Localizing subcategories for finite group schemes

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(joint work with D. Benson, S. Iyengar, H. Krause)

Let k be a field of positive characteristic p. An affine group scheme G over k is a representable functor from the category of commutative k-algebras to groups. The coordinate algebra, denoted by $k[G]$, is a commutative Hopf k-algebra. An affine group scheme G is *finite* if the coordinate algebra is finite dimensional over k. In that case we define the *group algebra* kG to the the linear dual of $k[G]$. Hence, kG is a finite dimensional cocommutative Hopf algebra. Likewise, starting with any finite dimensional cocommutative Hopf algebra, its linear dual is a coordinate algebra of a finite group scheme. One therefore has an equivalence of categories:

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional co-} \\ \text{commutative Hopf algebras} \end{array} \right\}$$

Via this equivalence, one can identify representations of G with kG-modules; for the rest of this note we shall refer to representations of G as *G-modules*. Examples of finite group schemes include finite groups, restricted Lie algebras and Frobenius kernels of algebraic groups. A finite group scheme is *unipotent* if the group algebra kG is local and is *abelian* if kG is commutative.

Let G will be a finite group scheme defined over k. Since the group algebra kG is Frobenius (see, for example, [16, I.6]), the projective modules are injective and, moreover, one can construct the stable module category $\mathsf{StMod}\,G$. Recall that the objects of $\mathsf{StMod}\,G$ are G-modules, whereas the Hom-sets are defined as follows:

$$\underline{\mathrm{Hom}}(M,N) := \frac{\mathrm{Hom}_G(M,N)}{\mathrm{PHom}_G(M,N)}$$

with $\mathrm{PHom}_G(M,N)$ being the subset of all G-maps between M and N which factor through a projective G-module. The category $\mathsf{StMod}\,G$ is a compactly generated tensor triangulated category with the compact objects being the stable module category of finite dimensional G-modules, denoted $\mathsf{stmod}\,G$.

A subcategory $\mathcal{C}$ of $\mathsf{StMod}\,G$ is *localizing* if it is a full triangulated subcategory closed under set-indexed direct sums. It is *tensor ideal* if for any $M \in \mathsf{StMod}G$, $C \in \mathcal{C}$, we have $M \otimes C \in \mathcal{C}$. A subcategory $\mathcal{C}$ of $\mathsf{stmod}\,G$ is thick (or épaisse) if it is a full triangulated subcategory closed under taking direct summands.

The cohomology ring $H^*(G,k) = \mathsf{Ext}^*_G(k,k)$ is a graded commutative k-algebra which is finitely generated by a fundamental result of Friedlander and Suslin [15]. The following is the main theorem of this note:

Theorem 1. *For any finite group scheme G, there is one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Localizing tensor-ideal} \\ \textit{subcategories of } \mathsf{StMod}\,G \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{subsets of} \\ \mathsf{Proj}\,H^*(G,k) \end{array} \right\}$$

which restricts to one-to-one correspondence

$$\left\{ \begin{array}{c} \textit{Thick tensor-ideal} \\ \textit{subcategories of } \mathsf{stmod}\,G \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \mathsf{Proj}\,H^*(G,k) \end{array} \right\}$$

The correspondence is given explicitly as follows:

$$\mathcal{C} \quad \longmapsto \quad V = \bigcup_{M \in \mathcal{C}} \operatorname{supp} M$$

$$\mathcal{C} = \{M \in \mathsf{StMod}\, G \mid \operatorname{supp} M \subset V\} \quad \longleftarrow\!\!\!\mid \quad V$$

This theorem generalizes the main result in [8] where it was proved for finite groups. An essential feature of the argument in [8] was the fact that various properties of modules for finite groups, such as projectivity, are detected upon restriction to elementary abelian p-subgroups. Unfortunately, this does not generalize to arbitrary finite group schemes. The approach we use to prove Theorem 1 for any finite group scheme is substantially different and relies heavily on the notion of cosupport introduced in [9]. In particular, it yields a completely new proof of the classification theorem even for finite groups. In fact, it yields two new proofs! In this note we'll sketch the strategy which yields the theorem in full generality. A simpler, and conceptually very pleasing, new proof which works only for finite groups is alluded to in S. Iyengar's note in the same volume.

The support of M, $\operatorname{supp} M$, is a geometric invariant associated to any G-module M which we now describe. In fact, to prove Theorem 1, we need to develop two notions of support, and parallel notions of cosupport. The first theory of support and cosupport is due to Benson-Iyengar-Krause [6], [7], [8], [9], building on the earlier work of Rickard in representation theory [19]. To each homogeneous prime ideal $\mathfrak{p}$ (strictly smaller than the irrelevant ideal) of $H^*(G,k)$ we associate a universal module (usually infinite dimensional) $\Gamma_{\mathfrak{p}}(k)$ (see [6]). Then the *cohomological* support and cosupport are defined as follows:

Definition 2 ([6], [9])**.**

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \Gamma_{\mathfrak{p}}(k) \otimes_k M \ \textit{is not projective}\}.$$

$$\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \mathsf{Hom}_k(\Gamma_{\mathfrak{p}}(k), M) \ \textit{is not projective}\}.$$

The general philosophy captured beautifully by Balmer in [1] prescribes that to classify tensor ideal subcategories in a tensor triangulated category one needs "good" theory of supports. Benson-Iyengar-Krause support and cosupport defined above satisfy many of the properties expected of a "good" theory but their cohomological nature renders them unsuitable for testing behavior with respect to tensor products and function objects. To repair this, we introduce another theory, that of π-supports and π-cosupports. For a field extension K/k, we denote by G_K the finite group scheme over K with the coordinate algebra $K[G_K] := K \otimes_k k[G]$.

Definition 3 ([13], [14])**.** *A* π-point *of G, defined over a field extension K of k, is a morphism of K-algebras*

$$\alpha : K[t]/(t^p) \to KG_K$$

which factors through the group algebra of a unipotent abelian subgroup scheme C of G_K, and such that KG_K is flat when viewed as a left (equivalently, as a right) module over $K[t]/(t^p)$ via α.

We say that a pair of π-points $\alpha : K[t]/(t^p) \to KG_K$ and $\beta : L[t]/(t^p) \to LG_L$ are equivalent if they satisfy the following condition: *for any finite dimensional kG-module M, the module $\alpha^*(K \otimes_k M)$ is projective if and only if $\beta^*(L \otimes_k M)$ is projective.* The set of equivalence classes of π-points is denoted $\Pi(G)$; it has a naturally defined Zariski topology. By [13, Theorem 3.6], there is a natural homeomorphism $\Pi(G) \simeq \mathsf{Proj}\, H^*(G, k)$, which allows us to identify these two spaces. Via this identification, we associate to each homogeneous prime ideal $\mathfrak{p} \subset H^*(G, k)$, strictly smaller than the irrelevant ideal, a π-point $\alpha_{\mathfrak{p}}$ whose equivalence class in $\Pi(G)$ coincides with the point $\mathfrak{p}$ on $\mathsf{Proj}\, H^*(G, k)$. By [14, 4.6], [10, 2.1], the definition given below is independent of which representative we choose.

Definition 4. *The* π-support *of M is the subset of* $\mathsf{Proj}\, H^*(G, k)$ *defined by*

$$\pi\text{-}\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G, k) \mid \alpha_{\mathfrak{p}}^*(K \otimes_k M) \textit{ is not projective}\}.$$

The π-cosupport *of M is the subset of* $\mathsf{Proj}\, H^*(G, k)$ *defined by*

$$\pi\text{-}\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G, k) \mid \alpha_{\mathfrak{p}}^*(\mathsf{Hom}_k(K, M)) \textit{ is not projective}\}.$$

The usefulness of π-support and π-cosupport is postulated in the following theorem which appears to be intractable for cohomological supports.

Theorem 5. *Let M and N be G-modules. Then there are equalities*

$$\pi\text{-}\operatorname{supp}(M \otimes_k N) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{supp}(N),$$
$$\pi\text{-}\operatorname{cosupp}(\mathsf{Hom}_k(M, N)) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{cosupp}(N).$$

To prove Theorem 1, we need to identify cohomological and π-supports. This can be done formally following the strategy developed in [5] once we know the following detection result:

Theorem 6. *Let G be a finite group scheme, and M be a G-module. Then M is projective if and only if $\pi\text{-}\operatorname{supp}(M) = \emptyset$.*

This detection theorem is an ultimate generalization of the famous Dade's lemma [12]. It builds on the work of many authors, see [5], [2], [17], [18]. In this generality the result was stated in [14] but the proof contained an error. The complete proof is to appear in [11].

Corollary 7. (1) $\pi\text{-}\operatorname{supp} \Gamma_{\mathfrak{p}}(k) = \mathfrak{p}$;
(2) *For any G-module M, $\pi\text{-}\operatorname{supp}(M) = \operatorname{supp}(M)$.*

By the work of Benson-Iyengar-Krause [6], Theorem 1 follows from the "stratification" of $\mathsf{StMod}\, G$ by $\mathsf{Proj}\, H^*(G, k)$. Explicitly, one needs to show the following:

For any point $\mathfrak{p} \in \mathsf{Proj}\, H^*(G, k)$, the tensor ideal localizing subcategory

$$\Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G) = \{M \in \mathsf{StMod}\, G \mid \operatorname{supp}(M) = \mathfrak{p}\}$$

is minimal, that is, does not contain any non-trivial proper tensor ideal localizing subcategories. This is equivalent to showing that for any non-zero objects $M, N \in \Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$, $\underline{\mathsf{Hom}}(M, N) \neq 0$. It is at this point that the function object formula for cosupport (5) becomes of utmost importance.

With these ingredients in place, the proof of Theorem 1 proceeds in two steps. First, we show that for any closed point $\mathfrak{m} \in \operatorname{Proj} H^*(G,k)$, the subcategory $\Gamma_{\mathfrak{m}}(\mathsf{StMod}\, G)$ is minimal. To reduce the problem from any point $\mathfrak{p}$ on $\operatorname{Proj} H^*(G,k)$ to a closed point on $\operatorname{Proj} H^*(G_K,K)$ for some field extension K/k, we use a commutative algebra calculation with Carlson modules (or, equivalently, Koszul objects) to show the following:

Theorem 8. *Let $\mathfrak{p}$ be a point on $\operatorname{Proj} H^*(G,k)$. Let K be the residue field at $\mathfrak{p}$ and let $\mathfrak{m}$ be a closed point in $\operatorname{Proj} H^*(G_K,K)$ "lying over" $\mathfrak{p}$. Then $\Gamma_{\mathfrak{p}}(k) \in \operatorname{Loc}^{\otimes}(\Gamma_{\mathfrak{m}}(K)\downarrow_G)$, where $\operatorname{Loc}^{\otimes}(\Gamma_{\mathfrak{m}}(K)\downarrow_G)$ is the minimal tensor ideal localizing subcategory containing $\Gamma_{\mathfrak{m}}(K)\downarrow_G$.*

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