lambda}}^* : \mathrm{H}^*(A,k) \to \mathrm{H}^{ev}(k[t]/(t^\ell), k)$ the map induced on cohomology by $\tau_{\underline{\lambda}} : k[t]/(t^\ell) \hookrightarrow A$ as defined in (2.1.1).

Recall the algebra $\widetilde{A} = k[X_1, \ldots, X_m] \rtimes G$ defined at the beginning of §2. We have a short exact sequence of augmented algebras

(5.0.1) $$k[X_1, \ldots, X_m] \xrightarrow{X_i \mapsto X_i^\ell} \widetilde{A} \to A$$

which induces a spectral sequence (see [12, XVI §6])

(5.0.2) $$\mathrm{H}^*(A, \mathrm{H}^*(k[X_1, \ldots, X_m], k)) \Rightarrow \mathrm{H}^*(\widetilde{A}, k).$$

Lemma 5.1. *The transgression map of the E_2 page of the spectral sequence (5.0.2) induces an isomorphism* $\mathrm{H}^1(k[X_1, \ldots, X_m], k) \cong \mathrm{H}^2(A,k)$.

Proof. We first show that the action of A on $\mathrm{H}^*(k[X_1, \ldots, X_m], k)$ is trivial. Since this action comes from tensoring the action of each factor $(k[X_i]/(X_i^\ell)) \rtimes \mathbb{Z}/\ell\mathbb{Z}$ of A on the corresponding $\mathrm{H}^*(k[X_i], k)$, it suffices to check this for $m = 1$. Let

$\widetilde{A}_1 = k[X] \rtimes \mathbb{Z}/\ell\mathbb{Z}$, $A_1 = (k[X]/(X^\ell)) \rtimes \mathbb{Z}/\ell\mathbb{Z}$. Let $\mathcal{L} = k[X^\ell] \subset \widetilde{A}_1$ denote the image of $k[X]$ in $\widetilde{A}_1$. Since $\mathrm{H}^i(k[X], k) = 0$ for $i > 1$ and $\mathrm{H}^0(k[X], k) = k$, the trivial module, we need only check that the action of A_1 on $\mathrm{H}^1(k[X], k)$ is trivial.

The Koszul resolution $K_* : 0 \to k[X] \xrightarrow{\cdot X} k[X] \xrightarrow{\varepsilon} k \to 0$ of k as a $k[X]$-module becomes a resolution as a $k[X] \rtimes \mathbb{Z}/\ell\mathbb{Z}$-module under the action $g \circ X^i = q^i X^i$ in degree 0 and $g \circ X^i = q^{i+1} X^i$ in degree 1. The spectral sequence (5.0.2) can be obtained as a Grothendieck spectral sequence associated to the composition of functors $\mathrm{Hom}_{\mathcal{L}}(-, k)$ and $\mathrm{Hom}_{A_1}(k, -)$. Hence, the action of A_1 on $\mathrm{H}^*(\mathcal{L}, k) = \mathrm{Ext}^*_{\mathcal{L}}(k, k)$ is induced by the action of A_1 on the complex (where $\delta = (\cdot X)^*$)

$$\mathrm{Hom}_{\mathcal{L}}(K_*, k) : \quad 0 \leftarrow \mathrm{Hom}_k(k[X]/(X^\ell), k) \xleftarrow{\delta} \mathrm{Hom}_k(k[X]/(X^\ell), k) \leftarrow 0$$

which, in turn, comes from the diagonal action of $\widetilde{A}_1$ on $\mathrm{Hom}_k(K_*, k)$ (see the appendix for the explicit formula). Computation yields that $H^1(\mathcal{L}, k)$ is generated by the cocycle $f : k[X]/(X^\ell) \to k$ specified by the condition $f(X^{\ell-1}) = 1, f(X^i) = 0$ for $i \neq \ell - 1$. We further compute that $X \circ f \in \mathrm{Im}\,\delta$ and $(g \circ f)(X^{\ell-1}) = f(g^{-1} \circ X^{\ell-1}) = f(g^{\ell-1} \circ X^{\ell-1}) = f(q^\ell X^{\ell-1}) = f(X^{\ell-1})$. Thus, the action of A_1 on $H^1(\mathcal{L}, k)$ is trivial.

Hence, A acts trivially on $\mathrm{H}^q(k[X_1, \ldots, X_m], k)$. Therefore,

$$E_2^{0,q} = \mathrm{H}^q(k[X_1, \ldots, X_m], k)^A = \mathrm{H}^q(k[X_1, \ldots, X_m], k).$$

Observe that $\mathrm{H}^1(\widetilde{A}, k) = 0$: We have $\mathrm{H}^1(\widetilde{A}, k) \cong \bigoplus_m \mathrm{H}^1(k[t] \rtimes \mathbb{Z}/\ell\mathbb{Z}, k)$ by the Künneth Theorem. Direct computation with the Koszul resolution K_* of k shows that $\mathrm{H}^1(k[t] \rtimes \mathbb{Z}/\ell\mathbb{Z}, k) = 0$. Thus $\mathrm{H}^1(\widetilde{A}, k) = 0$ as required.

It follows that $E_3^{0,1} = E_\infty^{0,1} = 0$, and so $\mathrm{Ker}\, d_2^{0,1} = 0$. Thus

$$d_2^{0,1} : \mathrm{H}^1(k[X_1, \ldots, X_m], k) \to \mathrm{H}^2(A, k)$$

is injective. Since $\dim_k \mathrm{H}^2(A, k) = m = \dim_k \mathrm{H}^1(k[X_1, \ldots, X_m], k)$, we conclude that $d_2^{0,1}$ is an isomorphism. □

The next lemma establishes that the map $\tau^*_{\underline{\lambda}}$ is "essentially surjective" and is invariant under the G-action on $\mathbb{P}^{m-1}$. Let $I = (X_1, X_2, \ldots, X_m)$ be the ideal of A generated by the X_i's, and let $z_1, \ldots, z_m$ be the basis of $(I/I^2)^\#$ dual to $X_1, \ldots, X_m$. As before, we denote by y_i the generators of $\mathrm{H}^*(A, k)$.

Lemma 5.2. *(i) For any $\underline{\lambda} \in \mathbb{P}^{m-1}$, $\tau^*_{\underline{\lambda}}$ is surjective onto* $\mathrm{H}^{ev}(k[t]/(t^\ell), k) \cong k[y]$.
(ii) For any $g \in G$, $\tau^*_{\underline{\lambda}} = \tau^*_{g \cdot \underline{\lambda}}$.

Proof. (i) Let $\underline{\lambda} = [\lambda_1 : \lambda_2 : \cdots : \lambda_m] \in \mathbb{P}^{m-1}$. Lemma 2.1 implies that the following diagram is commutative, where both rows are exact sequences of augmented algebras:

(5.2.1)
$$\begin{tikzcd} k[X_1,\dots,X_m] \arrow[r, "X_i\mapsto X_i^\ell"] & \widetilde{A} \arrow[r] & A \\ k[t] \arrow[u, "t\mapsto \lambda_1^\ell X_1+\cdots+\lambda_m^\ell X_m"'] \arrow[r, "t\mapsto t^\ell"] & k[t] \arrow[u, "t\mapsto \lambda_1 X_1 h_1+\cdots+\lambda_m X_m h_m"'] \arrow[r] & k[t]/(t^\ell) \arrow[u, "t\to \tau_{\underline{\lambda}}(t)"'] \end{tikzcd}$$

The rows of (5.2.1) induce compatible spectral sequences where the edge homomorphisms $d_2^{0,1}$ are isomorphisms by Lemma 5.1. Thus, we get another commutative diagram where vertical maps are restrictions.

(5.2.2)
$$\begin{tikzcd} (I/I^2)^{\#} \arrow[r, equal] \arrow[d] & \mathrm{H}^1(k[X_1,\dots,X_m],k) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^2(A,k) \arrow[d, "\tau_{\underline{\lambda}}^*"] \\ ((t)/(t^2))^{\#} \arrow[r, equal] & \mathrm{H}^1(k[t],k) \arrow[r, "\sim"] & \mathrm{H}^2(k[t]/(t^\ell),k) \end{tikzcd}$$

The leftmost column comes from the isomorphism $\mathrm{H}^1(k[X_1,\dots,X_m],k) \cong \mathrm{Hom}_{\mathrm{alg}}(k[X_1,\dots,X_m],k) = \mathrm{Hom}_k(I/I^2,k) = (I/I^2)^{\#}$.

By construction of the diagram (5.2.2), the leftmost vertical map is the dual to the map induced by $t \mapsto \lambda_1^\ell X_1 + \cdots + \lambda_n^\ell X_n$. Thus, it sends z_i to $\lambda_i^\ell z$, where z is the dual basis to t in $((t)/(t^2))^{\#}$. Therefore, the rightmost vertical map sends y_i to $\lambda_i^\ell y$. Since at least one of λ_i is nonzero, we conclude that $\tau_{\underline{\lambda}}^*$ is surjective onto $\mathrm{H}^2(k[t]/(t^\ell),k)$, and thus onto $\mathrm{H}^{ev}(k[t]/(t^\ell),k)$.

(ii) Let $\lambda' = g \cdot \lambda$. By the definition (2.4.1) of this action, as $q^\ell = 1$, we have $(\lambda_i')^\ell = \lambda_i^\ell$ for all i. It now follows from the proof of (i) that $\tau_{\underline{\lambda}}^* = \tau_{\underline{\lambda}'}^*$. □

The lemma implies that we can define a map

(5.2.3)
$$\Psi : V_A^r \to V_A^c$$

by sending $\underline{\lambda} \in V_A^r/G = \mathbb{P}^{m-1}/G$ to the homogeneous prime ideal $\mathrm{Ker}(\tau_{\underline{\lambda}}^*)$ of $\mathrm{H}^*(A,k)$. The following proposition is an immediate consequence of Lemma 5.2.

Proposition 5.3. $\Psi : V_A^r \to V_A^c$ *is a homeomorphism.*

Proof. As $V_A^r \cong \mathbb{P}^{m-1}/G$, first define $\widetilde{\Psi} : \mathbb{P}^{m-1} \to V_A^c$ by

$$\widetilde{\Psi}(\underline{\lambda}) = \mathrm{Ker}(\tau_{\underline{\lambda}}^*).$$

As it is shown in the proof of Lemma 5.2(i), $\tau_{\underline{\lambda}}^*(y_i) = \lambda_i^\ell y$, where y is the degree 2 generator of $\mathrm{H}^*(k[t]/(t^\ell),k)$. Thus, $\mathrm{Ker}(\tau_{\underline{\lambda}}^*)$ is generated by the elements $\sum a_i y_i \in \mathrm{H}^2(A,k)$ such that $\sum a_i \lambda_i^\ell = 0$. We get

$$\widetilde{\Psi}(\underline{\lambda}) = [\lambda_1^\ell : \lambda_2^\ell : \cdots : \lambda_m^\ell].$$

Finally, since $[\lambda_1^\ell : \lambda_2^\ell : \cdots : \lambda_m^\ell] = [\mu_1^\ell : \mu_2^\ell : \cdots : \mu_m^\ell]$ if and only if there exists $g \in G$ such that $[\lambda_1 : \cdots : \lambda_m] = g \cdot [\mu_1 : \cdots : \mu_m]$, we conclude that $\widetilde{\Psi} : \mathbb{P}^{m-1} \to V_A^c$ factors through $\Psi : V_A^r \to V_A^c$ and, moreover, that Ψ is a homeomorphism. $\square$

Remark 5.4. The rank variety V_A^r can be identified with $\operatorname{Proj} S^*((I/I^2)^\#)^G \cong (\operatorname{Proj} I/I^2)/G = \mathbb{P}^{m-1}/G$. The map Ψ is then given by the algebraic map

$$\psi : \mathrm{H}^*(A,k) \to (S^*((I/I^2)^\#))^G, \quad y_i \mapsto z_i^\ell. \tag{5.4.1}$$

We will need the following observation, which is well-known in the case of cyclic finite groups ([17, 3.2]). Let y be the degree 2 generator of $\mathrm{H}^*(k[t]/(t^\ell), k)$. Then multiplication by y induces an isomorphism

$$\cdot y : \mathrm{H}^n(k[t]/(t^\ell), N) \to \mathrm{H}^{n+2}(k[t]/(t^\ell), N) \tag{5.4.2}$$

for any $n > 0$ and any $k[t]/(t^\ell)$-module N. To see this, note that by the periodicity of the trivial module k arising from (4.0.5), we have $\mathrm{H}^n(k[t]/(t^\ell), N) \cong \mathrm{H}^{n+2}(k[t]/(t^\ell), N)$ for all $n > 0$. The element $y \in \mathrm{H}^2(k[t]/(t^\ell), k)$ corresponds to the identity map in $\underline{\mathrm{Hom}}_{k[t]/(t^\ell)}(k,k) \cong \underline{\mathrm{Hom}}_{k[t]/(t^\ell)}(\Omega^2(k), k)$, and its cup product with an element in $\mathrm{H}^n(k[t]/(t^\ell), N) \cong \underline{\mathrm{Hom}}_{k[t]/(t^\ell)}(\Omega^n(k), N)$ corresponds to composition with the identity map from $\Omega^n(k)$ to $\Omega^n(k)$.

Now we are able to determine the support varieties of the modules $V(\underline{\lambda})$ defined in (2.2.1), which will be used to obtain a connection between the rank and support varieties of an arbitrary finitely generated A-module.

Lemma 5.5. $V_A^c(V(\underline{\lambda})) = \Psi(\underline{\lambda})$.

Proof. Let $z \in I(k, V(\underline{\lambda}))$ be a homogeneous element of even degree in the annihilator of $\mathrm{Ext}_A^*(k, V(\underline{\lambda}))$ in $\mathrm{Ext}_A^*(k,k)$. Since the restriction map

$$\tau_{\underline{\lambda}}^* : \mathrm{Ext}_A^*(k, V(\underline{\lambda})) \to \mathrm{Ext}_{k[t]/(t^\ell)}^*(k, V(\underline{\lambda}))$$

is injective by Lemma 2.4, we can choose $v \in \mathrm{Ext}_A^*(k, V(\underline{\lambda}))$ such that $\tau_{\underline{\lambda}}^*(v) \neq 0$. Since $z \in I(k, V(\underline{\lambda}))$, we conclude that

$$\tau_{\underline{\lambda}}^*(z)\tau_{\underline{\lambda}}^*(v) = \tau_{\underline{\lambda}}^*(zv) = 0.$$

Due to the isomorphism (5.4.2) and as $\tau_{\underline{\lambda}}^*(v) \neq 0$, we get $\tau_{\underline{\lambda}}^*(z) = 0$. Thus, $z \in \mathrm{Ker}(\tau_{\underline{\lambda}}^*)$. Since all elements of $I(k, V(\underline{\lambda}))$ of odd degree are nilpotent and $\mathrm{Ker}(\tau_{\underline{\lambda}}^*)$ is a prime ideal, it follows that

$$I(k, V(\underline{\lambda})) \subset \mathrm{Ker}(\tau_{\underline{\lambda}}^*).$$

This implies that $\Psi(\underline{\lambda}) \in V_A^c(k, V(\underline{\lambda}))$. By Proposition 4.3(iv), $V_A^c(k, V(\underline{\lambda})) \subseteq V_A^c(V(\underline{\lambda}))$. Therefore $\Psi(\underline{\lambda}) \in V_A^c(V(\underline{\lambda}))$.

It remains to prove that $V_A^c(V(\underline{\lambda})) \subset \Psi(\underline{\lambda})$. Applying Proposition 4.3(iv) again, it suffices to show $V_A^c(S, V(\underline{\lambda})) \subset \Psi(\underline{\lambda})$ for any simple A-module S. This, in turn, will follow from the inclusion $\mathrm{Ker}(\tau_{\underline{\lambda}}^*) \subset I(S, V(\underline{\lambda}))$.

Let S be a simple A-module. We claim that the following diagram commutes:

(5.5.1)
$$\begin{tikzcd}
\operatorname{Ext}^*_A(k,k) \arrow[r, "-\otimes S"] \arrow[dr, "\tau^*_{\underline{\lambda}}"] & \operatorname{Ext}^*_A(S,S) \arrow[d, "\tau^*_{\underline{\lambda}}"] \\
& \operatorname{Ext}^*_{k[t]/(t^\ell)}(k,k)
\end{tikzcd}$$

Indeed, suppose $S = S_\chi$ where $\chi : G \to k^\times$ is a character, so that each $g \in G$ acts as multiplication by $\chi(g)$ and each X_i acts as 0. Under the map $-\otimes S$, an n-extension $k \to M_{n-1} \to \cdots \to M_0 \to k$ is sent to $S \to M_{n-1} \otimes S \to \cdots \to M_0 \otimes S \to S$. The action of $\tau_{\underline{\lambda}}(t)$ on each module $M_i \otimes S$ is as

$$\Delta(\tau_{\underline{\lambda}}(t)) = \sum_{i=1}^m \lambda_i (X_i h_i \otimes h_i + g_i h_i \otimes X_i h_i).$$

Since $X_i h_i$ acts by 0 on S, this is the same as the action of $\sum_{i=1}^m \lambda_i X_i h_i \otimes h_i$, which is $\sum_{i=1}^m \chi(h_i)\lambda_i X_i h_i \otimes 1$. Thus, when restricted to $k[t]/(t^\ell)$ via $\tau^*_{\underline{\lambda}}$, the n-extension is equivalent to $k \to M'_{n-1} \to \cdots \to M'_0 \to k$, where $M'_i = M_i$ as a vector space, and t acts on M'_i as $\sum_{i=1}^m \chi(h_i)\lambda_i X_i h_i = \tau_{\underline{\lambda}'}(t)$ with $\underline{\lambda}' = [\chi(h_1)\lambda_1 : \cdots : \chi(h_m)\lambda_m]$. Therefore there is a $g \in G$ with $\underline{\lambda}' = g \cdot \underline{\lambda}$. By Lemma 5.2(ii), $\tau^*_{\underline{\lambda}} = \tau^*_{\underline{\lambda}'}$, and so the diagram commutes.

Note that the map $-\otimes S$ in fact identifies $\operatorname{Ext}^*_A(k,k)$ and $\operatorname{Ext}^*_A(S,S)$ as graded vector spaces: An inverse map is given by $-\otimes S^\#$ since $S \otimes S^\# \cong k$.

Consider the following commutative diagram where the vertical arrows are restrictions via $\tau_{\underline{\lambda}}$ and horizontal arrows are actions via Yoneda product:

(5.5.2)
$$\begin{tikzcd}
\operatorname{Ext}^*_A(S,S) \times \operatorname{Ext}^*_A(S,V(\underline{\lambda})) \arrow[r] \arrow[d, "(\tau^*_{\underline{\lambda}}, \tau^*_{\underline{\lambda}})"] & \operatorname{Ext}^*_A(S,V(\underline{\lambda})) \arrow[d, "\tau^*_{\underline{\lambda}}"] \\
\operatorname{Ext}^*_{k[t]/(t^\ell)}(k,k) \times \operatorname{Ext}^*_{k[t]/(t^\ell)}(k,V(\underline{\lambda})) \arrow[r] & \operatorname{Ext}^*_{k[t]/(t^\ell)}(k,V(\underline{\lambda}))
\end{tikzcd}$$

The action of $\operatorname{Ext}^*_A(k,k)$ on $\operatorname{Ext}^*_A(S,V(\underline{\lambda}))$ factors through the action of $\operatorname{Ext}^*_A(S,S)$ via $-\otimes S : \operatorname{Ext}^*_A(k,k) \to \operatorname{Ext}^*_A(S,S)$. By Lemma 2.4, the rightmost vertical arrow of (5.5.2) is injective. Let $\alpha \in \operatorname{Ker}(\tau^*_{\underline{\lambda}})$ in $\operatorname{Ext}^*_A(k,k)$. Commutativity of (5.5.1) implies that $\alpha \otimes S$ is also in the kernel of $\tau^*_{\underline{\lambda}} : \operatorname{Ext}^*_A(S,S) \to \operatorname{Ext}^*_{k[t]/(t^\ell)}(k,k)$. This means that for every $\beta \in \operatorname{Ext}^*_A(S,V(\underline{\lambda}))$, we have $\tau^*_{\underline{\lambda}}((\alpha \otimes S)\cdot\beta) = 0$. As this $\tau^*_{\underline{\lambda}}$ is injective, this implies $(\alpha \otimes S)\cdot \beta = 0$, that is $\alpha \in I(S,V(\underline{\lambda}))$. Since this holds for any $\alpha \in \operatorname{Ker}(\tau^*_{\underline{\lambda}})$, we get $\operatorname{Ker}(\tau^*_{\underline{\lambda}}) \subset I(S,V(\underline{\lambda}))$ as required. □

Finally we use the modules $V(\underline{\lambda})$ and L_ζ to prove equivalence of the rank and support varieties.

Theorem 5.6. *Let M be a finitely generated A-module. Then*

$$\Psi(V^r_A(M)) = V^c_A(M).$$

Proof. Let $\underline{\lambda} \in V_A^r(M)$, where we abuse notation by identifying an element $\underline{\lambda} \in \mathbb{P}^{m-1}$ with its G-orbit. Then $\underline{\mathrm{Hom}}_A(V(\underline{\lambda}), M) \neq 0$ by Lemma 2.5(i). By periodicity (Lemma 2.3(v)),

$$\mathrm{Ext}_A^{2n}(V(\underline{\lambda}), M) = \underline{\mathrm{Hom}}_A(V(\underline{\lambda}), M) \neq 0$$

for every positive integer n. This implies that $V_A^c(V(\underline{\lambda}), M) \neq \emptyset$. Indeed, suppose $V_A^c(V(\underline{\lambda}), M) = \emptyset$. Then $\sqrt{I(V(\underline{\lambda}), M)} = \mathrm{H}^{>0}(A, k)$. Note that $\mathrm{Ext}_A^*(V(\underline{\lambda}), M) \cong \mathrm{H}^*(A, V(\underline{\lambda})^{\#} \otimes M)$ (see the appendix), and this is a finitely generated module over $\mathrm{H}^*(A, k)$ by Lemma 4.1. This implies that $\mathrm{Ext}_A^n(V(\underline{\lambda}), M) = 0$ for all sufficiently large n, contradicting what we found above. Thus, $V_A^c(V(\underline{\lambda}), M) \neq \emptyset$.

Proposition 4.3(iii) and Lemma 5.5 imply that $V_A^c(V(\underline{\lambda}), M) \subset V_A^c(V(\underline{\lambda})) \cap V_A^c(M) = \Psi(\underline{\lambda}) \cap V_A^c(M)$. Since $V_A^c(V(\underline{\lambda}), M) \neq \emptyset$, we conclude that $\Psi(\underline{\lambda}) \subset V_A^c(M)$. This proves the containment $\Psi(V_A^r(M)) \subseteq V_A^c(M)$.

Now suppose $\Psi(\underline{\lambda}) \in V_A^c(M)$. We can find a finite set of homogeneous elements $\zeta_1, \ldots, \zeta_{m-1}$ in $\mathrm{H}^*(A, k)$ (where $m-1$ is the projective dimension of V_A^c) such that $\Psi(\underline{\lambda}) = \bigcap \langle \zeta_i \rangle$. Letting $L_{\underline{\lambda}} = L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_n}$ (see (4.3.1)), Corollary 4.5 implies that $V_A^c(M \otimes L_{\underline{\lambda}}) = \Psi(\underline{\lambda})$. By the first part of the proof, it follows that $V_A^r(M \otimes L_{\underline{\lambda}}) \subset \{\underline{\lambda}\}$. Since $V_A^c(M \otimes L_{\underline{\lambda}}) = \Psi(\underline{\lambda}) \neq \emptyset$, Proposition 4.3(v) implies that $M \otimes L_{\underline{\lambda}}$ is not projective. Theorem 3.4 implies that $V_A^r(M \otimes L_{\underline{\lambda}}) \neq \emptyset$. Hence, $V_A^r(M \otimes L_{\underline{\lambda}}) = \{\underline{\lambda}\}$.

For each i, we have a short exact sequence $0 \to M \otimes L_{\zeta_i} \to M \otimes \Omega^n(k) \to M \to 0$ obtained by applying $M \otimes -$ to (4.3.1). By Proposition 3.3(iv), $V_A^r(M \otimes L_{\zeta_i}) \subset V_A^r(M) \cup V_A^r(M \otimes \Omega^n(k))$. Since $M \otimes \Omega^n(k) \cong \Omega^n(M)$ in the stable module category, Proposition 3.3(iii) implies that $V_A^r(M \otimes L_{\zeta_i}) \subset V_A^r(M)$. Proceeding by induction, we conclude that

$$V_A^r(M \otimes L_{\underline{\lambda}}) \subset V_A^r(M).$$

Thus, $\underline{\lambda} \in V_A^r(M)$. Since this holds for any $\underline{\lambda}$ such that $\Psi(\underline{\lambda}) \in V_A^c(M)$, we get the desired inclusion $V_A^c(M) \subseteq \Psi(V_A^r(M))$. $\square$

As a consequence of Lemma 5.5 and Theorem 5.6, we may now record the rank variety of $V(\underline{\lambda})$.

Corollary 5.7. $V_A^r(V(\underline{\lambda})) = \{\underline{\lambda}\}$.

6. Varieties for modules of truncated polynomial algebras

Our results have consequences for modules of the truncated polynomial algebra $\Lambda = k[X_1, \ldots, X_m]/(X_1^{\ell}, \ldots, X_m^{\ell})$, which we give in this section. We define the rank variety of a Λ-module M by

$$V_\Lambda^r(M) = V_A^r(M \uparrow^A), \tag{6.0.1}$$

where the rank variety of the induced A-module $M\uparrow^A = A\otimes_\Lambda M$ is given in Definition 3.2. Since A is free as a Λ-module, induction from Λ to A is well-behaved. The rank variety of the trivial Λ-module is $\mathbb{P}^{m-1}$ since $\tau(\underline{\lambda})$ acts trivially on $A\otimes_\Lambda k$, for any $\underline{\lambda}$. We will use the notation $V_\Lambda^r = V_\Lambda^r(k) = V_A^r(k\uparrow^A) \cong \mathbb{P}^{m-1}/G$.

Remarks 6.1. (i) An alternative definition of rank varieties for Λ-modules is given in [9, Rk. 4.7(2)]. We expect that our definition is equivalent to this one, however we only have a proof that they are equivalent in case $\ell = 2$ (see Remark 6.8 below).

(ii) Viewing the A-modules $V(\underline{\lambda}) = A\cdot\tau_{\underline{\lambda}}(t)^{\ell-1}$ (see Section 2) as Λ-modules by restriction, we have the following characterization of the rank variety for a Λ-module M: $V_\Lambda^r(M)$ consists of all $\underline{\lambda}\in\mathbb{P}^{m-1}/G$ such that $\underline{\mathrm{Hom}}_\Lambda(V(\underline{\lambda}), M)\neq 0$ (cf. [14, Lem. 3.7(2) and Defn. 4.1]). This is a consequence of Lemma 2.5(i) since the Eckmann-Shapiro Lemma implies that $\underline{\mathrm{Hom}}_A(V(\underline{\lambda}), M\uparrow^A)\cong\underline{\mathrm{Hom}}_\Lambda(V(\underline{\lambda}), M)$ as a result of an isomorphism between induced and coinduced modules. The isomorphism $\mathrm{Hom}_\Lambda(A, M)\cong M\uparrow^A$ is given by sending $f\in\mathrm{Hom}_\Lambda(A,M)$ to $\sum_{g\in G} g\otimes f(g^{-1})$. The left A-module structure of $\mathrm{Hom}_\Lambda(A,M)$ is standard, given by $(a\cdot f)(b) = f(ba)$ for all $a,b\in A$, $f\in\mathrm{Hom}_\Lambda(A,M)$.

We define the support variety $V_\Lambda^c(M)$ using the action of $\mathrm{H}^*(\Lambda,k) = \mathrm{Ext}_\Lambda^*(k,k)$ on $\mathrm{H}^*(\Lambda, M) = \mathrm{Ext}_\Lambda^*(k,M)$ by Yoneda composition. Note that $\mathrm{H}^*(\Lambda,k)$ is the tensor product of a symmetric algebra with an exterior algebra, each in m variables; this may be seen by using (4.0.5) and the Künneth formula. In particular, $\mathrm{H}^*(\Lambda,k)_{red}\cong\mathrm{H}^*(A,k)$. Let $\mathrm{H}^\bullet(\Lambda,k)$ denote the subalgebra of $\mathrm{H}^*(\Lambda,k)$ that is the sum of all even degree components when char $k\neq 2$, and $\mathrm{H}^\bullet(\Lambda,k) = \mathrm{H}^*(\Lambda,k)$ when char $k = 2$.

Definition 6.2. Let $V_\Lambda^c = \operatorname{Spec}\mathrm{H}^\bullet(\Lambda,k)$. For a finitely generated Λ-module M, define $V_\Lambda^c(M) = V_\Lambda^c(k,M)$ to be the closed subset of V_Λ^c defined by the annihilator ideal $\mathrm{Ann}_{\mathrm{H}^\bullet(\Lambda,k)}\mathrm{H}^*(\Lambda,M)$.

Remark 6.3. Since Λ does not have a Hopf algebra structure, we cannot consider the action of $\mathrm{H}^\bullet(\Lambda,k)$ on $\mathrm{Ext}_\Lambda^*(M,M)$ as it is usually done to define support varieties when the Hopf algebra structure is present. Nonetheless, since Λ has a unique trivial module k, Definition 6.2 is parallel to the standard definition in view of Proposition 4.3(iv). As Λ is a complete intersection, Avramov and Buchweitz have given another definition of support variety that applies [1, 2], and arguments in [30] combined with Theorem 6.6 below may be used to show our definition agrees with theirs.

The map Ψ of (5.2.3) can be identified with a map $\Psi: V_\Lambda^r\to V_\Lambda^c$ via $V_\Lambda^r = V_A^r$ and $V_\Lambda^c = V_A^c$; the second identification comes from the isomorphism $\mathrm{H}^*(\Lambda,k)_{red}\cong\mathrm{H}^*(A,k)$. We will show that Ψ takes the rank variety of a finitely generated Λ-module to its support variety.

Theorem 6.4. *Let M be a finitely generated Λ-module. Then*

$$\Psi(V^r_\Lambda(M)) = V^c_\Lambda(M).$$

Proof. By the definition (6.0.1) of $V^r_\Lambda(M)$ and Theorem 5.6, we need only check that $V^c_\Lambda(M) = V^c_A(M\uparrow^A)$. By Proposition 4.3(iv) it suffices to show that $V^c_\Lambda(M) = V^c_A(S, M\uparrow^A)$ for any simple A-module S. By arguments similar to those in Remark 6.1(ii), $\mathrm{Ext}^*_A(S, M\uparrow^A) \cong \mathrm{Ext}^*_\Lambda(S, M) \cong \mathrm{Ext}^*_\Lambda(k, M)$. This is an isomorphism of $\mathrm{H}^*(A,k)$-modules where $\mathrm{H}^*(A,k)$ acts on $\mathrm{Ext}^*_\Lambda(k,M)$ via the embedding $\mathrm{H}^*(A,k) \hookrightarrow \mathrm{H}^*(\Lambda,k)$. Since $\mathrm{H}^*(A,k) = \mathrm{H}^\bullet(\Lambda,k)_{red}$, the variety of the annihilator of $\mathrm{Ext}^*_\Lambda(k,M)$ is determined by the action of $\mathrm{H}^*(A,k)$. The statement follows. □

We now explain the connection between our results and the work on support varieties which was done from the point of view of Hochschild cohomology in [15], [16], [30].

We will show that the rank variety of a Λ-module is also equivalent to its *Hochschild* support variety defined as follows via a particular choice of subalgebra of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda,\Lambda)$, where $\Lambda^e = \Lambda \otimes \Lambda^{op}$. We use the definition of (Hochschild) support varieties for modules of finite dimensional algebras given in [30] and developed further in [16]. Let

$$H = \mathrm{H}^*(A,k).$$

We will show that H embeds canonically as a subalgebra of $\mathrm{HH}^*(\Lambda)$. To do this, define a map $\delta : A \to A^e = A \otimes A^{op}$ by $\delta(a) = \sum a_1 \otimes S(a_2)$. We will need to consider the following subalgebra of A^e,

$$\mathcal{D} = \Lambda^e \rtimes \delta(G) = \bigoplus_{g\in G} (\Lambda g \otimes \Lambda g^{-1}) \subset A^e,$$

where $\delta(G) = \{(g, g^{-1}) \mid g \in G\} \cong G$ acts on Λ^e by the left and right actions induced by the action of G on Λ. Note that $\mathcal{D}$ contains the subalgebra $\delta(A) \cong A$ (see Lemma 7.1 below). As noted in the proof of Lemma 7.2, A^e is projective as a right $\delta(A)$-module under multiplication. Note that $A^e = \oplus_{g\in G}((g\otimes 1)\mathcal{D})$ as a right $\delta(A)$-module, that is $\mathcal{D}$ is a direct summand of the projective $\delta(A)$-module A^e, and so is projective itself.

Let $P_\bullet \to k$ be an A-projective resolution of the trivial A-module k. The isomorphism $A \cong A^e \otimes_{\delta(A)} k$ of A^e-modules given in Lemma 7.1 restricts to an isomorphism $\Lambda \cong \mathcal{D} \otimes_{\delta(A)} k$ of $\mathcal{D}$-modules. Therefore induction to $\mathcal{D}$ yields a $\mathcal{D}$-projective resolution of the $\mathcal{D}$-module Λ:

$$\cdots \to \mathcal{D} \otimes_{\delta(A)} P_1 \to \mathcal{D} \otimes_{\delta(A)} P_0 \to \Lambda \to 0.$$

Induction further to A^e yields an A^e-projective resolution

$$\cdots \to A^e \otimes_{\delta(A)} P_1 \to A^e \otimes_{\delta(A)} P_0 \to A \to 0.$$

Now suppose $\xi \in \mathrm{H}^n(A,k)$, and identify ξ with a representative A-map $\xi : P_n \to k$. Induction yields a $\mathcal{D}$-map $\xi' : \mathcal{D} \otimes_{\delta(A)} P_n \to \Lambda$, and further induction yields an A^e-map $\xi'' : A^e \otimes_{\delta(A)} P_n \to A$. The induction from $\delta(A)$ to A^e results in precisely the embedding of $\mathrm{H}^n(A,k)$ into $\mathrm{HH}^n(A)$ given in Lemma 7.2. Therefore, the map sending ξ to ξ' is an embedding of $\mathrm{H}^n(A,k)$ into $\mathrm{Ext}^n_{\mathcal{D}}(\Lambda,\Lambda)$. Note that $\mathrm{Ext}^n_{\mathcal{D}}(\Lambda,\Lambda) \cong \mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda)^G = \mathrm{HH}^n(\Lambda)^G$ since the characteristic of k is relatively prime to the order of G. This provides the embedding of $H = \mathrm{H}^*(A,k) \hookrightarrow \mathrm{HH}^*(\Lambda)^G \hookrightarrow \mathrm{HH}^*(\Lambda)$.

Remark 6.5. The embedding $H = \mathrm{H}^*(A,k) \hookrightarrow \mathrm{HH}^*(\Lambda)$ can be described explicitly as follows. In case $m = 1$, identify Λ with $k[t]/(t^\ell)$. There is a periodic Λ^e-free resolution of Λ:

$$\cdots \xrightarrow{v\cdot} \Lambda^e \xrightarrow{u\cdot} \Lambda^e \xrightarrow{v\cdot} \Lambda^e \xrightarrow{u\cdot} \Lambda^e \xrightarrow{m} \Lambda \to 0$$

where $u = t\otimes 1 - 1\otimes t$ and $v = t^{\ell-1}\otimes 1 + t^{\ell-2}\otimes t + \cdots + 1\otimes t^{\ell-1}$ [34, Exer. 9.1.4]. Using this resolution, one computes $\mathrm{HH}^n(\Lambda) \cong \Lambda/(t^{\ell-1})$ and $\mathrm{HH}^n(\Lambda)^G \cong k$ for all $n > 0$. As $\mathrm{H}^n(A,k) = k$ for all even $n \geq 0$, and 0 for all odd $n > 0$, the embedding $\mathrm{H}^*(A,k) \hookrightarrow \mathrm{HH}^*(\Lambda)^G$ is forced to be an isomorphism onto $\mathrm{HH}^*(\Lambda)^G_{red}$ in case the characteristic is not 2. Apply the Künneth Theorem to obtain the embedding for all m.

We must verify that Λ and H satisfy the properties required by the theory of support varieties defined via Hochschild cohomology in [16]: As Λ is local, it is an indecomposable algebra. The cohomology algebra $H = \mathrm{H}^*(A,k)$ is a polynomial ring in m variables, each of degree 2 (4.0.4), so it is a commutative, noetherian, graded subalgebra of $\mathrm{HH}^*(\Lambda)$. The assumption $H^0 = Z(\Lambda) = \Lambda$ does not hold. However the generators of Λ are nilpotent and so we may consider H to be the reduced version of the subalgebra of $\mathrm{HH}^*(\Lambda)$ generated by H and Λ, and for the purpose of defining varieties it suffices just to consider H. Thus H essentially satisfies assumption (Fg1) of [16]. Since $\mathrm{H}^*(A,k) \cong \mathrm{Ext}^*_\Lambda(k,k)^G$, we get that $\mathrm{Ext}^*_\Lambda(k,k)$ is a finitely generated module over H, however we must check that the usual action agrees with that defined via the subalgebra $\mathcal{D}$ above. To see this, it suffices to check that $(\mathcal{D}\otimes_{\delta(A)} M)\otimes_\Lambda k \cong M$ as Λ-modules, for any A-module M. Using the techniques of Lemma 7.1, we may write an arbitrary element of $(\mathcal{D}\otimes_{\delta(A)} M)\otimes_\Lambda k$ as a linear combination of certain elements

$$\begin{aligned} a\otimes b\otimes m\otimes 1 &= \sum a_1\otimes \varepsilon(a_2)b\otimes m\otimes 1 \\ &= \sum a_1\otimes S(a_2)a_3b\otimes m\otimes 1 \\ &= \sum 1\otimes a_2b\otimes a_1m\otimes 1 \\ &= \sum 1\otimes 1\otimes a_1m\otimes \varepsilon(a_2b) = 1\otimes 1\otimes \varepsilon(b)am\otimes 1 \end{aligned}$$

$(a, b \in A,\ m \in M)$ since Λ is a left coideal subalgebra of Λ (that is $a_2 b \in \Lambda$ when $a \otimes b \in \mathcal{D}$). The desired isomorphism $(\mathcal{D} \otimes_{\delta(A)} M) \otimes_\Lambda k \to M$ is thus given by $a \otimes b \otimes m \otimes 1 \mapsto \varepsilon(b) am$, with inverse $m \mapsto 1 \otimes 1 \otimes m \otimes 1$. This proves that (Fg2) of [16] is satisfied.

As in [16, 30] we define the (Hochschild) support variety of a Λ-module M, with respect to $H = \mathrm{H}^*(A, k)$, as

$$V_\Lambda^H(M) = \operatorname{Proj} H / \operatorname{Ann}_H \operatorname{Ext}^*_\Lambda(M, M) \tag{6.5.1}$$

where the action of H on $\operatorname{Ext}^*_\Lambda(M, M)$ is by $- \otimes_\Lambda M$ (under the identification of elements of H with Λ^e-extensions of Λ by Λ) followed by Yoneda composition. We next show that $V_\Lambda^H(M)$ is homeomorphic to the support variety $V_\Lambda^c(M)$ given in Definition 6.2.

Theorem 6.6. *Let M be a finitely generated Λ-module. Then*

$$V_\Lambda^H(M) \cong V_\Lambda^c(M).$$

Proof. It was shown in the proof of Theorem 6.4 that $V_\Lambda^c(M) = V_A^c(M\uparrow^A)$. We will show here that $V_A^c(M\uparrow^A) \cong V_\Lambda^H(M)$. We analyze the following diagram:

$$\begin{tikzcd}
\mathrm{H}^*(A,k) \arrow[d, "A^e \otimes_A -"'] \arrow[dr, "- \otimes_k M\uparrow^A"] & \\
\mathrm{HH}^*(A) \arrow[r, "- \otimes_A M\uparrow^A"] & \operatorname{Ext}^*_A(M\uparrow^A, M\uparrow^A) \\
\operatorname{Ext}^*_{\mathcal{D}}(\Lambda, A) \arrow[u, "\cong"] & \operatorname{Ext}^*_\Lambda(M, M\uparrow^A) \arrow[u, "\cong"'] \\
\mathrm{HH}^*(\Lambda)^G \arrow[u] \arrow[r, "- \otimes_\Lambda M"] & \operatorname{Ext}^*_\Lambda(M, M) \arrow[u]
\end{tikzcd}$$

The lower left arrow going up is defined by applying the isomorphism $\operatorname{Ext}^*_{\mathcal{D}}(\Lambda, \Lambda) \cong \mathrm{HH}^*(\Lambda)^G$ discussed above and identifying Λ with the $\mathcal{D}$-submodule $\Lambda \rtimes 1$ of $A = \Lambda \rtimes G$. The lower right arrow going up is defined by identifying M with the Λ-submodule $1 \otimes M$ of $M\uparrow^A$. The embedding of $\mathrm{H}^*(A, k) = \operatorname{Ext}^*_A(k, k)$ into $\mathrm{HH}^*(A) = \operatorname{Ext}^*_{A^e}(A, A)$ in the diagram identifies it with a subalgebra of the image of $\mathrm{HH}^*(\Lambda)^G$ in $\mathrm{HH}^*(A)$ as a result of the discussions above on the algebra $\mathcal{D}$.

The top triangle commutes by Lemma 7.3. This implies that the annihilators of $\operatorname{Ext}^*_A(M\uparrow^A, M\uparrow^A)$ in $\operatorname{Ext}^*_A(k, k)$ and in the subalgebra H of $\operatorname{Ext}^*_{A^e}(A, A)$ coincide. We have also seen that H may be identified with a subalgebra of $\mathrm{HH}^*(\Lambda)^G$, compatible with these embeddings. Therefore it remains to check commutativity of the bottom part of the diagram, since the vertical arrows are injections, implying the appropriate annihilators will coincide. This may be checked directly at the chain level, using the identification $\mathrm{HH}^*(\Lambda)^G \cong \operatorname{Ext}^*_{\mathcal{D}}(\Lambda, \Lambda)$ given above. $\square$

We have an immediate consequence of Theorems 6.4 and 6.6.

Corollary 6.7. *Let M be a finitely generated Λ-module. Then*

$$\Psi(V_\Lambda^r(M)) \cong V_\Lambda^H(M).$$

Remark 6.8. In case $\ell = 2$, our support varieties for Λ-modules are equivalent to those defined by Erdmann and Holloway [15] since our choice of H is precisely $\mathrm{HH}^*(\Lambda)$ modulo nilpotent elements. As a consequence of Corollary 6.7 and the analogue of the Avrunin-Scott Theorem in [15], our rank varieties for Λ-modules also coincide with those defined in [15]. In case $\ell > 2$, we expect our rank varieties for Λ-modules to coincide with those defined by Benson, Erdmann, and Holloway [9, Rk. 4.7(2)], but the cohomological techniques are not yet available in this case and so perhaps a different approach is needed.

7. Appendix

Hochschild cohomology of Hopf algebras. Here we allow A to be any Hopf algebra over the field k, and record some general results. In particular, we give connections between the cohomology $\mathrm{H}^*(A,k)$ and the Hochschild cohomology $\mathrm{HH}^*(A) = \mathrm{Ext}_{A^e}(A,A)$, where $A^e = A \otimes A^{op}$ acts on A by left and right multiplication. An embedding of $\mathrm{H}^*(A,k)$ into $\mathrm{HH}^*(A)$ is deduced by Ginzburg and Kumar [22, Prop. 5.6 and Cor. 5.6]. Here we give an explicit map expressing such an embedding at the chain level and the resulting connections between actions on $\mathrm{Ext}_A^*(M,M)$ where M is an A-module (see Lemma 7.3).

Lemma 7.1. *There is an isomorphism of A^e-modules $A \cong (A^e) \otimes_A k$, where A is embedded as a subalgebra of A^e via the algebra homomorphism $\delta : A \to A^e$ defined by $\delta(a) = \sum a_1 \otimes S(a_2)$.*

Proof. First note that δ is indeed injective as $\pi \circ \delta = \mathrm{id}$ where the linear map $\pi : A^e \to A$ is defined by $\pi(a \otimes b) = a\,\varepsilon(b)$. As S is an algebra anti-homomorphism, δ is an algebra homomorphism. Next define $f : A^e \otimes_A k \to A$ by $f(a \otimes b \otimes 1) = ab$ and $g : A \to A^e \otimes_A k$ by $g(a) = a \otimes 1 \otimes 1$. We check that f and g are inverse maps: $f \circ g(a) = f(a \otimes 1 \otimes 1) = a$ for all $a \in A$, and

$$\begin{aligned} g \circ f(a \otimes b \otimes 1) &= ab \otimes 1 \otimes 1 \\ &= \sum ab_1 \otimes \varepsilon(b_2) \otimes 1 \\ &= \sum ab_1 \otimes S(b_2)b_3 \otimes 1 \\ &= \sum a \otimes b_2 \otimes \varepsilon(b_1) = a \otimes b \otimes 1, \end{aligned}$$

for all $a, b \in A$. Similar calculations show that f and g are both A^e-module homomorphisms. □

We will next recall some homological properties of modules for a Hopf algebra. For proofs, see [5, I. §3.1]. Let U be a finite dimensional projective left (respectively, right) A-module, and V any finite dimensional left (respectively, right) A-module. Then $V \otimes U$ is a projective left (respectively, right) A-module. If V, W are left A-modules, then $\mathrm{Hom}_k(V,W)$ is a left A-module under the action $(a \cdot f)(v) = \sum a_2 \cdot f(S(a_1) \cdot v)$. In this way, $\mathrm{Hom}_k(V,W) \cong V^{\#} \otimes W$ as A-modules, where $V^{\#} = \mathrm{Hom}_k(V,k)$ is an A-module similarly, with k taking the trivial action of A. If U, V, and W are left A-modules, then there is a natural isomorphism $\mathrm{Hom}_k(V \otimes U, W) \cong \mathrm{Hom}_k(U, V^{\#} \otimes W)$ of A-modules. This restricts to a natural isomorphism of vector spaces $\mathrm{Hom}_A(V \otimes U, W) \cong \mathrm{Hom}_A(U, V^{\#} \otimes W)$, further inducing an isomorphism of graded vector spaces

$$\mathrm{Ext}^*_A(V \otimes U, W) \cong \mathrm{Ext}^*_A(U, V^{\#} \otimes W).$$

Similar results hold if U, V, W are right A-modules, where $\mathrm{Hom}_k(V,W)$ is a right A-module under the action $(f \cdot a)(v) = \sum f(v \cdot S(a_1)) \cdot a_2$. It is helpful to view $\mathrm{Hom}_A(U,V)$ as the subspace of A-invariant elements of $\mathrm{Hom}_k(U,V)$, that is as

$$(\mathrm{Hom}_k(U,V))^A = \{f \in \mathrm{Hom}_k(U,V) \mid a \cdot f = \varepsilon(a) f \text{ for all } a \in A\},$$

equivalent by a straightforward computation [35, Lem. 1].

We will consider A to be a left A-module under the left adjoint action, that is if $a, b \in A$,

$$a \cdot b = \sum a_1 b S(a_2).$$

Denote this A-module by A^{ad}.

Lemma 7.2. *There is an isomorphism of graded algebras* $\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, A^{ad})$. *This induces an embedding of* $\mathrm{H}^*(A,k)$ *into* $\mathrm{HH}^*(A)$.

Proof. Since $A^e = A \otimes A^{op}$, the tensor product of the free A-module A with another A-module, A^e is projective as an A-module, the action of A being multiplication by $\delta(A)$. We may therefore apply the Eckmann-Shapiro Lemma, together with Lemma 7.1, to obtain an isomorphism of vector spaces $\mathrm{Ext}^*_A(k, A^{ad}) \cong \mathrm{Ext}^*_{A^e}(A,A)$. This is in fact an isomorphism of algebras. (The correspondence of cup products follows from a generalization of the proof of [29, Prop. 3.1] from group algebras to Hopf algebras. See also [22, §5.6].) Note that the trivial module k is a direct summand of A^{ad}, with complement the augmentation ideal $\mathrm{Ker}(\varepsilon)$. This results in identification of $\mathrm{H}^*(A,k) = \mathrm{Ext}^*_A(k,k)$ with a subalgebra of $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A,A)$. By construction, this identification is given explicitly by the map $A^e \otimes_A -$ on extensions. $\square$

The next lemma gives the connection between actions on $\mathrm{Ext}^*_A(M,M)$ that is used in Section 6.

Lemma 7.3. *Let M be a finitely generated left A-module. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{H}^*(A,k) \arrow[d, "A^e\otimes_A -"'] \arrow[dr, "-\otimes_k M"] & \\
\mathrm{HH}^*(A) \arrow[r, "-\otimes_A M"] & \mathrm{Ext}^*_A(M,M)
\end{tikzcd}$$

Proof. The leftmost map $A^e\otimes_A -$ is induced by the diagonal embedding $\delta : A \to A^e$ defined in Lemma 7.1. The bottommost map involves right multiplication of A^e by $A \cong 1\otimes A$. Note that $-\otimes_A M$ does indeed take n-extensions to n-extensions: We may assume all modules in an n-extension of A by A as A^e-modules are projective over A. Then $-\otimes_A M$ takes such an exact sequence to another exact sequence.

We define functorial A-module isomorphisms $f_X : (A^e \otimes_A X)\otimes_A M \to X\otimes_k M$ for all left A-modules X. If $0\to k\to P_n \to \cdots \to P_1 \to k \to 0$ is an n-extension of A-modules representing an element of $\mathrm{H}^n(A,k)$, the functions f_{P_*} will take the extension $(A^e\otimes_A P_*)\otimes_A M$ to $P_*\otimes_k M$. We define f_X as a composition of two functions, $g_X : (A^e\otimes_A X)\otimes_A M \to (X\otimes_k A)\otimes_A M$ and $h_X : (X\otimes_k A)\otimes_A M \to X\otimes_k M$. Let $g_X(a\otimes b\otimes x\otimes m) = \sum a_1 x\otimes a_2 b\otimes m$ and $h_X(x\otimes a\otimes m) = x\otimes am$ for all $a,b\in A$, $x\in X$, $m\in M$. Clearly h_X is a well-defined isomorphism. It may be checked that g_X is well-defined with inverse $g_X^{-1}(x\otimes a\otimes m) = 1\otimes a\otimes x\otimes m$. Therefore an n-extension of A-modules k by k is taken to the same extension of M by M, either way around the diagram. □

Computational lemma. The following result is needed in the proof of Lemma 2.4. Let $A = \Lambda \rtimes G$ as in Section 2, and recall the notation $\tau_{\underline{\lambda}}(t) = \sum_{i=1}^m \lambda_i X_i h_i$ where $\underline{\lambda} = (\lambda_1,\ldots,\lambda_m)\in k^m$. If S is a simple A-module, let e_S be the primitive central idempotent of kG corresponding to S.

Lemma 7.4. *Let $a\in A$, $\underline{\lambda}\in k^m\setminus\{0\}$, and S a simple A-module. If $\tau_{\underline{\lambda}}(t)a = 0$ and $a\tau_{\underline{\lambda}}(t)^{\ell-1}$ is a scalar multiple of $e_S X_1^{\ell-1}\cdots X_m^{\ell-1}$, then $a\tau_{\underline{\lambda}}(t)^{\ell-1} = 0$.*

Proof. Write $a = \sum_{0\le a_i,b_i\le \ell-1} \alpha_{a_1,\ldots,a_m,b_1,\ldots,b_m} X_1^{a_1}\cdots X_m^{a_m} g_1^{b_1}\cdots g_m^{b_m}$ for some scalars $\alpha_{a_1,\ldots,a_m,b_1,\ldots,b_m}\in k$. Assume without loss of generality that $\lambda_1 = 1$. We use a q-multinomial formula for $\tau_{\underline{\lambda}}(t)^{\ell-1}$, which may be obtained from the q-binomial formula (stated in the proof of Lemma 2.1) and induction on m. We need the notation

$$\binom{n}{s_1,\ldots,s_m}_q = \frac{(n)_q!}{(s_1)_q!\cdots(s_m)_q!}.$$

Assuming $\tau_{\underline{\lambda}}(t)a = 0$ and $\lambda_1 = 1$,

$$
\begin{aligned}
0 &= \tau_{\underline{\lambda}}(t)^{\ell-1}a \\
&= \left(\sum_{\substack{0\le s_i\le \ell-1\\ s_1+\cdots+s_m=\ell-1}} \binom{\ell-1}{s_1,\ldots,s_m}_q \lambda_2^{s_2}\cdots\lambda_m^{s_m} X_1^{s_1}\cdots X_m^{s_m} h_1^{s_1}\cdots h_m^{s_m}\right)\cdot \\
&\qquad \left(\sum_{0\le a_i,b_i\le \ell-1} \alpha_{a_1,\ldots,a_m,b_1,\ldots,b_m} X_1^{a_1}\cdots X_m^{a_m} g_1^{b_1}\cdots g_m^{b_m}\right) \\
&= \sum_{\substack{0\le s_i\le \ell-1\\ s_1+\cdots+s_m=\ell-1}} \sum_{0\le a_i,b_i\le \ell-1} \alpha_{a_1,\ldots,a_m,b_1,\ldots,b_m} \binom{\ell-1}{s_1,\ldots,s_m}_q \lambda_2^{s_2}\cdots\lambda_m^{s_m}\cdot \\
&\qquad q^{a_1s_2+(a_1+a_2)s_3+\cdots+(a_1+\cdots+a_{m-1})s_m} X_1^{a_1+s_1}\cdots X_m^{a_m+s_m} g_1^{b_1+s_2+\cdots+s_m}\cdots g_m^{b_m} \\
&= \sum_{0\le a_i,b_i\le \ell-1} \left(\sum_{\substack{0\le s_i\le \ell-1\\ s_1+\cdots+s_m=\ell-1}} \alpha_{a_1-s_1,\ldots,a_m-s_m,b_1-s_2-\cdots-s_m,\ldots,b_m} \binom{\ell-1}{s_1,\ldots,s_m}_q \lambda_2^{s_2}\cdots\lambda_m^{s_m}\cdot \right. \\
&\qquad \left. q^{(a_1-s_1)s_2+(a_1+a_2-s_1-s_2)s_3+\cdots+(a_1+\cdots+a_{m-1}-s_1-\cdots-s_{m-1})s_m}\right) X_1^{a_1}\cdots X_m^{a_m} g_1^{b_1}\cdots g_m^{b_m},
\end{aligned}
$$

where we define $\alpha_{a_1,\ldots,a_m,b_1,\ldots,b_m} = 0$ if any one of $a_1,\ldots,a_m$ is negative. Each coefficient is thus 0, that is

$$
(7.4.1)\quad 0 = \sum_{\substack{0\le s_i\le \ell-1\\ s_1+\cdots+s_m=\ell-1}} \alpha_{a_1-s_1,\cdots,a_m-s_m,b_1-s_2-\cdots-s_m,\cdots,b_m} \binom{\ell-1}{s_1,\cdots,s_m}_q \lambda_2^{s_2}\cdots\lambda_m^{s_m}\cdot
q^{(a_1-s_1)s_2+(a_1+a_2-s_1-s_2)s_3+\cdots+(a_1+\cdots+a_{m-1}-s_1-\cdots-s_{m-1})s_m}.
$$

By similar calculations we have

$$
\begin{aligned}
a\tau_{\underline{\lambda}}(t)^{\ell-1} = \sum_{0\le a_i,b_i\le \ell-1} &\left(\sum_{\substack{0\le s_i\le \ell-1\\ s_1+\cdots+s_m=\ell-1}} \alpha_{a_1-s_1,\ldots,a_m-s_m,b_1-s_2-\cdots-s_m,\ldots,b_m} \binom{\ell-1}{s_1,\ldots,s_m}_q \cdot \right. \\
&\left. \lambda_2^{s_2}\cdots\lambda_m^{s_m} q^{(b_1-s_2-\cdots-s_m)s_1+\cdots+b_ms_m}\right) X_1^{a_1}\cdots X_m^{a_m} g_1^{b_1}\cdots g_m^{b_m}.
\end{aligned}
$$

If this were a nonzero element of $ke_SX_1^{\ell-1}\cdots X_m^{\ell-1}$, then for any m-tuple $(b_1,\ldots,b_m)$, the coefficient of $X_1^{\ell-1}\cdots X_m^{\ell-1}g_1^{b_1}\cdots g_m^{b_m}$ would be nonzero, that is

$$
(7.4.2)\quad 0 \ne \sum_{\substack{0\le s_i\le \ell-1\\ s_1+\cdots+s_m=\ell-1}} \alpha_{\ell-1-s_1,\ldots,\ell-1-s_m,b_1-s_2-\cdots-s_m,\cdots,b_m} \binom{\ell-1}{s_1,\ldots,s_m}_q \lambda_2^{s_2}\cdots\lambda_m^{s_m}\cdot
q^{(b_1-s_2-\cdots-s_m)s_1+\cdots+b_ms_m}.
$$

Letting $a_1 = \cdots = a_m = \ell - 1$ in (7.4.1) and comparing with (7.4.2), we claim that we may choose $(b_1, \ldots, b_m)$ so that these coefficients are equal, a contradiction: We will find a solution to

$$q^{(\ell-1-s_1)s_2+\cdots+((m-1)(\ell-1)-s_1-\cdots-s_{m-1})s_m} = q^{(b_1-s_2-\cdots-s_m)s_1+\cdots+b_m s_m}.$$

This equation is equivalent to $q^{-s_2-2s_3-\cdots-(m-1)s_m} = q^{b_1 s_1+\cdots+b_m s_m}$, which has solution $b_1 = 0,\ b_2 = -1, \ldots,\ b_m = -(m-1)$. Thus if $a\tau_{\underline{\lambda}}(t)^{\ell-1}$ is a scalar multiple of $e_S X_1^{\ell-1} \cdots X_m^{\ell-1}$, it must be 0. $\square$

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