INFINITE DIMENSIONAL MODULES FOR FROBENIUS KERNELS

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Abstract. We prove that the projectivity of an arbitrary (possibly infinite dimensional) module for a Frobenius kernel can be detected by restrictions to one-parameter subgroups. Building upon this result, we introduce the support cone of such a module, extending the construction of support variety for a finite dimensional module, and show that such support cones satisfy most of the familiar properties of support varieties. We also verify that our representation-theoretic definition of support cones admits an interpretation in terms of Rickard idempotent modules associated to thick subcategories of the stable category of finite dimensional modules.

0. Introduction

The representation theory of finite dimensional Lie algebras is not only a subject of interest in its own right but reflects significant aspects of the representation theory of algebraic groups. There has been considerable interest in the study of geometric aspects of the finite dimensional representation theory of the restricted enveloping algebra of the Lie algebra of an algebraic group G over a field k of positive characteristic (e.g. [14],[15],[16],[21]), which can be viewed equivalently as the representation theory of the infinitesimal group scheme $G_{(1)}$. The cohomological approach, initiated by D. Quillen in his work in representation theory of finite groups ([24]), involves the action of the (even dimensional) cohomology algebra $H^*(A,k)$ on $\mathrm{Ext}^*_A(M,M)$ for a module M of a given cocommutative Hopf algebra A, whereas the local approach involves the representation-theoretic behavior of M restricted to a certain class of Hopf sub-algebras. In the modular representation theory of finite groups these two approaches were shown to be closely related with the Avrunin-Scott's proof ([3]) of Carlson's conjecture.

In [27],[28] both cohomological and local approaches were extended to the representation theory of Frobenius kernels $G_{(r)}$, infinitesimal approximations of the algebraic group G. The geometric objects associated to a finite dimensional $G_{(r)}$-module, resulting from these two approaches, were shown to be homeomorphic. At the same time, constructions of infinite dimensional representations of finite groups have been introduced and methods have been developed to extend the earlier geometric approach for finite dimensional representations to infinite dimensional representations of finite groups ([6],[7],[25]).

Following much earlier work for finite dimensional modules, we seek to associate to a possibly infinite dimensional module M of a Frobenius kernel $G_{(r)}$ a geometric object (its "support cone") $V(G_{(r)})_M$ which reflects some key properties of M. One criterion for such support cones is that they extend the existing construction

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of support varieties for finite dimensional modules. A second criterion is that these geometric objects satisfy the same properties for all modules that support varieties satisfy for finite dimensional modules. Whereas for finite dimensional modules one can define support varieties either in terms of cohomology or in terms of local representation behavior, in the infinite dimensional context these constructions give quite different objects. We find that the local representation theory approach leads to a much better generalization.

A fundamental property of finite dimensional modules is that projectivity can be detected locally on a family of "small" subgroups: elementary abelian subgroups for finite groups ([8]), cyclic $[p]$- nilpotent Lie subalgebras for Lie algebras ([15]) and one-parameter subgroups (i.e. subgroups of the form $\mathbb{G}_{a(r)}$) for arbitrary infinitesimal group schemes ([28]). The original proof of this local criterion for projectivity by Chouinard [8] is valid for arbitrary, not necessarily finite dimensional, modules of finite groups. However, the existing proof of such a criterion for infinitesimal group schemes is a consequence of the cohomological description of the support variety and only applies to the finite dimensional case (cf. [28, 7.6]). Section 1 is dedicated to proving a local criterion for projectivity for arbitrary modules for Frobenius kernels (cf. Th. 1.6), building upon a result of C. Bendel for infinitesimal unipotent group schemes ([4]).

With this local criterion for projectivity in mind, we formulate in section 2 our definition of the *support cone* for an arbitrary $G_{(r)}$-module. We use a representation-theoretic construction introduced in [28], which is parallel to Carlson's rank varieties for elementary abelian p-groups ([9]). The support cone is determined upon restriction of the module to a family of "subgroups", isomorphic to $\mathbb{G}_{a(1)}$. A combination of the local criterion for projectivity of the first section and a generalization of Dade's lemma [12] for infinite dimensional modules proved in [7] ensures that these support cones satisfy the key property of support varieties for finite dimensional modules: $V(G_{(r)})_M = 0$ if and only if M is projective. Another important property of support varieties for finite dimensional modules inherited by support cones is good behavior with respect to tensor products: $V(G_{(r)})_{M\otimes N} = V(G_{(r)})_M \cap V(G_{(r)})_N$. Theorem 2.6 verifies that these and other familiar properties of support varieties for finite dimensional modules are satisfied by support cones.

In section 3, we provide a different description of support cones using Rickard idempotent modules ([25]) which are infinite dimensional modules associated to certain thick subcategories of finite dimensional modules. In this manner, our approach to "supports" agrees with that of [7] for arbitrary modules for a finite group. We apply this description to show that any conical subset of $V(G_{(r)})$ can be realized as the support cone of some $G_{(r)}$-module. We further show that the complexity of an infinite dimensional module M as defined in [6] equals the "dimension" of the support cone of M. As a final remark we give an example of the failure of the tensor product property for a natural cohomological formulation of "support" for infinite dimension modules, thereby indicating a fundamental problem with extending the cohomological approach to infinite dimensional modules.

Throughout the paper k will denote an algebraically closed field of positive characteristic p.

The author is in great debt to Eric Friedlander without whom this paper would have never been written. We thank Andrei Suslin for very useful discussions on the first section of the paper, thank Chris Bendel for providing his insights on support varieties for infinite dimensional modules, and thank the referee for the helpful comments.

The author gratefully acknowledges the hospitality of the University of Heidelberg during the preparation of this paper.

1. Local projectivity test for Frobenius kernels

Let G_r be an infinitesimal finite group scheme of height r which is a closed normal subgroup of a smooth algebraic group G and let M be a G_r-module, not necessarily finite dimensional over k. The case of the most interest for us is when G_r is the rth Frobenius kernel of G, denoted $G_{(r)}$, which is defined to be the kernel of the rth power of the Frobenius map $F^r : G \to G^{(r)}$ (cf. [20]). We shall call a G_r-module M *locally projective* if it satisfies the hypothesis of Theorem 1.6 : for any field extension K/k and any one-parameter subgroup $\mathbb{G}_{a(r)} \otimes K \to G_r \otimes K$, the restriction of $M \otimes K$ to $\mathbb{G}_{a(r)} \otimes K$ is projective.

The purpose of this section is to prove a local criterion for projectivity: M is projective if and only if it is locally projective, which is the content of Theorem 1.6. This projectivity detection result was proved in [4] for unipotent infinitesimal group schemes.

First we prove that induction from G_r to G preserves local projectivity. This enables us to use Suslin-Friedlander-Bendel spectral sequence ([28]), which is an extension to $G_{(r)}$-modules of the spectral sequence introduced in [2], to pass from the known case of a unipotent infinitesimal group scheme to G_r. As a remark at the end of the section we show that the proof can be much simplified using the Anderson-Jantzen spectral sequence in the case when the module under consideration has the structure of a rational G-module.

Proposition 1.4 deals with understanding of the composition of induction and restriction functors: $Res_H^G \circ Ind_{G_r}^G$ for a subgroup scheme $H \subset G$. Roughly speaking, we are looking for some analogue of the double coset formula in the representation theory of finite groups. We only analyze this composition in the case when the action of H on the affine variety G/G_r via the left regular representation is trivial which leads to the requirement for G_r to be normal in G.

Fix a Borel subgroup $B \subset G$ and let T and U be the corresponding torus and unipotent subgroup. We shall use the following notation: $B_r = B \cap G_r$, $U_r = U \cap G_r$ and $T_r = T \cap G_r$.

Recall that a finite dimensional Artin algebra A is called *quasi-Frobenius* if it is self-injective. By a theorem of Faith-Walker ([17]) this is equivalent to the fact that any projective A-module is injective and vice versa.

For any finite group scheme H, $k[H]^{\#}$ is a quasi-Frobenius algebra (cf., for example, [20] or [22]). The equivalence of categories of H-modules and $k[H]^{\#}$-modules together with the preceding remark imply that projective H-modules coincide with injective ones.

Lemma 1.1. *Let A be a quasi-Frobenius algebra and M be an A-module. If M admits a finite injective resolution, then M is injective.*

Proof. Assume that M is not injective and let

$$M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \longrightarrow I^n \longrightarrow 0$$

be an injective resolution of M of minimal length. By our assumption $n > 0$.

Since A is quasi-Frobenius and I^n is injective, it is also projective. Then the last map $\delta^n : I^{n-1} \to I^n$ in the injective resolution above splits and $I^{n-1} = J^{n-1} \bigoplus I^n$ for some injective module J^{n-1}. Then

$$M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots \longrightarrow I^{n-2} \longrightarrow J^{n-1} \longrightarrow 0$$

is an injective resolution of M of smaller length than the original one. Thus, M is injective.

□

We shall denote by $\Omega^{-n}M$ the $-n$-th Heller operator of M. Precisely, if $M \to I^0 \to I^1 \to \dots$ is the minimal injective resolution of M, then $\Omega^{-n}M = coker(I^{n-2} \to I^{n-1})$ for $n > 1$ and $\Omega^{-1}M = coker(M \to I^0)$.

Lemma 1.2. *Let A be a quasi-Frobenius algebra and M be an A-module. If there exists an integer n_0 such that $Ext^n_A(S, M) = 0$ for all $n > n_0$ and any simple A-module S, then M is projective.*

Proof. For any simple A-module S and $n > n_0$, we have an isomorphism

$$\mathrm{Hom}_A(S, \Omega^{-n}M) \cong \mathrm{Ext}^n_A(S, M)$$

(cf. [5, v.1;2.5.4]). The latter is 0 for all $n > n_0$. Therefore, $\Omega^{-n}M = 0$ for all $n > n_0$ (since any non-trivial module has a simple submodule). This implies that the minimal injective resolution of M is finite. The statement now follows from Lemma 1.1. □

We will need the following algebraic lemma to finish the proof of Proposition 1.4.

Lemma 1.3. *Let A be a regular ring of finite Krull dimension d and $J^\bullet$ be a cochain complex of flat A-modules such that $J^\bullet \otimes_A k(\mu)$ is acyclic in positive degrees for any prime ideal $\mu \subset A$. Then $H^n(J^\bullet) = 0$ for all $n > d$.*

Proof. We proceed by induction on $d = \dim A$.

First note that $J^\bullet$ has zero cohomology in degrees greater than m if and only if $J^\bullet_\mu$ has zero cohomology in degrees greater than m for all prime ideals μ. Indeed, the only if part follows from the exactness of localization. To prove the opposite direction assume that $J^\bullet$ is not acyclic. Let $[\alpha] \in H^n(J^\bullet)$ be a non-zero cycle. Since $J^\bullet$ is a complex of A-modules, $H^n(J^\bullet)$ also has a structure of an A-module. Let μ be a prime ideal in A containing $\mathrm{Ann}_A[\alpha]$. Then $[\alpha]_\mu = [\alpha_\mu]$ is a non-zero cycle in $H^n(J^\bullet)_\mu = H^n(J^\bullet_\mu)$ or, equivalently, $H^n(J^\bullet_\mu) \neq 0$.

In view of the preceding remark it suffices to prove the assertion of the lemma for local rings.

Let $d = 1$.
In this case A is a discrete valuation ring. Denote by π a generator of the maximal ideal of A, and by K the fraction field of A. Consider the short exact sequence

$$0 \to A \to A \to A/\pi A \to 0$$

and tensor it with $J^\bullet$ over A. Since $J^\bullet$ is flat we get an exact sequence of cochain complexes:

$$0 \to J^\bullet \to J^\bullet \to J^\bullet/\pi J^\bullet \to 0$$

and, therefore, a long exact sequence in cohomology:

$$\cdots \to H^{n-1}(J^\bullet/\pi J^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet/\pi J^\bullet) \to \dots.$$

Note that $J^\bullet/\pi J^\bullet = J^\bullet \otimes_A A/\pi A$ is acyclic in degrees higher than 0 by the assumption of the lemma. Therefore, multiplication by π induces an isomorphism on $H^n(J^\bullet)$ for all $n > 1$, which implies that the action of A on $H^n(J^\bullet)$ extends to an action of $K = Frac(A)$. Thus, $H^n(J^\bullet) = H^n(J^\bullet) \otimes_A K = H^n(J^\bullet \otimes_A K) = 0$.

$d - 1 \Rightarrow d$

Denote by $\mathcal{M}$ the maximal ideal of A. Let $t \in \mathcal{M}$ but $t \notin \mathcal{M}^2$. To apply the induction hypothesis to $J^\bullet/tJ^\bullet$ as a module over A/tA we have to check:

(i) *$J^\bullet/tJ^\bullet$ is flat.*

This holds since tensoring preserves flatness (cf. [13, 6.6a]).

(ii) *"local acyclicity".*

Let $\mu \in$ Spec (A/tA). Denote by π^* the map induced on spectra $\operatorname{Spec} A/tA \to \operatorname{Spec} A$ and let $\nu = \pi^*(\mu)$. We have

$$J^\bullet/tJ^\bullet \otimes_{A/tA} k(\mu) = J^\bullet \otimes_A A/tA \otimes_{A/tA} k(\mu) = J^\bullet \otimes_A k(\nu)$$

which implies that $J^\bullet/tJ^\bullet$ is acyclic in positive degrees.

(iii) $\dim A/tA \leq \dim A - 1$.

This allows us to conclude that $H^n(J^\bullet/tJ^\bullet) = 0$ for $n > d-1$. Combining this observation with a long exact sequence in cohomology:

$$\cdots \to H^{n-1}(J^\bullet/tJ^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet/tJ^\bullet) \to \dots,$$

we get that multiplication by t induces an isomorphism on $H^n(J^\bullet)$ for $n > d$.

Let $S = \{t \in A :$ multiplication by t induces an isomorphism on $H^n(J^\bullet)$ for $n > d\}$. Then S is a multiplicative system in A which contains $\mathcal{M}\backslash\mathcal{M}^2$. Therefore, $\dim S^{-1}A < \dim A$ and we can apply induction hypothesis to $S^{-1}A$.

Let $[a] \in H^n(J^\bullet)$, $n > d$. $S^{-1}[a] \in S^{-1}H^n(J^\bullet) = H^n(S^{-1}J^\bullet) = 0$. So there exists $t \in S$ such that $t[a] = 0$. Since multiplication by any element in S induces an isomorphism on cohomology we conclude that $[a] = 0$ and, therefore, $H^n(J^\bullet) = 0$ for $n > d$ □

Proposition 1.4. *Let M be a locally projective G_r-module. Then $Ind_{G_r}^G(M)$ is locally projective as a G_r-module.*

Proof. We shall follow closely the proof of Theorem 4.1 of [28].

Let $H \otimes K \to G_r \otimes K$ be any one-parameter subgroup. We need to show that $\operatorname{Ind}_{G_r}^G(M) \otimes K$ restricted to $H \otimes K$ is projective. By extending scalars from k to K and by taking further the image of H in G we can assume that H is a k-subgroup scheme of G.

All invariants throughout the proof will be taken with respect to the action via the left regular representation of various subgroup schemes of G on $k[G]$ unless specified otherwise. To distinguish between right and left regular representations we shall use subscripts "l" or "r". Normality of G_r in G implies that $k[G]_r^{G_r} = k[G]_l^{G_r}$, so in this particular case we will just write $k[G]^{G_r}$.

Let $M \to I^\bullet$ be the standard G_r-injective resolution of M: $I^m = M \otimes k[G_r]^{\otimes m+1}$, where G_r acts on I^m via the right regular representation on the last tensor factor. Then $\mathrm{Ind}_{G_r}^G(M) \to \mathrm{Ind}_{G_r}^G(I^\bullet)$ is an injective resolution of $\mathrm{Ind}_{G_r}^G(M)$ as an H-module. ($\mathrm{Ind}_{G_r}^G$ is exact since G_r is a finite group scheme (cf. [20, I.5.13b)]) and Res_H^G takes injectives to injectives because any injective G-module is a direct summand of $k[G]\otimes$ <trivial G-module> and injectivity of $k[G]\downarrow_H$ itself is equivalent to the exactness of Ind_H^G (cf. [20, I.4.12]).)

If we set $J^\bullet = (\mathrm{Ind}_{G_r}^G(I^\bullet))^H$, then $H^*(J^\bullet) = H^*(H, \mathrm{Ind}_{G_r}^G(M))$. Note that $J^\bullet$ has a natural structure of a complex of $k[G/G_r]$-modules. Indeed, for any map $M_1 \otimes M_2 \to M_3$ of G_r-modules, we get a G-module map $\mathrm{Ind}_{G_r}^G(M_1) \otimes \mathrm{Ind}_{G_r}^G(M_2) \to \mathrm{Ind}_{G_r}^G(M_3)$. By taking $M_1 = k$ and $M_2 = M_3 = I^n$, we get a natural structure of an $\mathrm{Ind}_{G_r}^G k = k[G/G_r]$-module on $\mathrm{Ind}_{G_r}^G I^n$ compatible with the action of G. Since $k[G/G_r] \cong k[G]^{G_r}$ is H-invariant, $J^\bullet$ is a $k[G/G_r]$-subcomplex of $\mathrm{Ind}_{G_r}^G(I^\bullet)$.

We point out next that all J^n are flat $k[G/G_r]$-modules. Indeed,

$$J^n = (\mathrm{Ind}_{G_r}^G(I^n))^H = \mathrm{Ind}_{G_r}^G(Q \otimes k[G_r]))^H = Q \otimes (\mathrm{Ind}_{G_r}^G(k[G_r]))^H \cong Q \otimes k[G]_l^H$$

where $Q = M \otimes k[G_r]^{\otimes n}$ is a vector space with trivial G_r-action. We have an extension of rings $k[G/G_r] \cong k[G_r\backslash G] \to k[H\backslash G] \to k[G]$ where the composition and the second extension are faithfully flat since they correspond to a quotient by a finite group scheme acting freely (cf. [20, I.5.7]). Consequently, the first extension $k[G]_l^{G_r} \cong k[G_r\backslash G] \to k[H\backslash G] \cong k[G]_l^H$ is flat, which implies that $J^n = Q \otimes k[G]^H$ is flat over $k[G]^{G_r}$.

For any point $g \in G$ we are going to establish the following isomorphism:

$$J^\bullet \otimes_{k[G]^{G_r}} k(g) \cong (I^\bullet \otimes k(g))^{g^{-1}(H \otimes k(g))g}. \qquad (*)$$

First note that there is a natural isomorphism $(\mathrm{Ind}_{G_r\otimes k(g)}^{G\otimes k(g)}(N \otimes k(g)))^{H\otimes k(g)} \cong (\mathrm{Ind}_{G_r}^G(N))^H \otimes k(g)$. Furthermore, $J^\bullet \otimes k(g) \otimes_{k(g)[G/G_r]} k(g) \cong J^\bullet \otimes_{k[G/G_r]} k(g)$. Thus, it suffices to prove $(*)$ for a k-rational point g and then proceed by extension of scalars.

For a k-rational point $g \in G$ denote by $\overline{g}$ its image under the projection $G \to G/G_r$. For any G_r-module N we have a natural homomorphism

$$\epsilon_g : \mathrm{Ind}_{G_r}^G(N) \to N$$

given by evaluation at g, i.e. $\epsilon_g(n \otimes f) = f(g)n$. The restriction of ϵ_g to $(\mathrm{Ind}_{G_r}^G(N))^H$ lands in $N^{g^{-1}Hg}$. As it was noted above, $(\mathrm{Ind}_{G_r}^G(N))^H$ has a natural structure of a $k[G]^{G_r}$- module. If we make N into a $k[G]^{G_r}$-module via evaluation at $\overline{g}$, then ϵ_g becomes a homomorphism of $k[G]^{G_r}$-modules. Tensoring the left hand side with k over $k[G]^{G_r}$, we get a natural map of k-vector spaces:

$$\epsilon_g : (\mathrm{Ind}_{G_r}^G(N))^H \otimes_{k[G]^{G_r}} k \to N^{g^{-1}Hg}.$$

When $N = k[G_r]$ this is an isomorphism as one sees from the following Cartesian square:

$$\begin{tikzcd}
g^{-1}Hg\backslash G_r \arrow[r] \arrow[d, "x \to gx"'] & Spec\, k \arrow[d, "\overline{g}"'] \\
H\backslash G \arrow[r] & G_r\backslash G
\end{tikzcd}$$

Hence, ϵ_g is an isomorphism for any injective G_r-module. This implies the isomorphism of complexes $(*)$.

Computing cohomology of both sides of $(*)$ we get that $H^*(J^\bullet \otimes_{k[G]^{G_r}} k(g)) = H^*(g^{-1}(H \otimes k(g))g, M \otimes k(g))$ and the latter is trivial for $* > 0$, since $g^{-1}(H \otimes k(g))g$ is again a one-parameter subgroup of $G_r \otimes k(g)$ and M is locally projective. We conclude that $J^\bullet \otimes_{k[G]^{G_r}} k(g)$ is acyclic in positive degrees for any point $g \in G$.

We have $J^\bullet \otimes_{k[G]^{G_r}} k(g) = J^\bullet \otimes_{k[G]^{G_r}} k(\overline{g}) \otimes_{k(\overline{g})} k(g)$ and the extension of scalars $k(\overline{g}) \to k(g)$ gives an injective map on cohomology. Therefore, $J^\bullet \otimes_{k[G]^{G_r}} k(\overline{g})$ is also acyclic in positive degrees. Since the projection $G \to G/G_r$ is a bijection on points, we get that for any point $x \in G/G_r$ the complex $J^\bullet \otimes_{k[G]^{G_r}} k(x)$ is acyclic in positive degrees. Since G_r is a closed normal subgroup of G, G/G_r is a smooth affine scheme and hence $k[G/G_r]$ is a regular ring. Lemma 1.3 now implies that $J^\bullet$ is acyclic in all sufficiently large degrees. Hence, $H^*(H, \mathrm{Ind}_{G_r}^G(M)) = 0$ in all sufficiently large degrees. Since k is the only simple H-module, we get that $\mathrm{Ind}_{G_r}^G(M)$ is injective by applying Lemma 1.2. □

To prove Theorem 1.6 we are going to exploit one more construction introduced in [28, §3] which we briefly discuss below.

Let H be an affine k-group scheme, H' be a closed subgroup scheme, and X be the quotient scheme H/H' with the quotient map $p : H \to X$. There is an equivalence of categories between the category of quasi-coherent sheaves $\mathcal{M}$ on X and the category of rational H'-modules M provided with the structure of a left $k[H]$-module such that the multiplication $k[H] \otimes M \to M$ is a homomorphism of rational H'-modules, where H' acts on $k[H]$ via the right regular representation, given by the functor $\mathcal{M} \mapsto \Gamma(H, p^*\mathcal{M})$. Moreover, the sheaf cohomology $H^*_{Zar}(X, \mathcal{M})$ is naturally isomorphic to the rational cohomology $H^*(H', \Gamma(H, p^*\mathcal{M}))$.

Let $G' = G/G_r$ and $B' = B/B_r$. Since B' is a Borel subgroup of G', G'/B' is a projective variety. We are going to show that cohomology groups $H^n(B_r, \mathrm{Ind}_{G_r}^G(M))$ belong to the aforementioned category of rational B'-modules with the compatible structure of a left $k[G']$-module. Once this is done we can associate to $H^n(B_r, \mathrm{Ind}_{G_r}^G(M))$ a quasi-coherent sheaf on X, denoted $\mathcal{H}^q(\mathcal{B}_r, M)$, with the property

$$H^p(B/B_r, H^q(B_r, \mathrm{Ind}_{G_r}^G(M))) \cong H^p(X, \mathcal{H}^q(\mathcal{B}_r, M)). \qquad (**)$$

Lemma 1.5. *For any G_r-module M and any $n \geq 0$, the cohomology group $H^n(B_r, Ind_{G_r}^G(M))$ has the natural structures of a rational B/B_r-module and a left $k[G/G_r]$-module such that the action of $k[G/G_r]$ on M is a B/B_r-homomorphism.*

Proof. Let $M \to I^\bullet$ be the standard G_r-injective resolution of M. The cohomology groups $H^n(B_r, \mathrm{Ind}_{G_r}^G(M))$ can be computed via the complex $J^\bullet = (\mathrm{Ind}_{G_r}^G(I^\bullet))^{B_r}$, which has the natural structures of B/B_r and $k[G/G_r]$-modules. The action of $k[G/G_r]$ is given explicitly via

$$k[G/G_r] \otimes (\mathrm{Ind}_{G_r}^G(I^\bullet))^{B_r} \to (\mathrm{Ind}_{G_r}^G(I^\bullet))^{B_r}$$
$$\phi \otimes (f \otimes s) \longrightarrow \phi f \otimes s$$

which one easily checks to be a homomorphism of B/B_r-modules, where B/B_r acts on $k[G/G_r]$ via the *left* regular representation (since this is how the standard G-action on $\mathrm{Ind}_{G_r}^G(N) = (k[G] \otimes N)^{G_r}$ is defined).

To get the compatibility with B/B_r acting on $k[G/G_r]$ via the *right* regular representation we have to change the structure of $k[G/G_r]$ on $J^\bullet$ via the automorphism of G/G_r: $G/G_r \xrightarrow{\sigma} G/G_r$, $\sigma(x) = x^{-1}$. □

Theorem 1.6. *Let G_r be an infinitesimal k-group scheme of height r, which is a closed normal subgroup scheme of a smooth affine group scheme G. Let M be a G_r-module such that for any field extension K/k and any non-trivial one-parameter subgroup $\mathbb{G}_{a(r)} \otimes K \to G_r \otimes K$ the restriction of $M \otimes K$ to $\mathbb{G}_{a(r)} \otimes K$ is projective. Then M is projective as a G_r-module.*

Proof. Let X be the quotient scheme $(G/G_r)/(B/B_r)$. Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(B/B_r, H^q(B_r, \mathrm{Ind}_{G_r}^G(M))) \Longrightarrow H^{p+q}(B, \mathrm{Ind}_{G_r}^G(M)).$$

By Theorem 3.6 in [28], which is an extension to not necessarily reductive algebraic groups of a fundamental theorem of [11],

$$H^n(B, \mathrm{Ind}_{G_r}^G(M)) \cong H^n(G, \mathrm{Ind}_{G_r}^G(M)),$$

and by Shapiro's lemma

$$H^n(G, \mathrm{Ind}_{G_r}^G(M)) \cong H^n(G_r, M).$$

Since M is locally projective, Proposition 1.4 implies that $\mathrm{Ind}_{G_r}^G(M)$ is also locally projective as a G_r-module and thus as a U_r-module. Now, by a theorem of C. Bendel ([4]), which applies to unipotent infinitesimal group schemes, $\mathrm{Ind}_{G_r}^G(M)$ is projective as a U_r-module. We have a short exact sequence of group schemes: $1 \to U_r \to B_r \to T_r \to 1$, where T_r is diagonalizable and hence cohomologically trivial. Applying the Serre spectral sequence, we get an isomorphism: $H^*(B_r, \mathrm{Ind}_{G_r}^G(M)) \cong H^*(U_r, \mathrm{Ind}_{G_r}^G(M))^{T_r}$ and the latter is 0 in positive degrees, since $\mathrm{Ind}_{G_r}^G(M)$ is a projective U_r-module. Thus, $H^q(B_r, \mathrm{Ind}_{G_r}^G(M)) = 0$ for $q > 0$, so that the Hochschild-Serre spectral sequence above collapses and we get an isomorphism

$$H^p(B/B_r, H^0(B_r, \mathrm{Ind}_{G_r}^G(M))) \cong H^p(G_r, M).$$

Combining this with the isomorphism $(**)$ one gets:

$$H^p(X^{(r)}, \mathcal{H}^0(\mathcal{B}_r, M)) \cong H^p(G_r, M).$$

Let $x = \dim X$. Since X is a projective variety, its cohomology groups with coefficients in any quasi-coherent sheaf are trivial in degrees higher than x (cf. [18, III.2.7]). Thus, $H^p(G_r, M) = 0$ for $p > x$. Applying the same argument to $M \otimes N^\#$, we get $\mathrm{Ext}^p_{G_r}(N, M) = 0$ for all $p > x$ and all finite-dimensional modules N. By Lemma 1.2, M is projective.

□

By applying the preceding theorem to the special case of a Frobenius kernel, we get the following result:

Corollary 1.7. *Let $G_{(r)}$ be the r-th Frobenius kernel of a smooth algebraic group G and let M be a $G_{(r)}$-module such that for any field extension K/k and any non-trivial one-parameter subgroup $\mathbb{G}_{a(r)} \otimes K \to G_{(r)} \otimes K$ the restriction of $M \otimes K$ to $\mathbb{G}_{a(r)} \otimes K$ is projective. Then M is projective as a $G_{(r)}$-module.*

Remark 1.8. Let G be a semi-simple simply connected algebraic group. Assume all the hypotheses of Corollary 1.7 and also assume that the $G_{(r)}$-structure on M comes from a structure of a rational G-module. In this case we do not need to consider induced modules and can significantly simplify the proof of our local criterion for projectivity. Indeed, for a rational G-module M, we have the following spectral sequence ([2]):

$$H^p(G^{(r)}/B^{(r)}, \mathcal{L}(H^q(B_{(r)}, M))) \Longrightarrow H^{p+q}(G_{(r)}, M),$$

where $\mathcal{L}(H^q(B_{(r)}, M))$ is the sheaf on $G^{(r)}/B^{(r)}$ associated to $H^q(B_r, M)$ considered as a $B^{(r)}$-module (cf. [20, I.5]).

Local projectivity of M implies that M is a projective $B_{(r)}$-module which makes the spectral sequence collapse. Thus, we get an isomorphism:

$$H^p(G^{(r)}/B^{(r)}, \mathcal{L}(H^0(B_{(r)}, M))) \cong H^p(G_{(r)}, M).$$

Since $G^{(r)}/B^{(r)}$ is a projective variety, $H^p(G^{(r)}/B^{(r)}, \mathcal{L}(H^0(B_{(r)}, M))) = 0$ for $p > \dim G^{(r)}/B^{(r)}$. Thus, $H^p(G_{(r)}, M) = 0$ for $p > \dim G^{(r)}/B^{(r)}$. Applying the same argument to $M \otimes S^{\#}$, we get that

$$\mathrm{Ext}^p_{G_{(r)}}(S, M) = 0$$

for any simple G-module and any $p > \dim G^{(r)}/B^{(r)}$. Due to the assumptions made on G we know that all simple $G_{(r)}$-modules come from restricting simple G-modules corresponding to restricted dominant weights (cf. [20, II.3]). Thus, we have vanishing of Ext-groups in all sufficiently large degrees for all simple $G_{(r)}$-modules. By Lemma 1.2, M is projective as a $G_{(r)}$-module.

2. Support cones for Frobenius kernels

In this section G will denote an arbitrary infinitesimal k-group scheme of height r unless specified otherwise. We shall start with a brief summary of a few results about support varieties for finite dimensional modules and establishing notation which will be used through the rest of the paper.

In what follows p will be assumed to be greater than 2 to simplify notation although everything still holds for $p = 2$ if we change $H^{ev}(G, k)$ to $H^*(G, k)$.

We shall denote by $|G|$ the cohomological support scheme of G, $\mathrm{Spec}\, H^{ev}(G, k)$. For a finite dimensional G-module M the cohomological support variety of M (denoted $|G|_M$) is the Zariski closed subset of $\mathrm{Spec}\, H^{ev}(G, k)$ defined by the ideal $\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}^*_G(M, M))$. In [27],[28] these cohomological objects were given a representation-theoretic interpretation which is analogous to Quillen's stratification theorem and Carslon's rank varieties for finite groups. Namely, consider the functor

$$V(G) : (\text{comm } k\text{-alg}) \to (\text{sets})$$

defined by setting

$$V(G)(A) = \mathrm{Hom}_{Gr/A}(\mathbb{G}_{a(r)} \otimes_k A, G \otimes_k A).$$

This functor is representable by an affine scheme of finite type over k, which we will still denote $V(G)$. $V(G)$ is a cone or, which amounts to the same thing, the coordinate algebra $k[V(G)]$ is graded connected. We shall specify further the correspondence between one-parameter subgroups of G (i.e. group scheme homomorphisms $\mathbb{G}_{a(r)} \otimes K \to G \otimes K$) and points of $V(G)$.

Let $s \in V_r(G)$ be a point. This point defines a canonical $k(s)$-point of $V(G)$ and hence an associated group scheme homomorphism over $k(s)$:

$$\nu_s : \mathbb{G}_{a(r)} \otimes_k k(s) \to G \otimes_k k(s).$$

Note that if K/k is a field extension and $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ is a group scheme homomorphism, then this data defines a point $s \in V_r(G)$ and a field embedding $k(s) \hookrightarrow K$ such that ν is obtained from ν_s by extending scalars from $k(s)$ to K.

There is a natural homomorphism of graded commutative k-algebras

$$\psi : H^{ev}(G, k) \to k[V(G)]$$

which induces a homeomorphism of schemes

$$\Psi : V(G) \to |G|$$

([27, 1.14]; [28, 5.2]). Furthermore, restricted to $V(G)_M$, the "representation-theoretic" support variety of a *finite* dimensional G-module M, defined as in 2.1 below, Ψ is a homeomorphism onto $|G|_M$ ([28, 6.8]).

Looking for a good definition of a "support" for an infinite dimensional module it seems natural to establish the following criteria:

1. Restricted to the finite dimensional case our new construction should give the standard support variety for finite dimensional modules.

2. Standard properties of support varieties for finite dimensional modules should remain valid as properties of "supports" for all G-modules.

The natural extension of the cohomological definition of support variety does not satisfy the "tensor product property" for infinite dimensional modules. We will give an example of this failure as we look at Rickard idempotent modules in the next section. On the other hand, our extension of the representation-theoretic construction is not necessarily a closed subset of $V(G)$. This particular feature, though, shows that, extended to infinite dimensional modules, $V(G)_M$ gives a "finer" invariant than $|G|_M$. As it will be shown in the next section any conical subset of $V(G)$ can be realized as $V(G)_M$ for some G-module M.

Getting more sets as support cones also emphasizes the difference between finite and infinite dimensional case. The category of all modules is "richer" with respect to this invariant than the category of finite dimensional modules.

For these reasons we choose as our definition of "support" of an arbitrary G-module module M the representation-theoretic construction appearing below.

Let $v_0, \dots v_{p^r-1}$ be the basis of $k[\mathbb{G}_{a(r)}]^\# = (k[T]/T^{p^r})^\#$ dual to the standard basis of $k[T]/T^{p^r}$. Denote v_{p^i} by u_i. Then the algebra $k[\mathbb{G}_{a(r)}]^\#$ coincides with $k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$.

Definition 2.1. Let G be an infinitesimal k-group scheme of height r and let M be a rational G-module. The *support cone* of M is the following subset of $V(G)$:

$V(G)_M = \{s \in V(G) : M \otimes_k k(s)$ is not projective as a module for the subalgebra

$$k(s)[u_{r-1}]/(u_{r-1}^p) \subset k(s)[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) = k(s)[\mathbb{G}_{a(r)}]^\#\}.$$

We remark that by a "subset" of an affine scheme $X = \operatorname{Spec} A$ we would mean simply a set of prime ideals in A. We shall often use the same notation for a point in X and the corresponding prime ideal in A.

Let E_r be an elementary abelian p-group of rank r (i.e. $E_r = (\mathbb{Z}/p)^r$). If we view E_r as a commutative Lie algebra with trivial restriction, then its representation theory is equivalent to the representation theory of the infinitesimal group scheme $\mathbb{G}_{a(1)}^{\times r}$. ¿From this point of view our definition of support cone in the special case of $\mathbb{G}_{a(1)}^{\times r}$ agrees with the extension to infinite dimensional E_r-modules of the notion of rank variety given in [7].

We begin the study of support cones with a reformulation of the Theorem 5.2 in [7] from groups to algebras which enables us to apply the theorem to the representations of $\mathbb{G}_{a(r)}$, thanks to the fact that $kE_r \cong k[\mathbb{G}_{a(r)}]^{\#}$. This theorem is a generalization of Dade's lemma ([12]) for finite dimensional modules of elementary abelian p-groups to the infinite dimensional case.

Theorem 2.2. *Let $A = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$ and M be an A-module. M is projective if and only if for any field extension K/k and any element $z = c_0u_0 + \cdots + c_{r-1}u_{r-1}$, where $c_0, \ldots, c_{r-1} \in K$, the restriction of $M \otimes K$ to $K[z]/(z^p)$ is projective.*

The necessity of considering field extensions is what makes the statement of the theorem different from the classical Dade's lemma and is essential for infinite dimensional modules. It is possible to construct a $\mathbb{G}_{a(r)}$-module M (as we shall see in the next section) whose support cone $V(\mathbb{G}_{a(r)})_M$ is non-zero but does not have any k-rational points other than 0. For these reasons we work with the scheme $V(G)$ (i.e. prime ideal spectrum of $k[V(G)]$), as opposed to the variety of k-rational points (i.e. maximal ideal spectrum), which is sufficient in the finite dimensional case.

Note that any group scheme homomorphism $\mathbb{G}_{a(s)} \to G$, $s \leq r$, can be extended canonically to a one-parameter subgroup of height r, $\mathbb{G}_{a(r)} \to G$, via the projection $p_{r,s} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(s)}$ given by the natural embedding of coordinate algebras

$$k[\mathbb{G}_{a(s)}] = k[T_1]/(T_1^{p^s}) \xrightarrow{T_1 \to T^{p^{r-s}}} k[T]/(T^{p^r}) = k[\mathbb{G}_{a(r)}].$$

Conversely, any one-parameter subgroup $\mathbb{G}_{a(r)} \to G$ can be decomposed as

$$\mathbb{G}_{a(r)} \xrightarrow{p_{r,s}} \mathbb{G}_{a(s)} \hookrightarrow G$$

for some $s \leq r$.

Corollary 2.3. *Let M be a $\mathbb{G}_{a(r)}$-module. M is projective if and only if $V(\mathbb{G}_{a(r)})_M = 0$.*

Proof. The category of $\mathbb{G}_{a(r)}$-modules is equivalent to the category of $k[\mathbb{G}_{a(r)}]^{\#} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$-modules. To apply Theorem 2.2 we have to show that $V(\mathbb{G}_{a(r)})_M = 0$ is equivalent to the assumption of the theorem. Let $z = c_0u_0 + \cdots + c_{r-1}u_{r-1}$, where $c_0, \ldots, c_{r-1} \in K$, K is an extension of k, which we assume to be perfect (we can always extend scalars further). Consider an endomorphism α of $\mathbb{G}_{a(r)} \otimes K$ defined on the level of coordinate algebras via the formula:

$$\begin{tikzcd}
K[\mathbb{G}_{a(r)}] \arrow[r] \arrow[d, equal] & K[\mathbb{G}_{a(r)}] \arrow[d, equal] \\
K[T]/(T^{p^r}) \arrow[r, "T \to \sum_0^{r-1} c_i^{p^{-i}} T^{p^{r-1-i}}"] & K[T]/(T^{p^r})
\end{tikzcd}$$

Dual to this map is an endomorphism of $K[\mathbb{G}_{a(r)}]^\# = K[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$, which takes u_{r-1} to $c_0u_0 + \cdots + c_{r-1}u_{r-1}$. By definition of $V(\mathbb{G}_{a(r)})$, α corresponds to a point there defined over K.

Since $V(\mathbb{G}_{a(r)})_M$ is assumed to be 0, the restriction of $M \otimes K$ to $K[u_{r-1}]/(u_{r-1}^p) \subset K[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) = K[\mathbb{G}_{a(r)}]^\#$ is projective, where $M \otimes K$ is considered as a $K[\mathbb{G}_{a(r)}]^\#$-module via the pull-back of α. By the construction of α this is equivalent to $M \otimes K$ being projective when restricted to the subalgebra of $K[\mathbb{G}_{a(r)}]^\# = K[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$ generated by $z = c_0u_0 + \cdots + c_{r-1}u_{r-1}$. Thus we proved that for any z as above $M \otimes K$ is projective when restricted to $K[z]/(z^p) \subset K[\mathbb{G}_{a(r)}]^\#$. Now we can apply Theorem 2.2 to conclude that M is projective.

To prove the "only if" part it suffices to show that for any one-parameter subgroup $\mathbb{G}_{a(r)} \xrightarrow{\alpha} \mathbb{G}_{a(r)}$, $k[\mathbb{G}_{a(r)}]^\#$ is projective over $k[u_{r-1}]/(u_{r-1}^p)$, where the module structure on $k[\mathbb{G}_{a(r)}]^\#$ is given via the composition $k[u_{r-1}]/(u_{r-1}^p) \subset k[\mathbb{G}_{a(r)}]^\# \xrightarrow{\alpha_*} k[\mathbb{G}_{a(r)}]^\#$. Decompose α as $\mathbb{G}_{a(r)} \xrightarrow{p_{r,s}} \mathbb{G}_{a(s)} \hookrightarrow \mathbb{G}_{a(r)}$. Since $\mathbb{G}_{a(s)}$ is a finite group scheme, $k[\mathbb{G}_{a(r)}]^\#$ is injective (and hence projective) as a $\mathbb{G}_{a(s)}$-module (cf. [20, I.5.13b)]). The composition $k[u_{r-1}]/(u_{r-1}^p) \subset k[\mathbb{G}_{a(r)}]^\# \xrightarrow{(p_{r,s})_*} k[\mathbb{G}_{a(s)}]^\# = k[u_0, \dots, u_{s-1}]/(u_0^p, \dots, u_{s-1}^p)$ takes u_{r-1} to u_{s-1}, which clearly implies that $k[\mathbb{G}_{a(s)}]^\#$ (and, therefore, $k[\mathbb{G}_{a(r)}]^\#$) is free as a $k[u_{r-1}]/(u_{r-1}^p)$-module.

□

We shall call an affine k-scheme $X = \operatorname{Spec} A$ *conical* if A is a graded connected k-algebra. The data of a (non-negative) grading on A is equivalent to a right monoid action of $\mathbb{A}^1$ on X, where the monoid structure on $\mathbb{A}^1$ is just the usual multiplication. (The correspondence is given in the following way: the canonical k-algebra homomorphism $A \to A[T]$ defined by the grading on A induces a morphism of schemes $X \times \mathbb{A}^1 \to X$ which defines a monoid action of $\mathbb{A}^1$. Conversely, given an action we get a homomorphism $A \to A[T]$ which defines a non-negative grading on A).

Definition 2.4. (conical subset) Let $X = \operatorname{Spec} A$ be a conical affine scheme, where the conical structure is given by the map $\rho : X \times \mathbb{A}^1 \to X$. Denote by $\pi_X : X \times \mathbb{A}^1 \to X$ the canonical projection onto X. A subset W of X is said to be *conical* if it is stable under the action of $\mathbb{A}^1$ on X and if for any point $s \in X$ we have $\pi_X(\rho^{-1}(s)) \subset X$.

Note that if W is a closed subset, then it is conical if and only if it is defined by a graded ideal, or, equivalently, if it corresponds to a homogeneous subvariety. In fact, in this familiar case or even in the more general case of a subset closed under specialization, the second condition is redundant and implied by the first.

Next we give an example of a conical set which we find to be more illuminating. Since A is connected we can give a precise meaning to the 0-point: this is the point corresponding to the augmentation ideal in A and it belongs to any conical subset.

Example 2.5. Let $s \in X$ be a point corresponding to a graded prime ideal $\mu_s \subset A$. Denote $\pi_X(\rho^{-1}(s)) \subset X$ by $L(s)$. Then $L(s) \cup 0$ is the minimal conical subset containing s: by our definition of "conical", $s \in W$ implies $L(s) \subset W$ for any conical subset W. We give a description of $L(s)$ in terms of prime ideals:

$$L(s) = \{\mu \in \operatorname{Spec} A : \ \mu \text{ is not homogeneous}, \ \mu_s \subset \mu \text{ and } \operatorname{ht}(\mu) = \operatorname{ht}(\mu_s) + 1\} \cup \{s\}$$

To justify this claim we make three simple observations. Denote the action of $\mathbb{A}^1$ on X by $\bullet$.

First, the action of $\mathbb{A}^1$ cannot increase the height of the ideal and can lower it at most by one.

Second, since any set of the form $\{\text{homogeneous ideal}\} \cup 0$ is stable under the action, $L(s)$ does not contain any homogeneous ideals other than μ_s.

Third, let p be any point in X, c be the generic point of $\mathbb{A}^1$, and s be the point corresponding to the maximal homogeneous ideal contained in the ideal μ_p. Assume also that μ_s is strictly contained in μ_p (i.e. μ_p is not homogeneous), in which case $\mathrm{ht}(\mu_s) = \mathrm{ht}(\mu_p) - 1$. Then $p \bullet c = s$ which implies that $p \in L(s)$.

To see that $p \bullet c = s$ we note that if μ_p is the kernel of the map $A \to k(p)$, then the kernel of the induced map

$$A \xrightarrow{\sum_0^n a_i \to \sum_0^n a_i T^i} A[T] \longrightarrow k(p)(T)$$

is the maximal homogeneous ideal contained in μ_p, i.e. μ_s.

Next we describe how to give an action of $\mathbb{A}^1$ on $V(G)$ and, therefore, define a grading on $k[V(G)]$. All proofs can be found in [27].

We have a natural morphism of schemes defined by taking composition of morphisms

$$V(G) \times V(\mathbb{G}_{a(r)}) \to V(G).$$

Taking G to be $\mathbb{G}_{a(r)}$ we see that $V(\mathbb{G}_{a(r)})$ has a natural structure of a monoid scheme over k. Restricting the action to a submonoid of $V(\mathbb{G}_{a(r)})$ consisting of homomorphisms of $\mathbb{G}_{a(r)}$ given by linear maps of coordinate algebras, we get a right monoid action of $\mathbb{A}^1$ on $V(G)$, which, consequently, defines a grading on $k[V(G)]$. Moreover, $k[V(G)]$ becomes a graded *connected* k-algebra with respect to this grading which makes $V(G)$ into a conical k-scheme.

The following theorem establishes the list of properties satisfied by support cones. The most difficult one is 2.6.3, the detection of projectivity "on" support cones, which follows from the local projectivity detection theorem of section 1 and Corollary 2.3.

Theorem 2.6. *Let G be an infinitesimal k-group scheme of height r which is a closed normal subgroup of a smooth algebraic group and let M and N be G-modules. Support cones satisfy the following properties:*

0. For a finite dimensional module M, $V(G)_M \cong |G|_M$.

1. $V(G)_M$ is a conical subset of $V(G)$.

2. "Naturality." Let $f : H \to G$ be a homomorphism of infinitesimal group schemes of height $\leq r$. Denote by $f_* : V(H) \to V(G)$ the associated morphism of schemes. Then

$$f_*^{-1}(V(G)_M) = V(H)_M,$$

where M is considered as an H-module via f.

3. $V(G)_M = 0$ if and only if M is projective.

4. "Tensor product property." $V(G)_{(M \otimes N)} = V(G)_M \cap V(G)_N$.

5. $V(G)_{(M \bigoplus N)} = V(G)_M \cup V(G)_N$.

6. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of* G*-modules. Then for any permutation* (ijk) *of* (123) *we have*

$$V(G)_{M_i} \subset V(G)_{M_j} \cup V(G)_{M_k}.$$

Proof. Note that over the algebra $K[u]/(u^p)$ projective=free which we shall use without mention throughout the argument.

0. This is proved in [28], Cor.6.8.

1. The proof for finite dimensional modules given in [28], Prop.6.1, generalizes immediately to our case but we shall include it here for the completeness of the argument. Denote the action of $\mathbb{A}^1$ on $V(G)$, $V(G) \times \mathbb{A}^1 \to V(G)$, by $\bullet$.

Let $s \in V(G)$ and let $\nu_s : \mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ be the one-parameter subgroup determined by s. By the definition of $V(G)_M$, $s \in V(G)_M$ if and only if the restriction of $M \otimes k(s)$ to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[\mathbb{G}_{a(r)}]^{\#}$ via ν_s is not projective. Let c be a point in $\mathbb{A}^1$. We can extend the scalars to a field K/k such that both s and c are defined over K. Let $\nu_{s,K} : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ be the one-parameter subgroup which is obtained from ν_s by extending scalars from $k(s)$ to K. If $c = 0$, then the corresponding one-parameter subgroup is trivial and the restriction of the pull-back of M via the trivial subgroup to $K[u_{r-1}]/u_{r-1}^p$ is never projective. So, in this case $c \bullet s \in V(G)_M$. Assume $c \neq 0$. To prove that $V(G)_M$ is conical we have to show that $s \in V(G)_M$ if and only if $c \bullet s \in V(G)_M$. Considered as a point in $V(\mathbb{G}_{a(r)})$ defined over K, c determines a group scheme homomorphism $\nu_{c,K} : \mathbb{G}_{a(r)} \otimes K \to \mathbb{G}_{a(r)} \otimes K$, given by the multiplication by c^{-1} on the coordinate algebra $K[\mathbb{G}_{a(r)}]$. By definition of the action of $\mathbb{A}^1$ on $V(G)$, the group scheme homomorphism $\nu_{c \bullet s,K} : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ is defined via the composition

$$\mathbb{G}_{a(r)} \otimes K \xrightarrow{\nu_{c,K}} \mathbb{G}_{a(r)} \otimes K \xrightarrow{\nu_{s,K}} G \otimes K.$$

The homomorphism $(\nu_{c,K})_* : K[\mathbb{G}_{a(r)}]^{\#} \to K[\mathbb{G}_{a(r)}]^{\#}$ restricted to $K[u_{r-1}]/u_{r-1}^p$ is given by

$$K[u_{r-1}]/u_{r-1}^p \xrightarrow{u_{r-1} \to c^{p^{r-1}} u_{r-1}} K[u_{r-1}]/u_{r-1}^p$$

which is clearly a ring isomorphism. Consequently, M is not projective as a module over the right hand side of the above isomorphism if and only if M is not projective when restricted to the left hand side. The statement follows.

2. Follows immediately from the definition of $V(G)_M$.

3. Note that $V(G)_M = 0$ implies $V(G \otimes K)_{M \otimes K} = 0$ for any field extension K/k. Let $\mathbb{G}_{a(r)} \otimes K \to G \otimes K$ be any non-trivial one-parameter subgroup. By naturality $V_{\mathbb{G}_{a(r)} \otimes K}(M \otimes K) = 0$, which is equivalent, in view of Cor. 2.3, to the fact that the restriction of $M \otimes K$ to $\mathbb{G}_{a(r)} \otimes K$ is projective. Applying Theorem 1.6, we conclude that M is projective.

Now suppose that M is a projective G-module. Then M is a direct summand of $k[G] \otimes <$ trivial module $>$, and the support variety of $k[G]$ is trivial, since it is injective as a G-module.

4. The inclusion $V(G)_{M \otimes N} \subset V(G)_M \cap V(G)_N$ follows from the fact that tensor product of a projective module with anything is projective. Indeed, let $s \in V(G)_{M \otimes N}$. By the definition of support cone, $M \otimes N \otimes k(s)$ is not projective when restricted to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) = k(s)[\mathbb{G}_{a(r)}]^{\#}$,

where $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ is the one-parameter subgroup of $G \otimes k(s)$ corresponding to the point $s \in V(G)$. In view of the remark above, neither $M \otimes k(s)$ nor $N \otimes k(s)$ is projective and, therefore, $s \in V(G)_M \cap V(G)_N$.

To prove the other inclusion we have to show that if both $M \otimes k(s)$ and $N \otimes k(s)$ are not free as modules over $k(s)[u_{r-1}]/(u_{r-1}^p)$, then $M \otimes N \otimes k(s)$ is not free. Denote $k(s)[u_{r-1}]/(u_{r-1}^p)$ by A. Note that

$$M \otimes N \otimes k(s) \cong (M \otimes k(s)) \otimes_{k(s)} (N \otimes k(s)).$$

Since any A-module is a direct sum of finite dimensional indecomposables (cf. [17]), we can write $M \otimes k(s) = \bigoplus_I M_i$ and $N \otimes k(s) = \bigoplus_J N_j$ for some finite dimensional A-modules M_i and N_j. Consequently,

$$(M \otimes k(s)) \otimes_{k(s)} (N \otimes k(s)) = \bigoplus_{I,J} (M_i \otimes_{k(s)} N_j).$$

If both $M \otimes k(s)$ and $N \otimes k(s)$ are not free, then there exist i and j such that M_i and N_j are not free A-modules. The tensor product of two finite dimensional A-modules is free if and only if at least one of them is free, which implies that $M_i \otimes N_j$ is not free. Since over A projective=free, we get that $M \otimes N \otimes k(s)$ has a direct summand, namely $M_i \otimes N_j$, which is not projective. Therefore, $M \otimes N \otimes k(s)$ is not free.

5. The restriction of $(M \oplus N) \otimes k(s) = (M \otimes k(s)) \oplus (N \otimes k(s))$ to $k(s)[u_{r-1}]/(u_{r-1}^p)$ is not free if and only if the restriction of either $(M \otimes k(s))$ or $(N \otimes k(s))$ is not.

6. This follows immediately from the fact that when two $k(s)[u_{r-1}]/(u_{r-1}^p)$-modules out of three in a short exact sequence are free, then the third module has to be free.

□

Unlike the situation with finite dimensional modules, the support cone $V(G)_M$ for an infinite dimensional G-module M is typically not a closed subset of $V(G)$ and thus is not homeomorphic to $V(\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}^*_G(M, M)))$. As the following proposition shows, a much weaker relationship does hold.

In what follows we identify $V(G)$ and $\mathrm{Spec}\, H^{ev}(G, k)$ via the homeomorphism Ψ mentioned in the beginning of this section.

Proposition 2.7. *Let G be an infinitesimal k-group scheme of height r satisfying the hypotheses of Theorem 1.6 and M be a G-module. Then*

$$V(G)_M \subset V(Ann_{H^{ev}(G,k)}(Ext^*_G(M, M))).$$

Proof. Let $s \in V(G)_M$. Since both $V(G)_M$ and $V(\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}^*_G(M, M)))$ are conical (the latter corresponds to an annihilator of a graded module, i.e. is defined by a graded ideal), they are completely determined by their homogeneous ideals, so we can assume that the point s corresponds to a homogeneous prime ideal. To simplify notation denote $k(s)$ by K and $M \otimes K$ by M_K. Then s corresponds to a one-parameter subgroup $\nu_s : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ such that M_K restricted to $K[u_{r-1}]/u_{r-1}^p \subset K[\mathbb{G}_{a(r)}]^\#$ via ν_s is not projective. We have an equivalence of categories between the category of H-modules and $K[H]^\#$-modules for any finite group scheme H. Hence, the composition of algebra homomorphisms $K[u_{r-1}]/u_{r-1}^p \subset K[\mathbb{G}_{a(r)}]^\# \to K[G]^\#$ induces a map on cohomology which, by some abuse of notation, we denote ν_s^*: $H^{ev}(G \otimes K, K) \to H^{ev}(\mathbb{G}_{a(1)} \otimes K, K)$.

(Note that this map does not correspond to any "real" map of group schemes, but only to a map of coalgebras on the level of coordinate algebras.)

$\mathbb{G}_{a(1)}$ has representation theory equivalent to that of $\mathbb{Z}/p$. Recall that $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K) \cong K[x]$ where x is a generator in degree 2. Note that $H^{ev}(\mathbb{G}_{a(1)} \otimes K, M_K) = \operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(K, M_K)$ is naturally a left module for the algebra $\operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K, M_K)$ via Yoneda composition. Furthermore, the action of $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K)$ on $\operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(K, M_K)$ factors through the action of $\operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K, M_K)$ via the natural map of algebras $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K) \stackrel{\otimes M_K}{\to} \operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K, M_K)$. Since x induces a "periodicity" isomorphism on $H^*(\mathbb{G}_{a(1)} \otimes K, M_K)$ (cf. [5, v.1,3.5]) and the latter is non-trivial in positive degrees due to the fact that M_K is not projective restricted to $K[u_{r-1}]/u^p_{r-1}$, we conclude that the map $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K) \to \operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K, M_K)$ is injective.

Thinking of Ext-groups in terms of extensions one sees easily that the following diagram of algebra homomorphisms is commutative:

$$\begin{tikzcd}
H^{ev}(G,k) \arrow[r, "\otimes K"] \arrow[d, "\otimes M"'] & H^{ev}(G\otimes K,K) \arrow[r, "\nu_s^*"] \arrow[d, "\otimes M_K"'] & H^{ev}(\mathbb{G}_{a(1)}\otimes K,K) \arrow[d, "\otimes M_K"'] \\
\operatorname{Ext}^*_G(M,M) \arrow[r, "\otimes K"] & \operatorname{Ext}^*_{G\otimes K}(M_K,M_K) \arrow[r] & \operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K,M_K)
\end{tikzcd}$$

where the right lower map is again restriction via ν_s. By the construction of the homeomorphism $\Psi : V(G) \to \operatorname{Spec} H^{ev}(G,k)$ (cf. [27]), the point $s \in V(G)$ corresponds to the homogeneous prime ideal $\mu_s \subset H^{ev}(G,k)$ which is the kernel of the map $H^{ev}(G,k) \to H^{ev}(\mathbb{G}_{a(1)} \otimes K, K)$ appearing as the top row of the commutative diagram above. Now, the commutativity of the diagram together with the injectivity of the right vertical arrow imply that

$$\operatorname{Ker}(H^{ev}(G,k) \to \operatorname{Ext}^*_G(M,M) \to \operatorname{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K,M_K)) = \operatorname{Ker}(H^{ev}(G,k) \to H^{ev}(\mathbb{G}_{a(1)}\otimes K,K)).$$

Since $\operatorname{Ker}(H^{ev}(G,k) \to \operatorname{Ext}^*_G(M,M)) = \operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(M,M))$ is contained in the left hand side, and the right hand side equals μ_s, we conclude that $\operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(M,M)) \subset \mu_s$.

$\square$

The following result, which is immediately implied by the proposition above and Theorem 2.6.3, will be used in the next section to show that the "tensor product property" does not hold for the extension to infinite dimensional modules of the cohomological definition of support variety.

Corollary 2.8. *Let G be an infinitesimal group satisfying the hypotheses of Theorem 1.6 and M be a G-module. If $V(Ann_{H^{ev}(G,k)}(Ext^*_G(M,M))) \subset Spec\, H^{ev}(G,k)$ is 0, then M is projective.*

3. Support cones using Rickard idempotents

In this section we shall give a different description of support cones which is a translation into our situation of the approach to the extension of support varieties to infinite dimensional modules for finite groups taken in [7].

Throughout this section G will denote an infinitesimal k-group scheme of height r which satisfies the hypothesis of Theorem 2.6. In particular, it can be any Frobenius kernel.

We shall denote by StMod(G) the stable category of all G-modules. Recall that objects of StMod(G) are G-modules and maps are equivalence classes of G-module homomorphisms where two maps are equivalent if their difference factors through a projective G-module.

StMod(G) is a triangulated category due to the fact that projectives are injectives. The shift operator in StMod(G) is given by the Heller operator Ω^{-1} : StMod(G) $\to$ StMod(G) and distinguished triangles come from short exact sequences in Mod(G). We shall denote by stmod(G) the full triangulated subcategory of StMod(G) whose objects are represented by finite dimensional modules. This subcategory is equivalent to the usual stable module category of finite dimensional G-modules (i.e. the category of finite dimensional G-modules whose maps are equivalence classes of G-homomorphisms where two maps are equivalent if their difference factors through a finite dimensional projective G-module). A full triangulated subcategory $\mathcal{C}$ of stmod(G) (respectively StMod (G)) is called *thick* if it is closed under taking direct summands (respectively taking direct summands and arbitrary direct sums). It is called *tensor-ideal* if it is closed under taking tensor products with any G-module. We shall use the notation $\underline{\mathrm{Hom}}$ for $\mathrm{Hom}_{\mathrm{StMod}}$ and "$\cong$" for stable isomorphisms.

Two modules are stably isomorphic (i.e. isomorphic in StMod(G)) if and only if they become isomorphic after adding projective summands to them. This implies that support cones are well-defined in StMod(G).

Let $\mathcal{C}$ be a thick subcategory of stmod(G). Denote by $\vec{\mathcal{C}}$ the full triangulated subcategory of StMod(G) whose objects are filtered colimits of objects in $\mathcal{C}$. ($\vec{\mathcal{C}}$ coincides with the smallest full triangulated subcategory of StMod(G) which contains $\mathcal{C}$ and is closed under taking direct summands and arbitrary direct sums (cf. [25]).)

The following is a restatement of the existence of the simplest case of Bousfield localization - "finite localization" - in our situation. The reader can find a detailed discussion of Bousfield localization for any finite dimensional cocommutative Hopf algebra in [26] or [19]. Alternatively, the proofs given in [25] apply without change to prove Theorem 3.1 and Proposition 3.3.

Theorem 3.1. *(Bousfield localization)*

I. Let $\mathcal{C}$ be a thick subcategory of stmod(G). There exist exact functors $\mathcal{E}_{\mathcal{C}}, \mathcal{F}_{\mathcal{C}}$: StMod(G) $\to$ StMod(G) characterized by the following properties:

(i) For any $M \in$ StMod(G), the modules $\mathcal{E}_{\mathcal{C}}(M)$ and $\mathcal{F}_{\mathcal{C}}(M)$ fit in a distinguished triangle:

$$\mathcal{T}_{\mathcal{C}}(M) : \mathcal{E}_{\mathcal{C}}(M) \to M \to \mathcal{F}_{\mathcal{C}}(M) \to \Omega^{-1}\mathcal{E}_{\mathcal{C}}(M).$$

(ii) $\mathcal{E}_{\mathcal{C}}(M)$ belongs to $\vec{\mathcal{C}}$ and satisfies the following universal property: the map $\epsilon_M : \mathcal{E}_{\mathcal{C}}(M) \to M$, which occurs in the distinguished triangle $\mathcal{T}_{\mathcal{C}}(M)$, is the universal map in StMod(G) from an object in $\vec{\mathcal{C}}$ to M, i.e. for any $C \in \vec{\mathcal{C}}$, ϵ_m induces an isomorphism

$$\underline{Hom}(C, \mathcal{E}_{\mathcal{C}}(M)) \simeq \underline{Hom}(C, M).$$

(iii) The map $\eta_M : M \to \mathcal{F}_{\mathcal{C}}(M)$, which occurs in the distinguished triangle $\mathcal{T}_{\mathcal{C}}(M)$, is the universal map in StMod(G) from M to a $\mathcal{C}$-local object (where N is called a $\mathcal{C}$-local object iff $\underline{Hom}(M, N) = 0$ for any $M \in \mathcal{C}$)

II. Suppose C is also tensor-ideal. Then for any G-module M we have stable isomorphisms: $\mathcal{E}_{\mathcal{C}}(M) \cong \mathcal{E}_{\mathcal{C}}(k) \otimes M$, $\mathcal{F}_{\mathcal{C}}(M) \cong \mathcal{F}_{\mathcal{C}}(k) \otimes M$.

Remark 3.2. In fact, the distinguished triangle $\mathcal{T}_{\mathcal{C}}(M)$ is uniquely determined up to a stable isomorphism by the following properties: $\mathcal{E}_{\mathcal{C}}(M) \in \vec{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}(M)$ is $\mathcal{C}$-local.

The modules $\mathcal{E}_{\mathcal{C}}(k)$ and $\mathcal{F}_{\mathcal{C}}(k)$ were introduced by J. Rickard ([25]) for finite groups and are thereby called *Rickard idempotent modules*. We justify the name in the following proposition:

Proposition 3.3. *Let $\mathcal{C}$ be a tensor-ideal thick subcategory of stmod(G). Then*

(i) there are stable isomorphisms:

$$\mathcal{E}_{\mathcal{C}}(k) \otimes \mathcal{E}_{\mathcal{C}}(k) \cong \mathcal{E}_{\mathcal{C}}(k) \text{ and } \mathcal{F}_{\mathcal{C}}(k) \otimes \mathcal{F}_{\mathcal{C}}(k) \cong \mathcal{F}_{\mathcal{C}}(k);$$

(ii) $\mathcal{E}_{\mathcal{C}}(k) \otimes \mathcal{F}_{\mathcal{C}}(k)$ is projective;

(iii) for a finite dimensional G-module M, the following are equivalent:
- $M \in \mathcal{C}$
- $M \otimes \mathcal{E}_{\mathcal{C}}(k)$ is stably isomorphic to M
- $M \otimes \mathcal{F}_{\mathcal{C}}(k)$ is projective.

Lemma 3.4. *Let W be a subset in $V(G)$ and let $\mathcal{C}_W$ be the full subcategory of stmod(G) consisting of finitely generated modules M whose variety $V(G)_M$ is contained in W. Then $\mathcal{C}_W$ is a tensor-ideal thick subcategory of stmod(G).*

The statement of the lemma follows immediately from the standard properties of support varieties and implies the existence of the Rickard idempotents associated to the subcategory $\mathcal{C}_{\mathcal{W}}$. In this special case we shall use the following notation:

$$E(W) = \mathcal{E}_{\mathcal{C}_W}(k),\ F(W) = \mathcal{F}_{\mathcal{C}_W}(k) \text{ and } T(W) = \mathcal{T}_{\mathcal{C}_W}(k)$$

Definition 3.5. Let W be a subset in an affine scheme $X = \operatorname{Spec} A$. W is said to be *closed under specialization* if for any two primes $\mu \subset \nu \subset A$, $\mu \in W$ implies $\nu \in W$.

Being closed under specialization is equivalent to the fact that for any $s \in W$ the Zariski closure of s, denoted $\overline{s}$, is contained in W. For any $U \subset X$ we denote by $\mathrm{Cs}\,(U)$ the *closure under specialization* of U, i.e.

$$\mathrm{Cs}\,(U) = \bigcup_{s \in U} \overline{s}$$

Note that closure under specialization of a conical subset is again conical.

Let V be a closed conical subset of $V(G)$. Denote by V' the subset of V consisting of all points of V except for generic points of irreducible components of V. Define

$$\kappa(V) \overset{def}{=} E(V) \otimes F(V').$$

As a tensor product of idempotent modules, $\kappa(V)$ is again idempotent, i.e. $\kappa(V) \otimes \kappa(V) \cong \kappa(V)$.

Note that the generic point of an irreducible closed conical subvariety is a homogeneous prime ideal, so that there is a natural 1-1 correspondence between homogeneous prime ideals and closed irreducible conical subvarieties. For an irreducible closed conical set V with the generic point s we shall use $\kappa(s)$ to denote $\kappa(V)$. In particular, for any point $s \in V(G)$ corresponding to a homogeneous prime ideal, $\kappa(s)$ will substitute for $\kappa(\overline{s})$ to simplify notation.

Theorem 3.6. *Let W be a conical closed under specialization subset of $V(G)$. Then*

$$V(G)_{E(W)} = W.$$

Before proving the theorem we state an immediate corollary:

Corollary 3.7. *For any conical closed under specialization subset W of $V(G)$ there exists a G-module M whose support cone coincides with W.*

The statement of the corollary is an extension of the "realization" theorem for support varieties of finite dimensional modules (see [10] for finite groups, [15] for restricted Lie algebras, [28] for arbitrary infinitesimal groups). There are many different conical closed under specialization subsets with the same closure: for example, any union of infinitely many lines through the origin in $\mathbb{A}^2$ is a conical closed under specialization non-closed subset with the closure $\mathbb{A}^2$. The theorem, thus, demonstrates that $V(G)_M$ is a "finer" invariant than one taking values in closed subsets (e.g. $V(\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}^*_G(M,M)))$).

The proof of the theorem given below is adapted to our case from the proof of the analogous result for elementary abelian groups in [7].

Proof. Let s be a point in W. Since W is conical closed under specialization, the smallest closed conical subvariety of $V(G)$ containing s is contained in W. Denote this subvariety by V_s. By the "realization" theorem for finite dimensional modules (cf. [28, 7.5]), there exists a finite dimensional G-module M such that $V(G)_M = V_s$. By the definition of $\mathcal{C}_W$, we have that $M \in \mathcal{C}_W$, which is equivalent to the fact that $M \otimes E(W) \cong M$ in $\mathrm{StMod}(G)$ (cf. Prop. 3.3). The "tensor product property" implies that $V_s = V(G)_M \subset V(G)_{E(W)}$. Since $s \in V_s$ by the construction of V_s, the inclusion $W \subset V(G)_{E(W)}$ follows.

To prove the other inclusion, choose $s \notin W$. By Theorem 3.1, $E(W) = \varinjlim_{i\in I} M_i$, where M_i are finite dimensional modules such that $V(G)_{M_i} \subset W$ for all i. Let $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ be the one-parameter subgroup corresponding to the point s. Since $s \notin V(G)_{M_i}$ for any $i \in I$, the restriction of $M_i \otimes k(s)$ to $k(s)[u_{r-1}]/(u_{r-1}^p) \subset k(s)[\mathbb{G}_{a(r)}]^{\#}$ (see §2 for notation) is always projective. Then the restriction of $E(W) \otimes k(s)$ to the same subalgebra is projective as a filtered colimit of projective modules (we also use that restriction commutes with colimits). Thus, $E(W)\downarrow_{k(s)[u_{r-1}]/(u_{r-1}^p)}$ is projective, which implies that $s \notin V(G)_{E(W)}$. The statement follows. $\square$

Recall that for a non-zero point $s \in V(G)$ corresponding to a graded prime ideal $\mu_s \in k[V(G)]$, we denote by $L(s)$ the minimal conical subset containing s with 0 removed. Alternatively, $L(s) = \{\mu \in \operatorname{Spec} k[V(G)] : \mu$ is not homogeneous, $\mu_s \subset \mu$ and $\mathrm{ht}(\mu) = \mathrm{ht}(\mu_s)+1\} \cup \{s\}$ (cf. Ex. 2.5). Let s be the generic point of a closed irreducible conical subset V of $V(G)$. Let further $V' = V \backslash \{s\}$ and let $\widetilde{V'}$ be the maximal conical closed under specialization subset in V'. It is easy to see that

$$\widetilde{V'} = V' \backslash L(s).$$

The thick subcategory $C_{V'} \subset \mathrm{stmod}(G)$, corresponding to V', coincides with the thick subcategory $C_{\widetilde{V'}}$ and, therefore, we have stable isomorphisms:

$$E(V') \cong E(\widetilde{V'}),\ F(V') \cong F(\widetilde{V'}).$$

Applying the theorem above to $E(\widetilde{V'})$, we get

$$V(G)_{E(V')} = \widetilde{V'}.$$

Thanks to our description of $\widetilde{V'}$, we can rewrite the last formula as

$$V(G)_{E(V')} = V' \backslash L(s).$$

Now we can describe support cones of F and κ - modules. We shall denote by W^c the complement of any subset W of $V(G)$.

Corollary 3.8. *For a conical closed under specialization subset W of $V(G)$ we have*

$$V(G)_{F(W)} = W^c \cup 0.$$

Furthermore, if V is an irreducible closed conical subset of $V(G)$ and s is the generic point of V, then

$$V(G)_{\kappa(V)} = L(s) \cup 0.$$

Proof. The existence of the distinguished triangle $T(W) : E(W) \to k \to F(W) \to \Omega^{-1}E(W)$ implies that

$$V(G) \subset V(G)_{E(W)} \cup V(G)_{F(W)}$$

(cf. Th. 2.6.6). Proposition 3.3.2 asserts that $E(W) \otimes F(W)$ is projective and hence

$$V(G)_{E(W)} \cap V(G)_{F(W)} = 0.$$

We conclude that $V(G)_{F(W)} = W^c \cup 0$.

The second statement follows immediately from the "tensor product property" and the definition of $\kappa(V)$ as $E(V) \otimes F(V')$.

□

For a conical subset W in $V(G)$ we denote by **Proj W**, the "projectivization" of W, the set of points in W which correspond to homogeneous prime ideals of $k[V(G)]$ excluding the augmentation ideal. Proj W can be viewed as a subset of the scheme Proj $k[V(G)]$.

There is 1-1 correspondence between conical subsets of $V(G)$ and their "projectivizations", i.e. a conical subset is completely determined by its homogeneous ideals. Therefore, the standard properties of support cones, described in Th. 2.6, apply to their "projectivizations".

In view of this remark the next theorem is a straightforward application of the above corollary.

Theorem 3.9. *Let M be a G-module. Then*

$$\mathrm{Proj}\ V(G)_M = \{s \in \mathrm{Proj}\ k[V(G)] : M \otimes \kappa(s) \textit{ is not projective as a } G\textit{-module}\}.$$

Proof. Let s be a homogeneous prime ideal in $k[V(G)]$ such that $M \otimes \kappa(s)$ is not projective as a G-module. Then $\operatorname{Proj} V(G)_{M\otimes\kappa(s)} = \operatorname{Proj} V(G)_M \cap \operatorname{Proj} V(G)_{\kappa(s)}$ is non-empty. Since $\operatorname{Proj} V(G)_{\kappa(s)} = \operatorname{Proj}(L(s)\cup 0) = \{s\}$ in view of the Corollary 3.8 above, we conclude that $s \in \operatorname{Proj} V(G)_M$.

Conversely, if $s \in \operatorname{Proj} V(G)_M$, then $\operatorname{Proj} V(G)_{M\otimes\kappa(s)}$ is non-empty, which implies that $M \otimes \kappa(s)$ is not projective as a G-module. □

Remark 3.10. We can restate the previous theorem in terms of the affine support cones using the following notation: for any prime ideal $\mu \subset k[V(G)]$ denote by $hom(\mu)$ the maximal homogeneous prime ideal contained in μ. Note that $\operatorname{ht}(hom(\mu)) = \operatorname{ht}(\mu) - 1$ unless μ itself is homogeneous. Any conical subset containing μ contains $hom(\mu)$ and vice versa. Together with the theorem above this observation implies the following description of $V(G)_M$:

$$V(G)_M = \{s \in V(G) : M \otimes \kappa(\mathrm{hom}(s)) \text{ is not projective as a } G\text{-module}\}.$$

As another application of the Corollary 3.8, we can generalize our "realization" statement to arbitrary conical sets. We shall utilize the notation $\mathrm{hom}(s)$ introduced in the remark above.

Corollary 3.11. *Any conical subset of $V(G)$ can be realized as a support cone of a G-module.*

Proof. Let W be a conical subset of $V(G)$. For any $s \in W$, W contains $\mathrm{hom}(s)$. Furthermore, by the definition of conical subset, for any point s corresponding to a homogeneous prime ideal, W contains the entire set $L(s)$. We conclude that

$$W = \bigcup_{s\in \operatorname{Proj} W} L(s) \cup 0$$

and, therefore, W is the support cone of the module $\bigoplus\limits_{s\in\operatorname{Proj} W} \kappa(s)$. □

As an application of Theorem 3.9 we are going to show that $V(G)_{\mathrm{Ind}_H^G(M)} \subset V(H)_M$ for an arbitrary H-module M, where H is a subgroup scheme of G. Although for finite dimensional modules this follows from the cohomological description of the support variety of M and Generalized Frobenius reciprocity, in the infinite dimensional case this approach is not available due to the lack of the cohomological description.

We shall need the following general fact about Rickard idempotents. The proof is merely a repetition of the one in [7].

Lemma 3.12. *Let G be an infinitesimal group scheme, H be a closed subgroup scheme of G and W be a subset of $V(G)$. Let $i_* : V(H) \hookrightarrow V(G)$ be the embedding of schemes induced by the inclusion $i : H \hookrightarrow G$. Then the following two distinguished triangles in StMod(H) are stably isomorphic:*

$$T(i_*^{-1}(W)) : E(i_*^{-1}(W)) \to k \to F(i_*^{-1}(W)) \to \Omega^{-1}E(i_*^{-1}(W))$$

and

$$T(W)\downarrow_H : E(W)\downarrow_H \to k \to F(W)\downarrow_H \to \Omega^{-1}E(W)\downarrow_H .$$

Proof. We have to show that $T(W)\downarrow_H$ satisfies universal properties of the distinguished triangle $T(i_*^{-1}(W))$. Since $E(W) \in \vec{\mathcal{C}}_W$, Prop. 2.6.2 implies that $E(W)\downarrow_H$

$\in \vec{\mathcal{C}}_{i_*^{-1}(W)}$. To check that $F(W)\downarrow_H$ is $\mathcal{C}_{i_*^{-1}(W)}$-local we note that the fact that $V(G)_{\mathrm{Ind}_H^G(M)} \subset V(H)_M$ for a finite dimensional H-module M (see, for example, [23, 2.3.1(b)]) implies that for any H-module M in $C_{i_*^{-1}(W)}$, we have $\mathrm{Ind}_H^G(M) \in \mathcal{C}_W$. Recall that for any finite dimensional G-module N, $V(G)_N = V(G)_{N^\#}$, where $N^\#$ is the k-linear dual of N. Hence, an isomorphism $\mathrm{Coind}_H^G(M) = (\mathrm{Ind}_H^G(M^\#))^\#$ (cf. [20, I.8.15]) implies that $V(G)_{\mathrm{Coind}_H^G(M)} \subset V(H)_M$ for a finite dimensional H-module M. Applying the fact that $F(W)$ is $\mathcal{C}_W$-local, we get

$$\underline{\mathrm{Hom}}_H(M, F(W)\downarrow_H) = \underline{\mathrm{Hom}}_G(\mathrm{Coind}_G^H(M), F(W)) = 0.$$

for any finite dimensional H-module M. Thus, $F(W)\downarrow_H$ is $\mathcal{C}_{i_*^{-1}(W)}$-local. In view of Remark 3.2, we conclude that $T(i_*^{-1}(W)) \cong T(W)\downarrow_H$.

□

Corollary 3.13. *Let G be an infinitesimal group scheme and H be a closed subgroup scheme of G, both satisfying the hypotheses of Theorem 1.6. Let M be an H-module. Then*

$$V(G)_{Ind_H^G(M)} \subset V(H)_M.$$

Proof. The embedding of group schemes $i : H \subset G$ induces a closed embedding of affine schemes $i_* : V(H) \hookrightarrow V(G)$ (cf. [28, 5.4]). We identify $V(H)$ with its image in $V(G)$.

Let $M = \varinjlim_{i\in I} M_i$, where M_i are finite dimensional H-modules. Then

$$\mathrm{Ind}_H^G(M) = \varinjlim_{i\in I}\mathrm{Ind}_H^G(M_i)$$

and, therefore,

$$V(G)_{\mathrm{Ind}_H^G(M)} \subset \bigcup_{i\in I} V(G)_{\mathrm{Ind}_H^G(M_i)} \subset V(H).$$

The last inclusion holds because the assertion of the corollary is known for finite dimensional modules (cf. [23, 2.3.1(b)]).

To prove the corollary it now suffices to check that for any point $s \in V(G)_{\mathrm{Ind}_G^H(M)} \subset V(H)$, corresponding to a homogeneous prime ideal in $k[V(G)]$, s is contained in $V(H)_M$. Let V be the Zariski closure of s. Since $s \in V(H)$, and the latter is closed in $V(G)$, we have $i_*^{-1}(V) = V \cap V(H) = V$. Lemma 3.12 implies that $\kappa(i_*^{-1}(V))$ is stably isomorphic to $\kappa(V)\downarrow_H$.

Applying Theorem 3.9 we get that $\mathrm{Ind}_G^H(M) \otimes \kappa(V)$ is not projective (= not injective), since $s \in V(G)_{\mathrm{Ind}_G^H(M)}$. By the tensor identity,

$$\mathrm{Ind}_G^H(M) \otimes \kappa(V) \cong \mathrm{Ind}_H^G(M \otimes \kappa(V)\downarrow_H).$$

Since induction takes injectives to injectives, we conclude that $M \otimes \kappa(V)\downarrow_H \cong M \otimes \kappa(i_*^{-1}(V))$ is not injective. Since s is a point in $V(H)$, it is still the generic point of $i_*^{-1}(V)$. Thus, $M \otimes \kappa(s)$ (where $\kappa(s)$ is now constructed in $\mathrm{StMod}(H)$) is not projective which implies, using Theorem 3.9 once again, that $s \in V(H)_M$.

□

Proposition 3.14. *Let W be a conical closed under specialization subset in $V(G)$. Then $\vec{\mathcal{C}}_W = \{M \in StMod(G) : V(G)_M \subset W\}$.*

Proof. Suppose $M \in \vec{\mathcal{C}}_W$. We need to show that $V(G)_M \subset W$. It suffices to check this inclusion for the points corresponding to homogeneous prime ideals. By the definition of $\vec{\mathcal{C}}_W$, M is stably isomorphic to $\varinjlim_{i\in I} M_i$ for some finite dimensional modules M_i whose varieties are contained in W. Let s be a point in $\operatorname{Proj} V(G)$ which does not belong to W. Then the restriction of $M_i \otimes k(s)$ to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[\mathbb{G}_{a(r)}]^\#$, where $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ is the one-parameter subgroup defined by the point s, is projective for all i. Since restriction commutes with filtered colimits, and a filtered colimit of projective modules is projective, we conclude that M restricted to the same subalgebra $k(s)[u_{r-1}]/u_{r-1}^p$ is projective. Thus, $s \notin V(G)_M$. The inclusion $V(G)_M \subset W$ follows.

Next assume that $V(G)_M \subset W$. By the "tensor product property" and Corollary 3.8, $M \otimes F(W)$ is projective. This implies, by tensoring the distinguished triangle $T(W)$ with M, that $M \otimes E(W) \cong M$ in $\operatorname{StMod}(G)$. Let $M \cong \varinjlim_{i\in I} M_i$ for some finite dimensional modules M_i and $E(W) \cong \varinjlim_{j\in J} N_j$ for some finite dimensional modules N_j, whose support varieties $V(G)_{N_j}$ are contained in W (the latter being possible by Theorem 3.1.I.(ii)). Then $M \cong M \otimes E(W) \cong \varinjlim_{(i,j)\in I\times J} M_i \otimes N_j$ and the variety of $M_i \otimes N_j$, $V(G)_{M_i \otimes N_j}$, is contained in $V(G)_{N_j}$ which, in turn, is contained in W for all pairs $(i,j) \in I \times J$. Thus, $M \in \vec{\mathcal{C}}_W$.

□

The following corollary is an immediate application of the proposition above to the closure under specialization of $V(G)_M$, $\operatorname{Cs}(V(G)_M)$.

Corollary 3.15. *For any G-module M there exists a filtered system of finite dimensional G-modules $\{M_i\}_{i\in I}$ such that*
(i) $M \cong \varinjlim_{i\in I} M_i$
(ii) $V(G)_{M_i} \subset Cs(V(G)_M)$.

Recall that complexity of a finite dimensional module M is defined to be the growth of the minimal projective resolution of M. It is proved to be equal to the dimension of the support variety of M ([1]). In [6] the following extension of the definition of complexity for infinite dimensional modules is given:

Definition 3.16. An arbitrary G-module M is said to have complexity c, denoted $c(M)$, if it can be realized as a filtered colimit of finite dimensional modules of complexity c but not lower.

For a subset W of $V(G)$ we define the *subset dimension* of W as follows:

$$\operatorname{s.dim}(W) \stackrel{def}{=} \max_{s\in W} \dim(\overline{s}).$$

Note that $\operatorname{s.dim}(W) = \operatorname{s.dim}(\operatorname{Cs}(W))$. In particular, for a closed subvariety V, its "subset dimension" coincides with the usual Krull dimension.

Using the notion of "subset dimension" we can formulate an alternative description of the complexity of an infinite dimensional module similar to the one mentioned above for the finite dimensional case:

Corollary 3.17. $c(M) = s.\dim(V(G)_M)$

Proof. Let $d = \operatorname{s.dim}(V(G)_M)$. The inequality $c(M) \le d$ follows immediately from Corollary 3.15.

Suppose $c(M) < d$. By our definition of subset dimension there exists a point $s \in V(G)_M$ such that $\dim(\overline{s}) = d$. Let $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ be the one-parameter subgroup corresponding to s. According to our definition of complexity, we can realize M as $\varinjlim_{i \in I} M_i$ for some finite dimensional modules M_i whose varieties have dimension no greater than $c(M)$. Then, clearly, $s \notin V(G)_{M_i}$, which implies that $M_i \otimes k(s)$ restricted to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[\mathbb{G}_{a(r)}]^{\#}$ is projective for any $i \in I$. Hence, the restriction of $M \otimes k(s)$ to the same subalgebra is also projective as a filtered colimit of projective modules. By the definition of a support cone, $s \notin V(G)_M$. The inequality in question follows. $\square$

To conclude, we give as promised an example of the failure of the "tensor product property" for the extension of the cohomological definition of the "support", for which we employ the construction of Rickard idempotents in a special case of a hypersurface defined by a single homogeneous element.

Example 3.18. Let $\xi \in H^n(G,k)$, where n is a positive even integer. Assume further that ξ is not nilpotent. Denote by $< \xi >$ the ideal generated by ξ and by $V(< \xi >)$ the variety of this ideal, i.e. $V(< \xi >) = \{\mu \in \operatorname{Spec} H^{ev}(G,k) : \xi \in \mu\}$. Let F_ξ be the filtered colimit of the sequence

$$k \to \Omega^{-n} k \to \Omega^{-2n} k \to \dots$$

where each map corresponds to ξ via the natural isomorphism

$$H^n(G,k) \cong \underline{\operatorname{Hom}}(\Omega^{-rn} k, \Omega^{-(r+1)n} k).$$

F_ξ is well-defined up to a stable isomorphism and comes equipped with a natural map from k, $k \to F_\xi$. Complete this map to a distinguished triangle in $\operatorname{StMod}(G)$:

$$E_\xi \to k \to F_\xi \to \Omega^{-1} E_\xi.$$

It can be shown (cf. [25]) that this distinguished triangle is stably isomorphic to a distinguished triangle defined by the thick subcategory $C_{V(<\xi>)}$. Thus, $V(G)_{E_\xi} = V(< \xi >)$ and $V(G)_{F_\xi} = V(< \xi >)^c \cup 0$. In particular, E_ξ is not projective.

The cohomology of F_ξ can be computed as the filtered colimit of the sequence

$$H^*(G,k) \to H^*(G, \Omega^{-n} k) \to H^*(G, \Omega^{-2n} k) \to \dots$$

which is equivalent to

$$H^*(G,k) \xrightarrow{\times \xi} H^{*+n}(G,k) \xrightarrow{\times \xi} H^{*+2n}(G,k) \xrightarrow{\times \xi} \dots$$

The direct limit of this sequence is isomorphic to $H^*(G,k)[1/\xi]$. The inclusion

$$\operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(F_\xi, F_\xi)) \subset \operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(k, F_\xi)) = \operatorname{Ann}_{H^{ev}(G,k)}(H^*(G,k)[1/\xi]) = 0$$

implies that $|G|_{F_\xi} = V(\operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(F_\xi, F_\xi))) = |G|$.

Since $F_\xi \otimes E_\xi$ is projective, the "tensor product property" for "cohomological supports", if valid, would imply that

$$0 = |G|_{F_\xi \otimes E_\xi} = |G|_{F_\xi} \cap |G|_{E_\xi} = |G| \cap |G|_{E_\xi} = |G|_{E_\xi}$$

which, in view of Proposition 2.7, contradicts the fact that E_ξ is not projective.

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