SECOND FLIP IN THE HASSETT-KEEL PROGRAM: EXISTENCE OF GOOD MODULI SPACES

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Abstract. We prove a general criterion for an algebraic stack to admit a good moduli space. This result may be considered as a generalization of the Keel-Mori theorem, which guarantees the existence of a coarse moduli space for a separated Deligne-Mumford stack. We apply this result to prove that the moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ parameterizing α-stable curves introduced in [AFSv14] admit good moduli spaces.

Contents

1. Introduction

This is the second paper in a trilogy in which we construct the second flip in the log minimal model program for $\overline{M}_{g,n}$. In this paper, we prove that the moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ parameterizing α-stable curves introduced in [AFSv14, §2] admit good moduli spaces. Namely, we prove:

Theorem 1.1. *For every $\alpha \in (2/3-\epsilon, 1]$, $\overline{\mathcal{M}}_{g,n}(\alpha)$ admits a good moduli space $\overline{\mathbb{M}}_{g,n}(\alpha)$ which is a proper algebraic space over $\mathbb{C}$. Furthermore, for each critical value $\alpha_c \in \{2/3,\ 7/10,\ 9/11\}$, there exists a diagram*

$$\begin{array}{ccccc}
\overline{\mathcal{M}}_{g,n}(\alpha_c{+}\epsilon) & \hookrightarrow & \overline{\mathcal{M}}_{g,n}(\alpha_c) & \hookleftarrow & \overline{\mathcal{M}}_{g,n}(\alpha_c{-}\epsilon) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathbb{M}}_{g,n}(\alpha_c{+}\epsilon) & \longrightarrow & \overline{\mathbb{M}}_{g,n}(\alpha_c) & \longleftarrow & \overline{\mathbb{M}}_{g,n}(\alpha_c{-}\epsilon)
\end{array}$$

where $\overline{\mathcal{M}}_{g,n}(\alpha_c) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$, $\overline{\mathcal{M}}_{g,n}(\alpha_c{+}\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c{+}\epsilon)$ and $\overline{\mathcal{M}}_{g,n}(\alpha_c{-}\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c{-}\epsilon)$ are good moduli spaces, and where $\overline{\mathbb{M}}_{g,n}(\alpha_c{+}\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$ and $\overline{\mathbb{M}}_{g,n}(\alpha_c{-}\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$ are proper morphisms of algebraic spaces.

The goal of the final paper in the trilogy [AFS15] is to establish an isomorphism between $\overline{\mathbb{M}}_{g,n}(\alpha)$ and the projective variety

$$\overline{M}_{g,n}(\alpha) := \operatorname{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(\overline{\mathcal{M}}_{g,n}, \lfloor m(K_{\overline{\mathcal{M}}_{g,n}} + \alpha\delta + (1-\alpha)\psi)\rfloor), \tag{1.1}$$

which thereby proves that the good moduli spaces of $\overline{\mathcal{M}}_{g,n}(\alpha)$ are indeed the log canonical models defined in (1.1).

In the case where $\alpha \in (2/3, 7/10]$ and $n = 0$, the spaces $\overline{\mathbb{M}}_{g,n}(\alpha)$ has been constructed using Geometric Invariant Theory (GIT) by Hassett and Hyeon in [HH09, HH13]. There is no known GIT construction of the moduli spaces corresponding to $\alpha \leq 2/3$. Thus, this paper gives the first *intrinsic* construction of a moduli space associated to an algebraic stack parameterizing objects with infinite automorphism groups.

In order to prove Theorem 1.1, we establish three general existence results for good moduli spaces. These existence results make essential use of the notions of local quotient presentation and VGIT chambers of a local quotient presentation, introduced in Definition 2.1 and Definition 2.4, respectively. Our first existence result gives conditions under which one may use a local quotient presentation to construct a good moduli space.

Theorem 1.2. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field k. Suppose that:*

(1) *for every closed point $x \in \mathcal{X}$, there exists a local quotient presentation $f\colon \mathcal{W} \to \mathcal{X}$ around x such that:*
   (a) *the morphism f is stabilizer preserving at closed points of $\mathcal{W}$, and*
   (b) *the morphism f sends closed points to closed points; and*
(2) *for any point $x \in \mathcal{X}(k)$, the closed substack $\overline{\{x\}}$ admits a good moduli space.*

Then $\mathcal{X}$ admits a good moduli space.

As we explain below, this result may be considered as an analog of the Keel-Mori theorem [KM97] for algebraic stacks, but in practice the hypotheses of Theorem 1.2 are harder to verify than those of the Keel-Mori theorem. Nevertheless, we believe that the above theorem should be applicable to many additional moduli problems. In fact, it has come to our attention that it has already been applied to construct a good moduli space of Kähler-Einstein Fano varieties in [LWX14] and [Oda14].

Our second existence result, Theorem 1.3, gives one situation in which the hypotheses of Theorem 1.2 are satisfied. It says that if $\mathcal{X}$ is an algebraic stack and $\mathcal{X}^+ \hookrightarrow \mathcal{X} \hookleftarrow \mathcal{X}^-$ is a pair of open immersions locally cut out by VGIT chambers of a local quotient presentation, then $\mathcal{X}$ admits a good moduli space if $\mathcal{X}^+$, $\mathcal{X} \smallsetminus \mathcal{X}^+$, and $\mathcal{X} \smallsetminus \mathcal{X}^-$ do.

Theorem 1.3. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field k, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Let $\mathcal{X}^+, \mathcal{X}^- \subset \mathcal{X}$ be open substacks, and let $\mathcal{Z}^+ = \mathcal{X} \smallsetminus \mathcal{X}^+$ and $\mathcal{Z}^- = \mathcal{X} \smallsetminus \mathcal{X}^-$ be their reduced complements. Suppose that*

(1) *the algebraic stacks $\mathcal{X}^+$, $\mathcal{Z}^+$, $\mathcal{Z}^-$ admit good moduli spaces; and*
(2) *for all closed points $x \in \mathcal{Z}^+ \cap \mathcal{Z}^-$, there exists a local quotient presentation $f\colon \mathcal{W} \to \mathcal{X}$ around x and a Cartesian diagram*

$$\begin{array}{ccccc} \mathcal{W}^+_{\mathcal{L}} & \hookrightarrow & \mathcal{W} & \hookleftarrow & \mathcal{W}^-_{\mathcal{L}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}^+ & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}^- \end{array} \tag{1.2}$$

where $\mathcal{W}^+_{\mathcal{L}}, \mathcal{W}^-_{\mathcal{L}}$ are the VGIT chambers of $\mathcal{W}$ with respect to $\mathcal{L}$.

Then there exist good moduli spaces $\mathcal{X} \to X$ and $\mathcal{X}^- \to X^-$ such that $X^+ \to X$ and $X^- \to X$ are proper and surjective. In particular, if X^+ is proper over k, then X and X^- are also proper over k.

The third existence result, Proposition 1.4, proves that one can check existence of a good moduli space after passing to a finite cover by a quotient stack. Recall that an algebraic stack $\mathcal{X}$ is called a *global quotient stack* if $\mathcal{Z} \simeq [Z/\operatorname{GL}_n]$, where Z is an algebraic space with an action of GL_n.

Proposition 1.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks of finite type over an algebraically closed field k of characteristic 0. Suppose that:*

(1) *the morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is finite and surjective;*
(2) *there exists a good moduli space $\mathcal{X} \to X$ with X separated; and*
(3) *the algebraic stack $\mathcal{Y}$ is a global quotient stack and admits local quotient presentations.*

Then there exists a good moduli space $\mathcal{Y} \to Y$ with Y separated. Moreover, if X is proper, so is Y.

Both Theorem 1.3 and Proposition 1.4 are proved using Theorem 1.2.

1.1. **Motivation and sketch of proof of Theorem 1.2.** In order to motivate the statement of Theorem 1.2, let us give an informal sketch of the proof. If $\mathcal{X}$ admits local quotient presentations, then every closed point $x \in \mathcal{X}$ admits an étale neighborhood of the form

$$[\operatorname{Spec} A_x \,/\, G_x] \to \mathcal{X},$$

where A_x is a finite type k-algebra and G_x is the stabilizer of x. The union

$$\coprod_{x \in \mathcal{X}} [\operatorname{Spec} A_x \,/\, G_x]$$

defines an étale cover of $\mathcal{X}$; reducing to a finite subcover, we obtain an atlas $f\colon \mathcal{W} \to \mathcal{X}$ with the following properties:

(1) f is affine and étale; and
(2) $\mathcal{W}$ admits a good moduli space W.

Indeed, (2) follows simply by taking invariants $[\operatorname{Spec} A_x/G_x] \to \operatorname{Spec} A_x^{G_x}$ and since f is affine, the fiber product $\mathcal{R} := \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ admits a good moduli space R. We may thus consider the following diagram:

$$\begin{array}{ccccc} \mathcal{R} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \mathcal{W} & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \phi} & & \\ R & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & W & & \end{array} \tag{1.3}$$

The crucial question is: Can we choose $f\colon \mathcal{W} \to \mathcal{X}$ to guarantee that the projections $q_1, q_2\colon R \rightrightarrows W$ define an étale equivalence relation? If so, then the algebraic space quotient $X = W/R$ gives a good moduli space for $\mathcal{X}$.

If $\mathcal{X}$ is a separated Deligne-Mumford stack, we can always do this. Indeed, the atlas f may be chosen to be stabilizer preserving.[1] Thus, we may take the projections $\mathcal{R} \rightrightarrows \mathcal{W}$ to be stabilizer preserving and étale, and this implies that the projections $R \rightrightarrows W$ are étale.[2] This leads to a direct proof of the Keel-Mori theorem for separated Deligne-Mumford stacks of finite type over $\operatorname{Spec} k$ (one can show directly that such stacks always admit local quotient presentations). In general, of course, algebraic stacks need not be separated so we must find weaker conditions which ensure that the projections q_1, q_2 are étale. In particular, we must identify a set of sufficient conditions that can be directly verified for geometrically-defined stacks such as $\overline{\mathcal{M}}_{g,n}(\alpha)$.

Our result gives at least one plausible answer to this problem. To begin, note that if $\omega \in \mathcal{W}$ is a *closed* k-point with image $w \in W$, then the formal neighborhood $\widehat{\mathcal{O}}_{W,w}$ can be identified with the G_ω-invariants $D_\omega^{G_\omega}$ of the miniversal deformation space D_ω of ω. Thus, we may ensure that q_i is étale at a k-point $r \in R$, or equivalently that the induced map $\widehat{\mathcal{O}}_{W,q_i(r)} \to \widehat{\mathcal{O}}_{R,r}$ is an isomorphism, by manually imposing the following conditions: $p_i(\rho)$ should be a closed point, where $\rho \in \mathcal{R}$ is the unique closed point in the preimage of $r \in R$, and p_i should induce an isomorphism of stabilizer groups $G_\rho \simeq G_{p_i(\rho)}$. Indeed, we then have $\widehat{\mathcal{O}}_{W,q_i(r)} = D_{p_i(\rho)}^{G_{p_i(\rho)}} \simeq D_\rho^{G_\rho} = \widehat{\mathcal{O}}_{R,r}$, where the middle isomorphism follows from the hypothesis that p_i is étale and stabilizer preserving. In sum, we have identified two key conditions that will imply that $R \rightrightarrows W$ is an étale equivalence relation:

($\star$) The morphism $f\colon \mathcal{W} \to \mathcal{X}$ is stabilizer preserving at closed points.

($\star\star$) The projections $p_1, p_2\colon \mathcal{W} \times_{\mathcal{X}} \mathcal{W} \rightrightarrows \mathcal{W}$ send closed points to closed points.

Condition ($\star$) is precisely hypothesis (1a) of Theorem 1.2. In practice, it is difficult to directly verify condition ($\star\star$), but it turns out that it is implied by conditions (1b) and (2), which are often easier to verify.

1.2. **Roadmap.** In Section 2, we recall the necessary terminology and results from [AFSv14]. We also prove some preliminary lemmas concerning strongly étale morphisms. In Section 3, we prove Theorem 1.2, Theorem 1.3 and Proposition 1.4. Namely, in §3.1, we prove Theorem 1.2 along the lines described in §1.1. Then we prove Theorem 1.3 and Proposition 1.4 in §3.2 and §3.3, respectively, by showing that after suitable reductions, their hypotheses imply that conditions (1a), (1b) and (2) of Theorem 1.2 are satisfied. In Section 4, we apply Theorem 1.3 to prove Theorem 1.1. Appendix A provides various examples of algebraic stacks highlighting the necessity of conditions (1a), (1b) and (2) in Theorem 1.2.

Notation. Theorem 1.1 holds over an arbitrary algebraically closed field $\mathbb{C}$ of characteristic 0. In Sections 2 and 3, we work over an algebraically closed field k of arbitrary characteristic unless when stated otherwise. A *linearly reductive* group scheme over a field k is, by definition, an affine group scheme of finite type over k such that every representation is completely reducible.

[1]The set of points $\omega \in \mathcal{W}$ where f is not stabilizer preserving is simply the image of the complement of the open substack $I_{\mathcal{W}} \subset I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}$ in $\mathcal{W}$ and therefore is closed since $I_{\mathcal{X}} \to \mathcal{X}$ is proper. By removing this locus from $\mathcal{W}$, $f\colon \mathcal{W} \to \mathcal{X}$ may be chosen to be stabilizer preserving.

[2]To see this, note that if $r \in R$ is any closed point and $\rho \in \mathcal{R}$ is its closed preimage, then $\widehat{\mathcal{O}}_{R,r} \simeq D_\rho^{G_\rho}$, where D_ρ denotes the miniversal formal deformation space of ρ and G_ρ is the stabilizer of ρ; similarly $\widehat{\mathcal{O}}_{W,q_i(r)} \simeq D_{p_i(\rho)}^{G_{p_i(\rho)}}$. Now p_i étale implies $D_\rho \simeq D_{p_i(\rho)}$ and p_i stabilizer preserving implies $G_\rho \simeq G_{p_i(\rho)}$, so $\widehat{\mathcal{O}}_{R,r} \simeq \widehat{\mathcal{O}}_{W,q_i(r)}$, i.e. q_i is étale.

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2. Background and preliminary results

2.1. **Local quotient presentations.**

Definition 2.1. Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field k, and let $x \in \mathcal{X}(k)$ be a closed point. We say that $f\colon \mathcal{W} \to \mathcal{X}$ is a *local quotient presentation around* x if:

(1) the stabilizer G_x of x is linearly reductive;
(2) there is an isomorphism $\mathcal{W} \simeq [\operatorname{Spec} A \,/\, G_x]$, where A is a finite type k-algebra;
(3) the morphism f is étale and affine; and
(4) there exists a point $w \in \mathcal{W}(k)$ such that $f(w) = x$ and f induces an isomorphism $G_w \simeq G_x$.

We sometimes write $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ as a local quotient presentation to indicate the chosen preimage of x. We say that $\mathcal{X}$ *admits local quotient presentations* if there exist local quotient presentations around all closed points $x \in \mathcal{X}(k)$.

Remark 2.2. Note that if $\mathcal{X}$ admits local quotient presentations, then $\mathcal{X}$ necessarily has affine diagonal and every closed point necessarily has a linearly reductive stabilizer group.

For our purpose of applying Theorem 1.3 to the moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$, the following result suffices to guarantee the existence of local quotient presentations.

Proposition 2.3. [AK14, §3.3] *Let k be an algebraically closed field. Let $\mathcal{X}$ be a quotient stack $[U/G]$ where U is a normal and separated scheme of finite type over k and G is a smooth linear algebraic group over k. If $x \in \mathcal{X}(k)$ is a point with linearly reductive stabilizer, then there exists a local quotient presentation $f\colon \mathcal{W} \to \mathcal{X}$ around x.*

2.2. **Local VGIT chambers.**

Let G be a linearly reductive group acting on an affine scheme $X = \operatorname{Spec} A$ by $\sigma\colon G \times X \to X$. Let $\chi\colon G \to \mathbb{G}_m$ be a character. Set $A_n := \{f \in A \mid \sigma^*(f) = \chi^*(t)^{-n} f\}$. We define the *VGIT ideals with respect to* χ to be:

$$I_\chi^+ := (f \in A \mid f \in A_n \text{ for some } n > 0),$$
$$I_\chi^- := (f \in A \mid f \in A_n \text{ for some } n < 0).$$

The *VGIT* $(+)$*-chamber and* $(-)$*-chamber of* X *with respect to* χ are the open subschemes

$$X_\chi^+ := X \setminus \mathbb{V}(I_\chi^+) \hookrightarrow X, \qquad X_\chi^- := X \setminus \mathbb{V}(I_\chi^-) \hookrightarrow X.$$

and since the open subschemes X_χ^+, X_χ^- are G-invariant, we also have stack-theoretic open immersions

$$[X_\chi^+/G] \hookrightarrow [X/G] \hookleftarrow [X_\chi^-/G].$$

We will refer to these open immersions as the *VGIT* $(+)/(-)$*-chambers of* $[X/G]$ *with respect to* χ.

Finally, given the data of an algebraic stack $\mathcal{X}$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$, we can define the VGIT chambers of a local quotient presentation. In this situation, note that if $x \in \mathcal{X}(k)$ is any point, then there is a natural action of the automorphism group G_x on the fiber $\mathcal{L}|_{BG_x}$ that induces a character $\chi_{\mathcal{L}} \colon G_x \to \mathbb{G}_m$.

Definition 2.4 (VGIT chambers of a local quotient presentation)**.** Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field k. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Let $x \in \mathcal{X}$ be a closed point. If $f \colon \mathcal{W} \to \mathcal{X}$ is a local quotient presentation around x, we define *the chambers of* $\mathcal{W}$ *with respect to* $\mathcal{L}$ to be the VGIT $(+)/(-)$-chambers

$$\mathcal{W}^+_{\mathcal{L}} \hookrightarrow \mathcal{W} \hookleftarrow \mathcal{W}^-_{\mathcal{L}}$$

of $\mathcal{W}$ with respect to the character $\chi_{\mathcal{L}} \colon G_x \to \mathbb{G}_m$.

2.3. Strongly étale morphisms.

Definition 2.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks of finite type over an algebraically closed field k which have affine diagonal. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say that

- the morphism f *sends closed points to closed points* if for every closed point $x \in \mathcal{X}$, the point $f(x) \in \mathcal{Y}$ is closed.
- the morphism f is *stabilizer preserving at* $x \in \mathcal{X}(k)$ if $\mathrm{Aut}_{\mathcal{X}(k)}(x) \to \mathrm{Aut}_{\mathcal{Y}(k)}(f(x))$ is an isomorphism.
- for a closed point $x \in \mathcal{X}$, the morphism f is *strongly étale at* x if f is étale at x, f is stabilizer preserving at x, and $f(x) \in \mathcal{Y}$ is closed.
- the morphism f is *strongly étale* if f is strongly étale at all closed points of $\mathcal{X}$.

Definition 2.6. Let $\phi \colon \mathcal{X} \to X$ be a good moduli space. We say that an open substack $\mathcal{U} \subset \mathcal{X}$ is *saturated* if $\phi^{-1}(\phi(\mathcal{U})) = \mathcal{U}$.

The following proposition is simply a stack-theoretic formulation of Luna's well-known results in invariant theory [Lun73, Chapitre II] often referred to as Luna's fundamental lemma. It justifies the terminology *strongly étale* by showing that strongly étale morphisms induce étale morphisms of good moduli spaces. It also shows that for a morphism of algebraic stacks admitting good moduli spaces, strongly étale is an open condition.

Proposition 2.7. *Let* $\mathcal{W}$ *and* $\mathcal{X}$ *be algebraic stacks of finite type over an algebraically closed field* k *which have affine diagonal. Consider a commutative diagram*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\phi} \\ W & \xrightarrow{g} & X \end{array} \tag{2.1}$$

where f *is representable and separated, and both* φ *and* ϕ *are good moduli spaces. Then*

(1) *if* f *is strongly étale at* $w \in \mathcal{W}$*, then* g *is étale at* $\varphi(w)$*;*
(2) *if* f *is strongly étale, then* g *is étale and Diagram* (2.1) *is Cartesian; and*
(3) *there exists a saturated open substack* $\mathcal{U} \subset \mathcal{W}$ *such that:*

(a) *the morphism* $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}$ *is strongly étale and* $f(\mathcal{U}) \subset \mathcal{X}$ *is saturated, and*
(b) *if* $w \in \mathcal{W}$ *is a closed point such that* f *is strongly étale at* w*, then* $w \in \mathcal{U}$*.*

Proof. [Alp13, Theorem 5.1] gives part (1) and that g is étale in (2). The hypotheses in (2) imply that the induced morphism $\Psi\colon \mathcal{W} \to W \times_X \mathcal{X}$ is representable, separated, quasi-finite and sends closed points to closed points. [Alp13, Proposition 6.4] implies that Ψ is finite. Moreover, since f and g are étale, so is Ψ. But since $\mathcal{W}$ and $W \times_X \mathcal{X}$ both have W as a good moduli space, it follows that a closed point in $W \times_X \mathcal{X}$ has a unique preimage under Ψ. Therefore, Ψ is an isomorphism and the diagram is Cartesian. Statement (3) follows from [Alp10, Theorem 6.10]. □

Lemma 2.8. *Let* $\mathcal{X}$ *be an algebraic stack of finite type over an algebraically closed field* k *which has affine diagonal, and let* $\phi\colon \mathcal{X} \to X$ *be a good moduli space. Let* $x \in \mathcal{X}$ *be a closed point and* $\mathcal{U} \subset \mathcal{X}$ *be an open substack containing* x*. Then there exists a saturated open substack* $\mathcal{U}_1 \subset \mathcal{U}$ *containing* x*. Moreover, if* $\mathcal{X} \simeq [\operatorname{Spec} A/G]$ *with* G *linearly reductive, then* $\mathcal{U}_1$ *can be chosen to be of the form* $[\operatorname{Spec} B/G]$ *for a* G*-invariant open affine subscheme* $\operatorname{Spec} B \subset \operatorname{Spec} A$*.*

Proof. The substacks $\{x\}$ and $\mathcal{X} \setminus \mathcal{U}$ are closed and disjoint. By [Alp13, Theorem 4.16], $\phi(\{x\})$ and $Z := \phi(\mathcal{X} \setminus \mathcal{U})$ are closed and disjoint. For the first statement, take $\mathcal{U}_1 = \phi^{-1}(X \setminus Z)$. For the second statement, take $\mathcal{U}_1 = \phi^{-1}(U_1)$ for an affine open subscheme $U_1 \subset X \setminus Z$. □

Lemma 2.9. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be algebraic stacks of finite type over an algebraically closed field* k *which have affine diagonal. Let* $f\colon \mathcal{X} \to \mathcal{Y}$ *be a strongly étale morphism. Suppose that for points* $x \in \mathcal{X}(k)$ *and* $y \in \mathcal{Y}(k)$*, the closed substacks* $\overline{\{x\}} \subset \mathcal{X}$ *and* $\overline{\{y\}} \subset \mathcal{Y}$ *admit good moduli spaces. Then for any finite type morphism* $g\colon \mathcal{Y}' \to \mathcal{Y}$ *with affine diagonal, the base change* $f'\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ *is strongly étale.*

Proof. Clearly, the morphism f' is étale. Let $x' \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ be a closed point. To check that f' is stabilizer preserving at x' and $f'(x') \in \mathcal{Y}'$ is closed, we may replace $\mathcal{Y}$ with $\overline{\{g(f'(x'))\}}$ and $\mathcal{X}$ with $\overline{\{g'(x')\}}$ where $g'\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$. Since f is strongly étale, Proposition 2.7(2) implies that f is in fact an isomorphism, and the desired properties of f' are immediate. □

3. General existence results

In this section, we prove Theorem 1.2, Theorem 1.3 and Proposition 1.4.

3.1. Existence via local quotient presentations.

We now prove Theorem 1.2.

Proposition 3.1. *Let* $\mathcal{X}$ *be an algebraic stack of finite type over an algebraically closed field* k *which has affine diagonal. Suppose that:*

(1) *there exists an affine, strongly étale, and surjective morphism* $f\colon \mathcal{X}_1 \to \mathcal{X}$ *from an algebraic stack* $\mathcal{X}_1$ *admitting a good moduli space* $\phi_1\colon \mathcal{X}_1 \to X_1$*; and*
(2) *for any* k*-point* $x \in \mathcal{X}$*, the closed substack* $\overline{\{x\}}$ *admits a good moduli space.*

Then $\mathcal{X}$ *admits a good moduli space* $\phi\colon \mathcal{X} \to X$*.*

Proof. Set $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. By Lemma 2.9, the projections $p_1, p_2\colon \mathcal{X}_2 \to \mathcal{X}_1$ are strongly étale. As f is affine, there exists a good moduli space $\phi_2\colon \mathcal{X}_2 \to X_2$ with projections $q_1, q_2\colon X_2 \to X_1$. Similarly, the algebraic stack $\mathcal{X}_3 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ admits a good moduli space $\phi_3\colon \mathcal{X}_3 \to X_3$

and the three projections $\mathcal{X}_3 \to \mathcal{X}_2$ are strongly étale. By Proposition 2.7(2), the induced diagram

$$\begin{array}{ccccccc} \mathcal{X}_3 & \Rrightarrow & \mathcal{X}_2 & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \mathcal{X}_1 & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \\ X_3 & \Rrightarrow & X_2 & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & X_1 & & \end{array}$$

is Cartesian. Moreover, by the universality of good moduli spaces, there is an induced identity map $X_1 \to X_2$, an inverse $X_2 \to X_2$ and a composition $X_2 \times_{q_1, X_1, q_2} X_2 \to X_2$ giving $X_2 \rightrightarrows X_1$ an étale groupoid structure.

To check that $\Delta\colon X_2 \to X_1 \times X_1$ is a monomorphism, it suffices to check that there is a unique pre-image of $(x_1, x_1) \in X_1 \times X_1$ where $x_1 \in X_1(k)$. Let $\xi_1 \in \mathcal{X}_1$ be the unique closed point in $\phi_1^{-1}(x_1)$. Since $\mathcal{X}_1 \to \mathcal{X}$ is stabilizer preserving at ξ_1, we can set $G := \mathrm{Aut}_{\mathcal{X}_1(k)}(\xi_1) \simeq \mathrm{Aut}_{\mathcal{X}(k)}(f(\xi_1))$. There are diagrams

$$\begin{array}{ccc} BG & \longrightarrow & BG \times BG \\ \downarrow & & \downarrow \\ \mathcal{X}_2 & \longrightarrow & \mathcal{X}_1 \times \mathcal{X}_1 \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{(p_1, p_2)} & \mathcal{X}_1 \times \mathcal{X}_1 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1 \times \phi_1} \\ X_2 & \xrightarrow{\Delta} & X_1 \times X_1 \end{array}$$

where the squares in the left diagram are Cartesian. Suppose $x_2 \in X_2(k)$ is a preimage of (x_1, x_1) under $\Delta\colon X_2 \to X_1 \times X_1$. Let $\xi_2 \in \mathcal{X}_2$ be the unique closed point in $\phi_2^{-1}(x_2)$. Then $(p_1(\xi_2), p_2(\xi_2)) \in \mathcal{X}_1 \times \mathcal{X}_1$ is closed and is therefore the unique closed point (ξ_1, ξ_1) in the $(\phi_1 \times \phi_1)^{-1}(x_1, x_1)$. But by Cartesianness of the left diagram, the point ξ_2 is the unique preimage of (ξ_1, ξ_1) under $\mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_1$. It follows that the point x_2 is the unique preimage of (x_1, x_1).

Since $X_2 \times_{q_1, X_1, q_2} X_2 \to X_2$ is an étale equivalence relation, there exists an algebraic space quotient X and induced maps $\phi\colon \mathcal{X} \to X$ and $X_1 \to X$. Consider

$$\begin{array}{ccccc} \mathcal{X}_2 & \longrightarrow & \mathcal{X}_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_1 & \longrightarrow & \mathcal{X} & \longrightarrow & X \end{array}$$

Since $\mathcal{X}_2 \simeq \mathcal{X}_1 \times_{X_1} X_2$ and $X_2 \simeq X_1 \times_X X_1$, the left and outer squares above are Cartesian. Since $\mathcal{X}_1 \to \mathcal{X}$ is étale and surjective, it follows that the right square is Cartesian. By descent ([Alp13, Proposition 4.7]), $\phi\colon \mathcal{X} \to X$ is a good moduli space. □

Proof of Theorem 1.2. After taking a disjoint union of finitely many local quotient presentations, there exists a strongly étale, affine and surjective morphism $f\colon \mathcal{W} \to \mathcal{X}$ where $\mathcal{W}$ admits a good moduli space. The theorem now follows from Proposition 3.1.

3.2. **Existence via local VGIT.** To prove Theorem 1.3, we will need some preliminary lemmas.

Lemma 3.2. *Let G be a linearly reductive group acting on an affine variety X of finite type over an algebraically closed field k. Let $\chi\colon G \to \mathbb{G}_m$ be a non-trivial character. Let $\lambda\colon \mathbb{G}_m \to G$ be a one-parameter subgroup and $x \in X^-_\chi(k)$ such that $x_0 = \lim_{t\to 0} \lambda(t)\cdot x \in X^G$ is fixed by G. Then $\langle \chi, \lambda\rangle > 0$.*

Proof. As $x \in X^-_\chi$, we have $\langle \chi, \lambda\rangle \geq 0$ by the Hilbert-Mumford criterion (c.f. [AFSv14, Proposition 3.5]). Suppose $\langle \chi, \lambda\rangle = 0$. Considering the action of G on $X \times \mathbb{A}^1$ induced by χ via $g\cdot(x,s) = (g\cdot x, \chi(g)^{-1}\cdot s)$, we obtain

$$\lim_{t\to 0} \lambda(t)\cdot(x,1) = (x_0, 1) \in X^G \times \mathbb{A}^1.$$

But X^G is contained in the unstable locus $X \setminus X^-_\chi$ since χ is a nontrivial linearization. It follows that $\overline{G\cdot(x,1)} \cap (X^G \times \{0\}) \neq \emptyset$ which contradicts $x \in X^-_\chi$ (c.f. [AFSv14, Remark 3.2]). ☐

Lemma 3.3. [AFSv14, Proposition 3.9] *Let G be a linearly reductive group with character $\chi\colon G \to \mathbb{G}_m$ and $h\colon \operatorname{Spec} A = X \to Y = \operatorname{Spec} B$ be a G-invariant morphism of affine schemes of finite type over an algebraically closed field k. Assume that $A = B \otimes_{B^G} A^G$. Then $h^{-1}(Y^+_\chi) = X^+_\chi$ and $h^{-1}(Y^-_\chi) = X^-_\chi$.*

Remark 3.4. While [AFSv14, Propositions 3.5 and 3.9] have an underlying characteristic 0 hypothesis, it is immediate that the proofs carry over to any characteristic.

Lemma 3.5. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field k, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Let $\mathcal{X}^+, \mathcal{X}^- \subset \mathcal{X}$ be open substacks, and let $\mathcal{Z}^+ = \mathcal{X} \smallsetminus \mathcal{X}^+$, $\mathcal{Z}^- = \mathcal{X} \smallsetminus \mathcal{X}^-$ be their reduced complements. Suppose that for all closed points $x \in \mathcal{X}$, there exists a local quotient presentation $f\colon \mathcal{W} \to \mathcal{X}$ around x and a Cartesian diagram*

$$\begin{array}{ccccc} \mathcal{W}^+ & \hookrightarrow & \mathcal{W} & \hookleftarrow & \mathcal{W}^- \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \mathcal{X}^+ & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}^- \end{array} \tag{3.1}$$

where $\mathcal{W}^+ = \mathcal{W}^+_{\mathcal{L}}$ and $\mathcal{W}^- = \mathcal{W}^-_{\mathcal{L}}$ are the VGIT chambers of $\mathcal{W}$ with respect to $\mathcal{L}$. Then

(1) *if $z \in \mathcal{X}^+(k) \cap \mathcal{X}^-(k)$, then the closure of z in $\mathcal{X}$ is contained in $\mathcal{X}^+ \cap \mathcal{X}^-$, and*
(2) *if $z \in \mathcal{X}(k)$ is a closed point, then either $z \in \mathcal{X}^+ \cap \mathcal{X}^-$ or $z \in \mathcal{Z}^+ \cap \mathcal{Z}^-$.*

Proof. For (1), if the closure of z in $\mathcal{X}$ is not contained in $\mathcal{X}^+ \cap \mathcal{X}^-$, then there exists an isotrivial specialization $z \rightsquigarrow x$ to a closed point in $\mathcal{X} \setminus (\mathcal{X}^+ \cap \mathcal{X}^-)$. Choose a local quotient presentation $f\colon \mathcal{W} = [W/G_x] \to \mathcal{X}$ around x such that (3.1) is Cartesian. Since $f^{-1}(x) \not\subset \mathcal{W}^+ \cap \mathcal{W}^-$, the character $\chi = \mathcal{L}|_{BG_x}$ is non-trivial. By the Hilbert-Mumford criterion ([Mum65, Theorem 2.1]), there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G_x$ such that $\lim_{t\to 0} \lambda(t)\cdot w = w_0$ where $w \in W$ and $w_0 \in W^{G_x}$ are points over z and x, respectively. As $w \in W^+_\chi \cap W^-_\chi$ and $w_0 \in W^{G_x}$, by applying Lemma 3.2 twice with the characters χ and χ^{-1}, we see that both $\langle \chi, \lambda\rangle < 0$ and $\langle \chi, \lambda\rangle > 0$, a contradiction.

For (2), choose a local quotient presentation $f\colon (\mathcal{W}, w) \to (\mathcal{X}, z)$ around z with $\mathcal{W} = [W/G_x]$. Let $\chi = \mathcal{L}|_{BG_x}$ be the character of $\mathcal{L}$. Since $w \in W^{G_x}$, the point w is semistable with respect to χ if and only if χ is trivial. It follows that $w \in \mathcal{W}^+ \cap \mathcal{W}^-$ if χ is trivial and $w \notin \mathcal{W}^+ \cup \mathcal{W}^-$ otherwise. ☐

Proof of Theorem 1.3. We show that $\mathcal{X}$ has a good moduli space by verifying the hypotheses of Theorem 1.2. Let $x_0 \in \mathcal{X}$ be a closed point. By Lemma 3.5(2), we have either $x_0 \in \mathcal{X}^+ \cap \mathcal{X}^-$ or $x_0 \in \mathcal{Z}^+ \cap \mathcal{Z}^-$. Suppose first that $x_0 \in \mathcal{X}^+ \cap \mathcal{X}^-$. Since $\mathcal{X}^+$ admits a good moduli space, Proposition 2.7(3) implies we may choose a local quotient presentation $f\colon \mathcal{W} \to \mathcal{X}^+$ around x_0 which is strongly étale. By applying Lemma 2.8, we may shrink further to assume that $f(\mathcal{W}) \subset \mathcal{X}^+ \cap \mathcal{X}^-$. Then Lemma 3.5(1) implies that the composition $f\colon \mathcal{W} \to \mathcal{X}^+ \hookrightarrow \mathcal{X}$ is also strongly étale.

On the other hand, suppose $x_0 \in \mathcal{Z}^+ \cap \mathcal{Z}^-$. Choose a local quotient presentation $f\colon (\mathcal{W}, w_0) \to (\mathcal{X}, x_0)$ around x_0 inducing a Cartesian diagram

$$\begin{array}{ccccc} \mathcal{W}^+ & \hookrightarrow & \mathcal{W} & \hookleftarrow & \mathcal{W}^- \\ \downarrow & & \downarrow f & & \downarrow \\ \mathcal{X}^+ & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}^- \end{array} \tag{3.2}$$

where $\mathcal{W}^+ = \mathcal{W}^+_{\mathcal{L}}$ and $\mathcal{W}^- = \mathcal{W}^-_{\mathcal{L}}$. We claim that, after shrinking suitably, we may assume that f is strongly étale. In proving this claim, we make implicit repeated use of Lemma 2.8 in conjunction with Lemma 3.3 to argue that if $\mathcal{W}' \subset \mathcal{W}$ is an open substack containing w_0, there exists open substack $\mathcal{W}'' \subset \mathcal{W}'$ containing w_0 such that $\mathcal{W}'' \to \mathcal{X}$ is a local quotient presentation around x_0 inducing a Cartesian diagram as in (3.2).

Using the hypothesis that $\mathcal{Z}^+, \mathcal{Z}^-$, and $\mathcal{X}^+$ admit good moduli spaces, we will first show that f may be chosen to satisfy:

(A) $f|_{f^{-1}(\mathcal{Z}^+)}$, $f|_{f^{-1}(\mathcal{Z}^-)}$ is strongly étale, and
(B) $f|_{\mathcal{W}^+}\colon \mathcal{W}^+ \to \mathcal{X}^+$ is strongly étale.

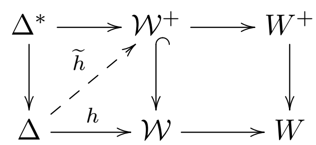
If f satisfies (A) and (B), then f is also strongly étale. Indeed, if $w \in \mathcal{W}$ is a closed point, then either $w \in f^{-1}(\mathcal{Z}^+) \cup f^{-1}(\mathcal{Z}^-)$ or $w \in f^{-1}(\mathcal{X}^+) \cap f^{-1}(\mathcal{X}^-)$. In the former case, (A) immediately implies that f is stabilizer preserving at w and $f(w)$ is closed in $\mathcal{X}$. In the latter case, (B) implies that f is stabilizer preserving at w and that $f(w)$ is closed in $\mathcal{X}^+$. Since $f(w) \in \mathcal{X}^+ \cap \mathcal{X}^-$, Lemma 3.5(1) implies that $f(w)$ remains closed in $\mathcal{X}$.

It remains to show that f can be chosen to satisfy (A) and (B). For (A), Proposition 2.7(3) implies the existence of an open substack $\mathcal{Q} \subset f^{-1}(\mathcal{Z}^+)$ containing w_0 such that $f|_{\mathcal{Q}}$ is strongly étale. After shrinking $\mathcal{W}$ suitably, we may assume $\mathcal{W} \cap f^{-1}(\mathcal{Z}^+) \subset \mathcal{Q}$. One argues similarly for $f|_{f^{-1}(\mathcal{Z}^-)}$.

For (B), Proposition 2.7(3) implies there exists an open substack $\mathcal{U} \subset \mathcal{W}^+$ such that $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}^+$ is strongly étale and, moreover, that $\mathcal{U}$ contains all closed points $w \in \mathcal{W}^+$ such that $f|_{\mathcal{W}^+}\colon \mathcal{W}^+ \to \mathcal{X}^+$ is strongly étale at w. Let $\mathcal{V} = \mathcal{W}^+ \setminus \mathcal{U}$ and let $\overline{\mathcal{V}}$ be the closure of $\mathcal{V}$ in $\mathcal{W}$. We claim that $w_0 \notin \overline{\mathcal{V}}$. Once this is established, we may replace $\mathcal{W}$ by an appropriate open substack of $\mathcal{W} \setminus \overline{\mathcal{V}}$ to obtain a local quotient presentation satisfying (B). Suppose, by way of contradiction, that $w_0 \in \overline{\mathcal{V}}$. Then there exists a specialization diagram

$$\begin{array}{ccc} \operatorname{Spec} K = \Delta^* & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R = \Delta & \xrightarrow{h} & \mathcal{W} \end{array}$$

such that $h(0) = w_0$. There exist good moduli spaces $\mathcal{W} \to W$ and $\mathcal{W}^+ \to W^+$, and the induced morphism $W^+ \to W$ is proper. Since the composition $\mathcal{W}^+ \to W^+ \to W$ is universally closed, there exists, after an extension of the fraction field K, a diagram



and a lift $\widetilde{h}\colon \Delta \to \mathcal{W}^+$ that extends $\Delta^* \to \mathcal{W}^+$ with $\widetilde{w} = \widetilde{h}(0) \in \mathcal{W}^+$ closed. There is an isotrivial specialization $\widetilde{w} \rightsquigarrow w_0$. It follows from Lemma 3.5(1) that $\widetilde{w} \in f^{-1}(\mathcal{Z}^-)$. By assumption (A), $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}^+$ is strongly étale at $\widetilde{w}$ so that $\widetilde{w} \in \mathcal{U}$. On the other hand, the generic point of the specialization $\widetilde{h}\colon \Delta \to \mathcal{W}^+$ lands in $\mathcal{V}$ so that $\widetilde{w} \in \mathcal{V}$, a contradiction. Thus, $w_0 \notin \overline{\mathcal{V}}$ as desired.

We have now shown that $\mathcal{X}$ satisfies condition (1) in Theorem 1.2, and it remains to verify condition (2). Let $x \in \mathcal{X}(k)$. If $x \in \mathcal{Z}^+$ (resp., $x \in \mathcal{Z}^-$), then $\overline{\{x\}} \subset \mathcal{Z}^+$ (resp., $\overline{\{x\}} \subset \mathcal{Z}^-$). Therefore, since $\mathcal{Z}^+$ (resp., $\mathcal{Z}^-$) admits a good moduli space, so does $\overline{\{x\}}$. On the other hand, if $x \in \mathcal{X}^- \cap \mathcal{X}^+$, then Lemma 3.5(1) implies the closure of x in $\mathcal{X}$ is contained in $\mathcal{X}^+$. Since $\mathcal{X}^+$ admits a good moduli space, so does $\overline{\{x\}}$. Now Theorem 1.2 implies that $\mathcal{X}$ admits a good moduli space $\phi\colon \mathcal{X} \to X$.

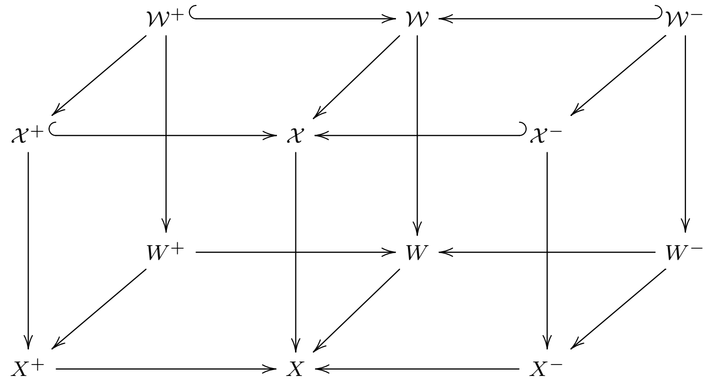
Next, we will apply Proposition 3.1 to show that $\mathcal{X}^-$ admits a good moduli space. Let $x \in \mathcal{X}^-$ be a closed point and $x \rightsquigarrow x_0$ be the isotrivial specialization to the unique closed point $x_0 \in \mathcal{X}$ in its closure. By Proposition 2.7, there exists a strongly étale local quotient presentation $f\colon \mathcal{W} \to \mathcal{X}$ around x_0 inducing a Cartesian diagram as in (1.2). By Lemma 2.9, the base change $f^-\colon \mathcal{W}^- \to \mathcal{X}^-$ is strongly étale. As $\mathcal{W}^-$ admits a good moduli space, we may shrink $\mathcal{W}^-$ further so that $f^-\colon \mathcal{W}^- \to \mathcal{X}^-$ is a strongly étale neighborhood of x.

It remains to check that if $x \in \mathcal{X}^-(k)$ is any point, then its closure $\overline{\{x\}}$ in $\mathcal{X}^-$ admits a good moduli space. Let $x \rightsquigarrow x_0$ be the isotrivial specialization to the unique closed point $x_0 \in \mathcal{X}$ in the closure of x. We claim in fact that $\phi^{-1}(\phi(x_0)) \cap \mathcal{X}^-$ admits a good moduli space, which in turn clearly implies that $\overline{\{x\}} \subset \mathcal{X}^-$ does as well. We can choose a local quotient presentation $f\colon (\mathcal{W}, w_0) \to (\mathcal{X}, x_0)$ around x_0 inducing a Cartesian diagram as in (1.2). After shrinking, we may assume by Proposition 2.7(3) that f is strongly étale and we may also assume that w_0 is the unique preimage of x_0. If we set $\mathcal{Z} = \phi^{-1}(\phi(x_0))$, then $f|_{f^{-1}(\mathcal{Z})}\colon f^{-1}(\mathcal{Z}) \to \mathcal{Z}$ is in fact an isomorphism as both $f^{-1}(\mathcal{Z})$ and $\mathcal{Z}$ have $\operatorname{Spec} k$ as a good moduli space. As $\mathcal{W}^-$ admits a good moduli space, so does $\mathcal{W}^- \cap f^{-1}(\mathcal{Z}) = \mathcal{X}^- \cap \mathcal{Z}$. This establishes that $\mathcal{X}^-$ admits a good moduli space.

Finally, we argue that $X^+ \to X$ and $X^- \to X$ are proper and surjective. By taking a disjoint union of local quotient presentations and applying Proposition 2.7(3), there exists an affine, strongly étale, and surjective morphism $f\colon \mathcal{W} \to \mathcal{X}$ from an algebraic stack admitting a good moduli space $\mathcal{W} \to W$ such that $\mathcal{W} = \mathcal{X} \times_X W$. Moreover, if we set $\mathcal{W}^+ := f^{-1}(\mathcal{X}^+)$ and $\mathcal{W}^- := f^{-1}(\mathcal{X}^-)$, then $\mathcal{W}^+$ and $\mathcal{W}^-$ admit good moduli spaces W^+ and W^- such that

$W^- \to W$ and $W^+ \to W$ are proper and surjective. This gives commutative cubes

(3.3)



The same argument as in the proof of the claim that $\mathcal{X}^-$ admits a good moduli space shows that $f|_{\mathcal{W}^+}\colon \mathcal{W}^+ \to \mathcal{X}^+$ and $f|_{\mathcal{W}^-}\colon \mathcal{W}^- \to \mathcal{X}^-$ send closed points to closed points. By Proposition 2.7(2), the left and right faces are Cartesian squares. Since the top faces are also Cartesian, we have $\mathcal{W}^+ \simeq \mathcal{X}^+ \times_X W$ and $\mathcal{W}^- \simeq \mathcal{X}^- \times_X W$. In particular, $\mathcal{W}^+ \to X^+ \times_X W$ and $\mathcal{W}^- \to X^- \times_X W$ are good moduli spaces. By uniqueness of good moduli spaces, we have $X^+ \times_X W \simeq W^+$ and $X^- \times_X W \simeq W^-$. Since $W^+ \to W$ and $W^- \to W$ are proper and surjective, $X^+ \to X$ and $X^- \to X$ are proper and surjective by étale descent.

3.3. **Existence via finite covers.** In proving Proposition 1.4, we will appeal to the following lemma:

Lemma 3.6. *Consider a commutative diagram*

$$\mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow X \qquad (\mathcal{X} \to X)$$

of algebraic stacks of finite type over an algebraically closed field k of characteristic 0 *where X is an algebraic space. Suppose that:*

(1) *the morphism $\mathcal{X} \to \mathcal{Y}$ is finite and surjective;*
(2) *the morphism $\mathcal{X} \to X$ is cohomologically affine; and*
(3) *the algebraic stack $\mathcal{Y}$ is a global quotient stack with affine diagonal.*

Then $\mathcal{Y} \to X$ is cohomologically affine.

Proof. We may write $\mathcal{Y} = [V/\operatorname{GL}_n]$, where V is an algebraic space with an action of GL_n. Since $\mathcal{X} \to \mathcal{Y}$ is affine, $\mathcal{X}$ is the quotient stack $\mathcal{X} = [U/G]$ where $U = \mathcal{X} \times_{\mathcal{Y}} V$. Since $U \to \mathcal{X}$ is affine and $\mathcal{X} \to X$ is cohomologically affine, the morphism $U \to X$ is affine by Serre's criterion. The morphism $U \to V$ is finite and surjective and therefore by Chevalley's theorem, we can conclude that $V \to X$ is affine. Therefore $\mathcal{Y} \to X$ is cohomologically affine. □

Proof of Proposition 1.4. Let $\mathcal{Z}$ be the scheme-theoretic image of $\mathcal{X} \to X \times \mathcal{Y}$. Since $\mathcal{X} \to \mathcal{Y}$ is finite and X is separated, $\mathcal{X} \to \mathcal{Z}$ is finite. Since $\mathcal{Z}$ is a global quotient stack (as $\mathcal{Y}$ is), we may apply Lemma 3.6 to conclude that the projection $\mathcal{Z} \to X$ is cohomologically affine which implies that $\mathcal{Z}$ admits a separated good moduli space. The composition $\mathcal{Z} \hookrightarrow X \times \mathcal{Y} \to \mathcal{Y}$ is finite, surjective and stabilizer preserving at closed points. Therefore, by replacing $\mathcal{X}$ with $\mathcal{Z}$, to prove the proposition, we may assume that $f\colon \mathcal{X} \to \mathcal{Y}$ is stabilizer preserving at closed points.

We will now show that the hypotheses of Theorem 1.2 are satisfied. Let $y_0 \in \mathcal{Y}$ be a closed point and $g\colon (\mathcal{Y}', y_0') \to (\mathcal{Y}, y_0)$ be a local quotient presentation around y_0. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

We first note that g' is strongly étale at each point $x' \in f'^{-1}(y_0')$. Indeed, g' is stabilizer preserving at x' as g is stabilizer preserving at y_0', and $g'(x') \in \mathcal{X}$ is a closed as $f(g'(x')) \in \mathcal{Y}$ is closed. By Proposition 2.7, there exists an open substack $\mathcal{U}' \subset \mathcal{X}'$ containing the fiber of y_0' such that $g'|_{\mathcal{U}'}$ is strongly étale. Therefore, $y_0' \notin \mathcal{Z} = \mathcal{Y}' \setminus f'(\mathcal{X}' \setminus \mathcal{U}')$ and $g|_{\mathcal{Y}' \setminus \mathcal{Z}}$ is strongly étale. By shrinking further using Lemma 2.8, we obtain a local quotient presentation $g\colon \mathcal{Y}' \to \mathcal{Y}$ around y_0 which is strongly étale.

Finally, let $y \in \mathcal{Y}(k)$ and $x \in \mathcal{X}(k)$ be any preimage. Set $\mathcal{X}_0 = \overline{\{x\}} \subset \mathcal{X}$ and $\mathcal{Y}_0 = \overline{\{y\}} \subset \mathcal{Y}$. As $\mathcal{X}_0 \to \mathcal{Y}_0$ is finite and surjective, $\mathcal{X}_0 \to \operatorname{Spec} k$ is a good moduli space and $\mathcal{Y}_0$ is a global quotient stack, we may conclude using Lemma 3.6 that $\mathcal{Y}_0$ admits a good moduli space. Therefore, we may apply Theorem 1.2 to establish the proposition.

Remark. The hypothesis that X is separated in Proposition 1.4 is necessary. For example, let X be the affine line with 0 doubled and let $\mathbb{Z}_2$ act on X by swapping the points at 0 and fixing all other points. Then $X \to [X/\mathbb{Z}_2]$ satisfies the hypotheses but $[X/\mathbb{Z}_2]$ does not admit a good moduli space.

4. Application to $\overline{\mathcal{M}}_{g,n}(\alpha)$

In this section, we apply Theorem 1.3 to prove that the algebraic stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ admit good moduli spaces (Theorem 1.1). We already know that the inclusions

$$\overline{\mathcal{M}}_{g,n}(\alpha+\epsilon) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\alpha) \hookleftarrow \overline{\mathcal{M}}_{g,n}(\alpha-\epsilon)$$

arise from local VGIT with respect to $\delta - \psi$ ([AFSv14, Theorem 3.17]). Thus, it only remains to show that for each critical value $\alpha_c \in \{9/11, 7/10, 2/3\}$, the closed substacks

$$\overline{\mathcal{S}}_{g,n}(\alpha_c) := \overline{\mathcal{M}}_{g,n}(\alpha_c) \setminus \overline{\mathcal{M}}_{g,n}(\alpha_c+\epsilon)$$
$$\overline{\mathcal{H}}_{g,n}(\alpha_c) := \overline{\mathcal{M}}_{g,n}(\alpha_c) \setminus \overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon)$$

admit good moduli spaces. We will prove this statement by induction on g. As with the boundary strata of $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{H}}_{g,n}(\alpha_c)$ can be described (up to a finite cover) as a product of moduli spaces of α_c-stable curves of lower genus. Likewise, $\overline{\mathcal{S}}_{g,n}(\alpha_c)$ can be described (up to a finite cover) as stacky projective bundles over moduli spaces of α_c-stable curves of lower genus. We use induction to deduce that these products and projective bundles admit good moduli spaces, and then apply Proposition 1.4 to conclude that $\overline{\mathcal{S}}_{g,n}(\alpha_c)$ and $\overline{\mathcal{H}}_{g,n}(\alpha_c)$ admit good moduli spaces.

4.1. **Properties of α-stability.** We will make repeated use of the following elementary properties of α-stability.

Lemma 4.1 ([AFSv14, Lemma 2.17])**.**

(1) *If* $(\widetilde{C}_1, \{p_i\}_{i=1}^n, q_1)$ *and* $(\widetilde{C}_2, \{p_i\}_{i=1}^n, q_2)$ *are* α*-stable curves, then*

$$(\widetilde{C}_1, \{p_i\}_{i=1}^n, q_1) \cup (\widetilde{C}_2, \{p_i\}_{i=1}^n, q_2)/(q_1 \sim q_2)$$

is α*-stable.*

(2) *If* $(\widetilde{C}, \{p_i\}_{i=1}^n, q_1, q_2)$ *is an* α*-stable curve, then*

$$(\widetilde{C}, \{p_i\}_{i=1}^n, q_1, q_2)/(q_1 \sim q_2)$$

is α*-stable provided one of the following conditions hold:*

- q_1 *and* q_2 *lie on disjoint irreducible components of* $\widetilde{C}$*,*
- q_1 *and* q_2 *lie on distinct irreducible components of* $\widetilde{C}$*, and at least one of these components is not a smooth rational curve.*

Lemma 4.2 ([AFSv14, Lemma 2.19])**.** *Let* $(C, \{p_i\}_{i=1}^n)$ *be an* n*-pointed curve with* $\omega_C(\sum_i p_i)$ *ample, and suppose* $q \in C$ *is an* α_c*-critical singularity. Then the stable pointed normalization of* $(C, \{p_i\}_{i=1}^n)$ *at* $q \in C$ *is* α_c*-stable if and only if* $(C, \{p_i\}_{i=1}^n)$ *is* α_c*-stable.*

4.2. **Existence for** $\overline{\mathcal{S}}_{g,n}(\alpha_c)$**.** In this section, we use induction on g to prove that $\overline{\mathcal{S}}_{g,n}(\alpha_c)$ admits a good moduli space. The base case is handled by the following lemma.

Lemma 4.3. *We have:*

$$\begin{aligned} \overline{\mathcal{S}}_{1,1}(9/11) &\simeq B\mathbb{G}_m, \\ \overline{\mathcal{S}}_{1,2}(7/10) &\simeq B\mathbb{G}_m, \ \textit{and} \\ \overline{\mathcal{S}}_{2,1}(2/3) &\simeq [\mathbb{A}^1/\mathbb{G}_m], \ \textit{where } \mathbb{G}_m \textit{ acts with weight } 1. \end{aligned}$$

In particular, the algebraic stacks $\overline{\mathcal{S}}_{1,1}(9/11)$*,* $\overline{\mathcal{S}}_{1,2}(7/10)$*,* $\overline{\mathcal{S}}_{2,1}(2/3)$ *admit good moduli spaces.*

Proof. The algebraic stacks $\overline{\mathcal{S}}_{1,1}(9/11)$ and $\overline{\mathcal{S}}_{1,2}(7/10)$ each contain a unique $\mathbb{C}$-point, namely the $\frac{9}{11}$-atom and the $\frac{7}{10}$-atom, and each of these curves have a $\mathbb{G}_m$-automorphism group. The stack $\overline{\mathcal{S}}_{2,1}(2/3)$ contains two isomorphism classes of curves, namely the $\frac{2}{3}$-atom, and the rational ramphoid cuspidal curve with non-trivial crimping. We construct this stack explicitly as follows: Start with the constant family $(\mathbb{P}^1 \times \mathbb{A}^1, \infty \times \mathbb{A}^1)$, let c be a coordinate on $\mathbb{A}^1$, and t a coordinate on $\mathbb{P}^1 - \infty$. Now let $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathcal{C}$ be the map defined by the inclusion of algebras $\mathbb{C}[t^2 + ct^3, t^5] \subset \mathbb{C}[c, t]$ on the complement of the infinity section, and defined as an isomorphism on the complement of the zero section. Then $(\mathcal{C} \to \mathbb{A}^1, \infty \times \mathbb{A}^1)$ is a family of rational ramphoid cuspidal curves whose fiber over $0 \in \mathbb{A}^1$ is a $\frac{2}{3}$-atom. Furthermore, $\mathbb{G}_m$ acts on the base and total space of this family by $t \to \lambda^{-1}t, c \to \lambda c$, since the subalgebra $\mathbb{C}[t^2 + ct^3, t^5] \subset \mathbb{C}[c, t]$ is invariant under this action. Thus, the family descends to $[\mathbb{A}^1/\mathbb{G}_m]$ and there is an induced map $[\mathbb{A}^1/\mathbb{G}_m] \to \overline{\mathcal{M}}_{2,1}(2/3)$. This map is a locally closed immersion by [vdW10, Theorem 1.109], and the image is precisely $\overline{\mathcal{S}}_{2,1}(2/3)$. Thus, $\overline{\mathcal{S}}_{2,1}(2/3) \simeq [\mathbb{A}^1/\mathbb{G}_m]$ as desired. □

For higher values of (g, n), the key observation is that every curve in $\overline{\mathcal{S}}_{g,n}(\alpha_c)$ can be obtained from an α_c-stable curve by 'sprouting' an appropriate singularity. We make this precise in the following definition.

Definition 4.4. If (C, p_1) is a 1-pointed curve, we say that C' is a *(ramphoid) cuspidal sprouting* of (C, p_1) if C' contains a (ramphoid) cusp $q \in C'$, and the pointed normalization of C' at q is isomorphic to one of:

(a) (C, p_1); or
(b) $(C \cup \mathbb{P}^1, \infty)$ where C and $\mathbb{P}^1$ are glued nodally by identifying $p_1 \sim 0$.

If (C, p_1, p_2) is a 2-pointed curve, we say that C' is a *tacnodal sprouting* of (C, p_1, p_2) if C' contains a tacnode $q \in C'$, and the pointed normalization of C' at q is isomorphic to one of:

(a) (C, p_1, p_2);
(b) $(C \cup \mathbb{P}^1, p_1, \infty)$ where C and $\mathbb{P}^1$ are glued nodally by identifying $p_2 \sim 0$;
(c) $(C \cup \mathbb{P}^1, p_2, \infty)$ where C and $\mathbb{P}^1$ are glued nodally by identifying $p_1 \sim 0$; or
(d) $(C \cup \mathbb{P}^1 \cup \mathbb{P}^1, \infty_1, \infty_2)$ where C is glued nodally to two copies of $\mathbb{P}^1$ along $p_1 \sim 0, p_2 \sim 0$.

In this definition, we allow the possibility that $(C, p_1, p_2) = (C_1, p_1) \coprod (C_2, p_2)$ is disconnected, with one marked point on each connected component.

If (C, p_1) is a 1-pointed curve, we say that C' is a *one-sided tacnodal sprouting* of (C, p_1) if C' contains a tacnode $q \in C'$, and the pointed normalization of C' at q is isomorphic to one of:

(a) $(C, p_1) \coprod (\mathbb{P}^1, 0)$; or
(b) $(C \cup \mathbb{P}^1, \infty) \coprod (\mathbb{P}^1, 0)$ where C and $\mathbb{P}^1$ are glued nodally by identifying $p_1 \sim 0$.

Remark. Suppose C' is a cuspidal sprouting, one-sided tacnodal sprouting or ramphoid cuspidal sprouting of (C, p_1) (resp., tacnodal sprouting of (C, p_1, p_2)) with α_c-critical singularity $q \in C'$. Then (C, p_1) (resp., (C, p_1, p_2)) is the stable pointed normalization of C' along q. By Lemma 4.2, C' is α_c-stable if and only if (C, p_1) (resp., (C, p_1, p_2)) is α_c-stable.

Lemma 4.5. *Fix* $\alpha_c \in \{9/11, 7/10, 2/3\}$*, and suppose* $(C, \{p_i\}_{i=1}^n) \in \overline{\mathcal{S}}_{g,n}(\alpha_c)$.

(1) *If* $(g,n) \neq (1,1)$*, then* $(C, \{p_i\}_{i=1}^n)$ *is a cuspidal sprouting of a* $9/11$*-stable curve in* $\overline{\mathcal{M}}_{g-1,n+1}(9/11)$.
(2) *If* $(g,n) \neq (1,2)$*, then one of the following holds:*
   (a) $(C, \{p_i\}_{i=1}^n)$ *is a tacnodal sprouting of a* $7/10$*-stable curve in* $\overline{\mathcal{M}}_{g-2,n+2}(7/10)$*;*
   (b) $(C, \{p_i\}_{i=1}^n)$ *is a tacnodal sprouting of a* $7/10$*-stable curve in*
   $$\overline{\mathcal{M}}_{g-i-1,n-m+1}(7/10) \times \overline{\mathcal{M}}_{i,m+1}(7/10); \quad \textit{or}$$
   (c) $(C, \{p_i\}_{i=1}^n)$ *is a one-sided tacnodal sprouting of a* $7/10$*-stable curve in* $\overline{\mathcal{M}}_{g-1,n}(7/10)$.
(3) *If* $(g,n) \neq (2,1)$*, then* $(C, \{p_i\}_{i=1}^n)$ *is a ramphoid cuspidal sprouting of a* $2/3$*-stable curve in* $\overline{\mathcal{M}}_{g-2,n+2}(2/3)$.

Proof. If $(C, \{p_i\}_{i=1}^n) \in \overline{\mathcal{S}}_{g,n}(\alpha_c)$, then $(C, \{p_i\}_{i=1}^n)$ contains an α_c-critical singularity $q \in C$. The stable pointed normalization of $(C, \{p_i\}_{i=1}^n)$ along q is well-defined by our hypothesis on (g,n), and is α_c-stable by Lemma 4.2. □

Lemma 4.5 gives a set-theoretic description of $\overline{\mathcal{S}}_{g,n}(\alpha_c)$, and we must now augment this to a stack-theoretic description. This means constructing universal families of cuspidal, tacnodal, and ramphoid cuspidal sproutings. A nearly identical construction was carried out in [Smy11b] for elliptic m-fold points (in particular, cusps and tacnodes), and for all curve singularities in [vdW10]. The only key difference is that here we allow *all* branches to sprout $\mathbb{P}^1$'s rather than a restricted subset. Therefore, we obtain non-separated, stacky compactifications (rather than Deligne-Mumford compactifications) of the associated crimping stack of the singularity.

In what follows, if $\mathcal{C} \to T$ is any family of curves with a section τ, we say that $\mathcal{C}$ *has an A_k-singularity along* τ if, étale locally on the base, the Henselization of $\mathcal{C}$ along τ is isomorphic to the Henselization of $T \times \mathbb{C}[x, y]/(y^2 - x^{k+1})$ along the zero section (cf. [vdW10, Definition 1.64]).

Definition 4.6. Let $\text{Sprout}_{g,n}(A_k)$ denote the stack of flat families of curves $(\mathcal{C} \to T, \{\sigma_i\}_{i=1}^{n+1})$ satisfying

(1) $(\mathcal{C} \to T, \{\sigma_i\}_{i=1}^{n})$ is a T-point of $\mathcal{U}_{g,n}(A_k)$; and
(2) $\mathcal{C}$ has an A_k-singularity along σ_{n+1}.

The fact that $\text{Sprout}_{g,n}(A_k)$ is an algebraic stack over (Schemes/$\mathbb{C}$) is verified in [vdW10]. There are obvious forgetful functors

$$F_k \colon\ \text{Sprout}_{g,n}(A_k) \to \mathcal{U}_{g,n}(A_k),$$

given by forgetting the section σ_{n+1}.

Proposition 4.7. *F_k is representable and finite.*

Proof. It is clear that F_k is representable. The fact that F_k is quasi-finite follows from the observations that a curve $(C, \{p_i\}_{i=1}^{n})$ in $\mathcal{U}_{g,n}(A_k)$ has only a finite number of A_k-singularities and that for a $\mathbb{C}$-point $x \in \text{Sprout}_{g,n}(A_k)$, the induced map $\text{Aut}_{\text{Sprout}_{g,n}(A_k)}(x) \to \text{Aut}_{\mathcal{U}_{g,n}(A_k)}(F_k(x))$ on automorphism groups has finite cokernel. To show that F_k is finite, it now suffices to verify the valuative criterion for properness: Let Δ be the spectrum of a discrete valuation ring, let Δ^* denote the spectrum of its fraction field, and suppose we are given a diagram

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \text{Sprout}_{g,n}(A_k) \\ \downarrow & & \downarrow{\scriptstyle F_k} \\ \Delta & \longrightarrow & \mathcal{U}_{g,n}(A_k) \end{array}$$

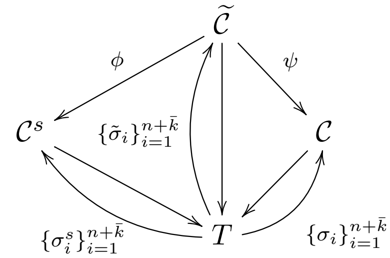
This corresponds to a diagram of families,

$$\begin{array}{ccc} \mathcal{C}_{\Delta^*} & \longrightarrow & \mathcal{C} \\ {\scriptstyle \sigma_{n+1}}\uparrow\ \downarrow{\scriptstyle \pi_{\Delta^*}} & & {\scriptstyle \pi}\downarrow \\ \Delta^* & \longrightarrow & \Delta \end{array}$$

such that $\mathcal{C}_{\Delta^*}$ has A_k-singularity along σ_{n+1}. Since $\mathcal{C} \to \Delta$ is proper, σ_{n+1} extends uniquely to a section of π, and since the limit of an A_k-singularity in $\mathcal{U}_{g,n}(A_k)$ is necessarily an A_k-singularity, $\mathcal{C}$ has an A_k-singularity along σ_{n+1}. This induces a unique lift $\Delta \to \text{Sprout}_{g,n}(A_k)$, cf. [vdW10, Theorem 1.109]. □

The algebraic stacks $\text{Sprout}_{g,n}(A_k)$ also admit stable pointed normalization functors, given by forgetting the crimping data of the singularity along σ_{n+1}. To be precise, if $(\mathcal{C} \to T, \{\sigma\}_{i=1}^{n+1})$

is a T-point of $\mathrm{Sprout}_{g,n}(A_k)$, there exists a commutative diagram



satisfying:

(1) $(\widetilde{\mathcal{C}} \to T, \{\tilde{\sigma}_i\}_{i=1}^{n+\bar{k}})$ is a family of $(n+\bar{k})$-pointed curves, where $\bar{k} \in \{1,2\}$;
(2) ψ is the pointed normalization of $\mathcal{C}$ along σ_{n+1}, i.e. ψ is finite and restricts to an isomorphism on the open set $\widetilde{\mathcal{C}} - \cup_{i=1}^{\bar{k}} \tilde{\sigma}_{n+i}$; and
(3) ϕ is the stabilization of $(\widetilde{\mathcal{C}}, \{\tilde{\sigma}_i\}_{i=1}^{n+\bar{k}})$, i.e. ϕ is the morphism associated to a high multiple of the line bundle $\omega_{\widetilde{\mathcal{C}}/T}(\Sigma_{i=1}^{n+\bar{k}} \tilde{\sigma}_i)$.

Remark 4.8. Issues arise when defining the stable pointed normalization for (g,n) small relative to k. From now on, we assume $k \in \{2,3,4\}$, and that $(g,n) \neq (1,1),(1,2),(2,1)$ when $k = 2,3,4$, respectively. This ensures that the stabilization morphism ϕ is well-defined. Indeed, under these hypotheses, $\omega_{\widetilde{\mathcal{C}}}(\Sigma_i \tilde{\sigma}_i)$ will be relatively big and nef, and the only components of fibers of $(\widetilde{\mathcal{C}}, \{\tilde{\sigma}_i\}_{i=1}^{n+\bar{k}})$ on which $\omega_{\widetilde{\mathcal{C}}}(\Sigma_i \tilde{\sigma}_i)$ has degree zero will be $\mathbb{P}^1$'s that meet the rest of the curve in a single node and are marked by one of the sections $\{\tilde{\sigma}_{n+i}\}_{i=1}^{\bar{k}}$. The effect of ϕ is simply to blow-down these $\mathbb{P}^1$'s.

Since normalization and stabilization are canonically defined, the association

$$(\mathcal{C} \to T, \{\sigma_i\}_{i=1}^{n}) \mapsto (\mathcal{C}^s \to T, \{\sigma_i^s\}_{i=1}^{n+\bar{k}})$$

is functorial, and we obtain normalization functors:

$$N_2\colon \mathrm{Sprout}_{g,n}(A_2) \to \mathcal{U}_{g-1,n+1}(A_2),$$
$$N_3\colon \mathrm{Sprout}_{g,n}(A_3) \to \coprod_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \left(\mathcal{U}_{g_1,n_1+1}(A_3) \times \mathcal{U}_{g_2,n_2+1}(A_3)\right) \coprod \mathcal{U}_{g-2,n+2}(A_3) \coprod \mathcal{U}_{g-1,n+1}(A_3),$$
$$N_4\colon \mathrm{Sprout}_{g,n}(A_4) \to \mathcal{U}_{g-2,n+1}(A_4).$$

The connected components of the range of N_3 correspond to the different possibilities for the stable pointed normalization of $\mathcal{C}$ along σ_{n+1}. Note that the last case $\mathcal{U}_{g-1,n+1}(A_3)$ corresponds to a one-sided tacnodal sprouting, i.e. one connected component of the pointed normalization of $\mathcal{C}$ along σ_{n+1} is a family of 2-pointed $\mathbb{P}^1$'s. It is convenient to distinguish these possibilities by defining:

$$\mathrm{Sprout}_{g,n}^{ns}(A_3) = N_3^{-1}(\mathcal{U}_{g-2,n+2}(A_3)),$$
$$\mathrm{Sprout}_{g,n}^{g_1,n_1}(A_3) = N_3^{-1}\left(\mathcal{U}_{g_1,n_1+1}(A_3) \times \mathcal{U}_{g_2,n_2+1}(A_3)\right), \text{ and}$$
$$\mathrm{Sprout}_{g,n}^{0,2}(A_3) = N_3^{-1}(\mathcal{U}_{g-1,n+1}(A_3)).$$

The following key proposition shows that N_k makes $\mathrm{Sprout}_{g,n}(A_k)$ a stacky projective bundle over the moduli stack of pointed normalizations.

We will use the following notation: If $\mathcal{E}$ is a locally free sheaf on an algebraic stack $\mathcal{X}$, we let $V(\mathcal{E})$ denote the total space of the associated vector bundle, $[V(\mathcal{E})/\mathbb{G}_m]$ the quotient stack for the natural action of $\mathbb{G}_m$ on the fibers of $V(\mathcal{E})$, and $p\colon [V(\mathcal{E})/\mathbb{G}_m] \to T$ the natural projection.

Proposition 4.9. *In the following statements, we let* $(\pi\colon \mathcal{C} \to \mathcal{U}_{g,n}(A_k), \{\sigma_i\}_{i=1}^n)$ *denote the universal family over* $\mathcal{U}_{g,n}(A_k)$*, and* $(\pi\colon \mathcal{C} \to \mathcal{U}_{g_1,n_1}(A_k) \times \mathcal{U}_{g_2,n_2}(A_k), \{\sigma_i\}_{i=1}^{n_1}, \{\tau_i\}_{i=1}^{n_2})$ *the universal family over* $\mathcal{U}_{g_1,n_1}(A_k) \times \mathcal{U}_{g_2,n_2}(A_k)$*.*

(1) *Let* $\mathcal{E}$ *be the invertible sheaf on* $\mathcal{U}_{g-1,n+1}(A_2)$ *defined by*
$$\mathcal{E} := \pi_*\left(\mathcal{O}_{\mathcal{C}}(-2\sigma_{n+1})/\mathcal{O}_{\mathcal{C}}(-3\sigma_{n+1})\right)$$
Then there exists an isomorphism
$$\gamma\colon [V(\mathcal{E})/\mathbb{G}_m] \simeq \mathrm{Sprout}_{g,n}(A_2)$$
such that $N_2 \circ \gamma = p$*.*

(2) *Let* $\mathcal{E}$ *be the locally free sheaf on* $\mathcal{U}_{g-2,n+2}(A_3)$ *defined by*
$$\mathcal{E} := \pi_*\left(\mathcal{O}_{\mathcal{C}}(-\sigma_{n+1})/\mathcal{O}_{\mathcal{C}}(-2\sigma_{n+1}) \oplus \mathcal{O}_{\mathcal{C}}(-\sigma_{n+2})/\mathcal{O}_{\mathcal{C}}(-2\sigma_{n+2})\right)$$
Then there exists an isomorphism
$$\gamma\colon [V(\mathcal{E})/\mathbb{G}_m] \simeq \mathrm{Sprout}_{g,n}^{ns}(A_3)$$
such that $N_3 \circ \gamma = p$*.*

(3) *Let* $\mathcal{E}$ *be the locally free sheaf on* $\mathcal{U}_{g_1,n_1+1}(A_3) \times \mathcal{U}_{g_2,n_1+1}(A_3)$ *defined by*
$$\mathcal{E} := \pi_*\left(\mathcal{O}_{\mathcal{C}}(-\sigma_{n_1+1})/\mathcal{O}_{\mathcal{C}}(-2\sigma_{n_1+1}) \oplus \mathcal{O}_{\mathcal{C}}(-\tau_{n_2+1})/\mathcal{O}_{\mathcal{C}}(-2\tau_{n_2+1})\right)$$
Then there exists an isomorphism
$$\gamma\colon [V(\mathcal{E})/\mathbb{G}_m] \simeq \mathrm{Sprout}_{g,n}^{g_1,n_1}(A_3)$$
such that $N_3 \circ \gamma = p$*.*

(4) *Let* $\mathcal{E}$ *be the locally free sheaf on* $\mathcal{U}_{g-1,n+1}(A_3)$ *defined by*
$$\mathcal{E} := \pi_*\left(\mathcal{O}_{\mathcal{C}}(-\sigma_{n+1})/\mathcal{O}_{\mathcal{C}}(-2\sigma_{n+1})\right)$$
Then there exists an isomorphism
$$\gamma\colon [V(\mathcal{E})/\mathbb{G}_m] \simeq \mathrm{Sprout}_{g,n}^{0,2}(A_3)$$
such that $N_3 \circ \gamma = p$*.*

(5) *Let* $\mathcal{E}$ *be the locally free sheaf on* $\mathcal{U}_{g-2,n+1}(A_4)$ *defined by*
$$\mathcal{E} := \pi_*\left(\mathcal{O}_{\mathcal{C}}(-2\sigma_{n+1})/\mathcal{O}_{\mathcal{C}}(-4\sigma_{n+1})\right)$$
Then there exists an isomorphism
$$\gamma\colon [V(\mathcal{E})/\mathbb{G}_m] \simeq \mathrm{Sprout}_{g,n}(A_4)$$
such that $N_4 \circ \gamma = p$*.*

Proof. We prove the hardest case (5), and leave the others as an exercise to the reader. To construct a map $\gamma\colon [V(\mathcal{E})/\mathbb{G}_m] \to \mathrm{Sprout}_{g,n}(A_4)$, we start with a family $(\pi\colon \mathcal{C} \to X, \{\sigma_i\}_{i=1}^{n+1})$ in $\mathcal{U}_{g-2,n+1}(A_4)$, and construct a family of ramphoid cuspidal sproutings over $[V(\mathcal{E}_X)/\mathbb{G}_m]$, where

$$\mathcal{E}_X := \pi_*\left(\mathcal{O}_{\mathcal{C}}(-2\sigma_{n+1})/\mathcal{O}_{\mathcal{C}}(-4\sigma_{n+1})\right).$$

Let $V := V(\mathcal{E}_X)$, $p\colon V \to X$ the natural projection, and $(\mathcal{C}_V \to V, \sigma_V)$ the family obtained from $(\mathcal{C} \to X, \sigma_{n+1})$ by base change along p. As the construction is local around σ_{n+1}, we will not keep track of $\{\sigma_i\}_{i=1}^{n}$ for the remainder of the argument. If we set $\mathcal{E}_V = p^*\mathcal{E}_X$, there exists a tautological section $e\colon \mathcal{O}_V \to \mathcal{E}_V$. Let $Z \subset V$ denote the divisor along which the composition

$$\mathcal{O}_V \to \mathcal{E}_V \to (\pi_V)_*\left(\mathcal{O}_{\mathcal{C}_V}(-2\sigma_V)/\mathcal{O}_{\mathcal{C}_V}(-3\sigma_V)\right)$$

vanishes, and let $\phi\colon \widetilde{\mathcal{C}} \to \mathcal{C}_V$ be the blow-up of $\mathcal{C}_V$ along $\sigma_V(Z)$. Since $\sigma_V(Z) \subset \mathcal{C}_V$ is a regular subscheme of codimension 2, the exceptional divisor E of the blow-up is a $\mathbb{P}^1$-bundle over $\sigma_V(Z)$. In other words, for all $z \in Z$, we have

$$\widetilde{\mathcal{C}}_z = \mathcal{C}_z \cup E_z = \mathcal{C}_z \cup \mathbb{P}^1.$$

Let $\tilde{\sigma}$ be the strict transform of σ_V on $\widetilde{\mathcal{C}}$, and observe that $\tilde{\sigma}$ passes through a smooth point of the $\mathbb{P}^1$ component in every fiber over Z. We will construct a map $\widetilde{\mathcal{C}} \to \mathcal{C}'$ which crimps $\tilde{\sigma}$ to a ramphoid cusp, and $\mathcal{C}' \to X$ will be the desired family of ramphoid cuspidal sproutings.

Setting $\widetilde{\pi}\colon \widetilde{\mathcal{C}} \to \mathcal{C}_V \to V$ and

$$\widetilde{\mathcal{E}} = \widetilde{\pi}_*\left(\mathcal{O}_{\widetilde{\mathcal{C}}}(-2\tilde{\sigma})/\mathcal{O}_{\widetilde{\mathcal{C}}}(-4\tilde{\sigma})\right)$$

we claim that e induces a section $\widetilde{e}\colon \mathcal{O}_V \to \widetilde{\mathcal{E}}$ with the property that the composition

$$\mathcal{O}_V \to \widetilde{\mathcal{E}} \to \widetilde{\pi}_*\left(\mathcal{O}_{\widetilde{\mathcal{C}}}(-2\tilde{\sigma})/\mathcal{O}_{\widetilde{\mathcal{C}}}(-3\tilde{\sigma})\right)$$

is never zero. To see this, let $U = \operatorname{Spec} R \subset X$ be an open affine along which $\mathcal{E}$ is trivial, and choose local coordinates on a, b on $p^{-1}(U) = \operatorname{Spec} R[a, b]$ such that the tautological section e is given by $at^2 + bt^3$, where t is a local equation for σ_V on $\mathcal{C}_V$. In these coordinates, ϕ is the blow-up along $a = t = 0$. Let $\tilde{a}, \tilde{t}$ be homogeneous coordinates for the blow-up and note that on the chart $\tilde{a} \neq 0$, $t' := \tilde{t}/\tilde{a}$ gives a local equation for $\tilde{\sigma}_V$. In these coordinates, ϕ is given by

$$(a, b, t) \to (a, b, at')$$

The section at^2+bt^3 pulls back to $a^3(t'^2+bt'^3)$, and $t'^2+bt'^3$ defines a section $\widetilde{e}$ of $\widetilde{\mathcal{E}}$ over $p^{-1}(U)$ with the stated property.

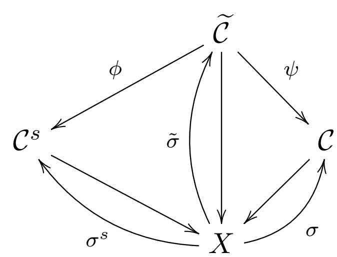
We will use $\widetilde{e}$ to construct a map $\psi\colon \widetilde{\mathcal{C}} \to \mathcal{C}'$ such that $\mathcal{C}'$ has a ramphoid cusp along $\psi \circ \tilde{\sigma}$. It is sufficient to define ψ locally around $\tilde{\sigma}$, so we may assume $\widetilde{\pi}$ is affine, i.e. $\widetilde{\mathcal{C}} := \operatorname{Spec}_V \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}$. We specify a sheaf of $\mathcal{O}_V$-subalgebras of $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}$ as follows: Consider the exact sequence

$$0 \to \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}_V}(-4\tilde{\sigma}) \to \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}_V}(-2\tilde{\sigma}) \to \widetilde{\mathcal{E}} \to 0$$

and let $\mathcal{F} \subset \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}$ be the sheaf of $\mathcal{O}_V$-subalgebras generated by any inverse image of $\widetilde{e}$ and all functions in $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}(-4\tilde{\sigma})$. We let $\psi\colon\ \operatorname{Spec}_V \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}} \to \mathcal{C}' := \operatorname{Spec}_V \mathcal{F}$ be the map corresponding to the inclusion $\mathcal{F} \subset \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}$. By construction, the inclusion of the complete local rings $\widehat{\mathcal{O}}_{C'_v,(\psi\circ\tilde{\sigma})(v)} \subset \widehat{\mathcal{O}}_{\widetilde{C}_v,\tilde{\sigma}(v)} \simeq \mathbb{C}[[t]]$ is of the form $\mathbb{C}[[t^2+bt^3, t^5]] \subset \mathbb{C}[[t]]$ for every $v \in V$. Observe that $\mathbb{C}[[t^2+bt^3, t^5]]$ is isomorphic to $\mathbb{C}[[x, y]]/(y^2 - x^5)$ as a $\mathbb{C}$-algebra.

Finally, we claim that $\mathcal{C}' \to V$ descends to a family of ramphoid cuspidal sproutings over the quotient stack $[V/\mathbb{G}_m]$. It suffices to show that the subsheaf $\mathcal{F} \subset \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}$ is invariant under the natural action of $\mathbb{G}_m$ on V. Using the same local coordinates introduced above, the sheaf $\mathcal{F}$ is given over the open set $\operatorname{Spec} R[a,b]$ by the $R[a,b]$-algebra generated by $t'^2+bt'^3$ and t'^5, where t' is a local equation for $\tilde{\sigma}$ on $\widetilde{\mathcal{C}}$. To see that this algebra is $\mathbb{G}_m$-invariant, note that the $\mathbb{G}_m$-action on $V = \operatorname{Spec} R[a,b]$ (acting with weight 1 on a and b) extends canonically to a $\mathbb{G}_m$-action on the blow-up, where $\mathbb{G}_m$ acts on $\tilde{a}$, $\tilde{t}$ with weight 1 and 0, respectively. Thus, $\mathbb{G}_m$ acts on $t' = \tilde{t}/\tilde{a}$ with weight -1, so that $t'^2 + bt'^3$ is a semi-invariant. It follows that the algebra generated by $t'^2 + bt'^3$ and t'^5 is $\mathbb{G}_m$-invariant as desired. Thus, we obtain a family $(\mathcal{C}' \to [V/\mathbb{G}_m], \psi \circ \tilde{\sigma})$ in $\mathrm{Sprout}_{g,n}(A_4)$ as desired.

To define an inverse map $\gamma^{-1}\colon\ \mathrm{Sprout}_{g,n}(A_4) \to [V/\mathbb{G}_m]$, we start with a family $(\mathcal{C} \to X, \sigma)$ in $\mathcal{U}_{g,n}(A_4)$ such that $\mathcal{C}$ has an A_4-singularity along σ. We must construct a map $X \to [V(\mathcal{E})/\mathbb{G}_m]$. By taking the stable pointed normalization of $\mathcal{C}$ along σ, we obtain a diagram



satisfying

(1) $(\widetilde{\mathcal{C}} \to X, \tilde{\sigma})$ is a family of $(n+1)$-pointed curves;
(2) ψ is the pointed normalization of $\mathcal{C}$ along σ, i.e. ψ is finite and restricts to an isomorphism on the open set $\widetilde{\mathcal{C}} - \tilde{\sigma}$; and
(3) ϕ is the stabilization of $(\widetilde{\mathcal{C}}, \tilde{\sigma})$, i.e. ϕ is the morphism associated to a high multiple of the relatively nef line bundle $\omega_{\widetilde{\mathcal{C}}/X}(\tilde{\sigma})$.

The family $(\mathcal{C}^s \to X, \sigma_i^s)$ induces a map $X \to \mathcal{U}_{g-2,n+1}(A_4)$, and we must show that this lifts to define a map $X \to [V(\mathcal{E})/\mathbb{G}_m]$. To see this, let $\mathcal{F}$ be the coherent sheaf defined by the following exact sequence

$$0 \to \pi_*\mathcal{O}_{\mathcal{C}} \cap \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}(-4\tilde{\sigma}) \subset \pi_*\mathcal{O}_{\mathcal{C}} \cap \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}(-2\tilde{\sigma}) \to \mathcal{F} \to 0.$$

The condition that $\mathcal{C}$ has a ramphoid cusp along $\psi \circ \tilde{\sigma}$ implies that $\mathcal{F} \subset \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathcal{C}}}(-2\sigma)/\mathcal{O}_{\widetilde{\mathcal{C}}}(-4\sigma)$ is a rank one sub-bundle. In particular, $\mathcal{F}$ induces a sub-bundle of $\pi^s_*\mathcal{O}_{\mathcal{C}^s}(-2\sigma^s)/\mathcal{O}_{\mathcal{C}^s}(-4\sigma^s)$ over the locus of fibers on which ϕ is an isomorphism. A local computation, similar to the one performed in the definition of γ, shows that $\mathcal{F}$ extends to a subsheaf of $\pi^s_*\mathcal{O}_{\mathcal{C}^s}(-2\sigma^s)/\mathcal{O}_{\mathcal{C}^s}(-4\sigma^s)$ over all of X (though not a sub-bundle; the morphism on fibers is zero precisely where ϕ fails to be an isomorphism). The subsheaf $\mathcal{F} \subset \mathcal{E}$ induces the desired morphism $X \to [V(\mathcal{E})/\mathbb{G}_m]$. □

Proposition 4.10. *Let $\alpha_c \in \{9/11, 7/10, 2/3\}$ and suppose that $\overline{\mathcal{M}}_{g',n'}(\alpha_c)$ admits a proper good moduli space for all (g',n') with $g' < g$. Then $\overline{\mathcal{S}}_{g,n}(\alpha_c)$ admits a proper good moduli space.*

Proof. Let $\alpha_c = 9/11$. By Lemma 4.3, we may assume $(g,n) \neq (1,1)$. By Proposition 4.9(1), there is a locally free sheaf $\mathcal{E}$ on $\overline{\mathcal{M}}_{g-1,n+1}(9/11)$ such that $[V(\mathcal{E})/\mathbb{G}_m]$ is the base of the universal family of cuspidal sproutings of curves in $\overline{\mathcal{M}}_{g-1,n+1}(9/11)$. By Lemma 4.2, the fibers of this family are 9/11-stable so there is an induced map

$$\Psi\colon\ [V(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{g,n}(9/11).$$

By Lemma 4.5, Ψ maps surjectively onto $\overline{\mathcal{S}}_{g,n}(9/11)$. Furthermore, Ψ is finite by Proposition 4.7. By hypothesis, $\overline{\mathcal{M}}_{g-1,n+1}(9/11)$ and therefore $[V(\mathcal{E})/\mathbb{G}_m]$ admits a proper good moduli space. Thus, $\overline{\mathcal{S}}_{g,n}(9/11)$ admits a proper good moduli space by Proposition 1.4.

Let $\alpha_c = 7/10$. By Lemma 4.3, we may assume $(g,n) \neq (1,2)$. If $g \geq 2$, Proposition 4.9(2) provides a locally free sheaf $\mathcal{E}$ on $\overline{\mathcal{M}}_{g-2,n+2}(7/10)$ such that $[V(\mathcal{E})/\mathbb{G}_m]$ is the base of the universal family of tacnodal sproutings of curves in $\overline{\mathcal{M}}_{g-2,n+2}(7/10)$, and there is an induced map $[V(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{g,n}(7/10)$. Similarly, for every pair of integers (i,m) such that $\overline{\mathcal{M}}_{g-i-1,n-m+1}(7/10) \times \overline{\mathcal{M}}_{i,m+1}(7/10)$ is defined, by Proposition 4.9(3), there is a locally free sheaf $\mathcal{E}$ on $\overline{\mathcal{M}}_{g-i-1,n-m+1}(7/10) \times \overline{\mathcal{M}}_{i,m+1}(7/10)$ such that $[V(\mathcal{E})/\mathbb{G}_m]$ is the universal family of tacnodal sproutings. By Lemma 4.2, there are induced maps $[V(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{g,n}(7/10)$. Finally, Proposition 4.9(4) provides a locally free sheaf on $\overline{\mathcal{M}}_{g-1,n}(7/10)$ such that $[V(\mathcal{E})/\mathbb{G}_m]$ is the base of the universal family of one-sided tacnodal sproutings of curves in $\overline{\mathcal{M}}_{g-1,n}(7/10)$. By Lemma 4.2, there is an induced map $[V(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{g,n}(7/10)$. The union of the maps $[V(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{g,n}(7/10)$ cover $\overline{\mathcal{S}}_{g,n}(7/10)$ by Lemma 4.5. Furthermore, each map is finite by Proposition 4.7. By hypothesis, each of the stacky projective bundles $[V(\mathcal{E})/\mathbb{G}_m]$ admits a proper good moduli space, and therefore so does $\overline{\mathcal{S}}_{g,n}(7/10)$ by Proposition 1.4.

Let $\alpha_c = 2/3$. By Lemma 4.3, we may assume $(g,n) \neq (2,1)$. By Proposition 4.9(5), there is a locally free sheaf $\mathcal{E}$ on $\overline{\mathcal{M}}_{g-2,n+1}(2/3)$ such that $[V(\mathcal{E})/\mathbb{G}_m]$ is the base of the universal family of ramphoid cuspidal sproutings of curves in $\overline{\mathcal{M}}_{g-2,n+1}(2/3)$. By Lemma 4.2, there is an induced map $\Psi\colon\ [V(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{g,n}(2/3)$ which maps surjectively onto $\overline{\mathcal{S}}_{g,n}(2/3)$ by Lemma 4.5. Furthermore, Ψ is finite by Proposition 4.7. Thus, $\overline{\mathcal{S}}_{g,n}(2/3)$ admits a proper good moduli space by Proposition 1.4. □

4.3. Existence for $\overline{\mathcal{H}}_{g,n}(\alpha_c)$.

In this section, we use induction on g to prove that $\overline{\mathcal{H}}_{g,n}(\alpha_c)$ admits a good moduli space. The base case is handled by the following easy lemma.

Lemma 4.11. *We have:*

$$\overline{\mathcal{H}}_{1,1}(9/11) = [\mathbb{A}^2/\mathbb{G}_m],\ \textit{with weights } 4,6;$$
$$\overline{\mathcal{H}}_{1,2}(7/10) = [\mathbb{A}^3/\mathbb{G}_m],\ \textit{with weights } 2,3,4; \textit{ and}$$
$$\overline{\mathcal{H}}_{2,1}(2/3) = [\mathbb{A}^4/\mathbb{G}_m],\ \textit{with weights } 4,6,8,10.$$

In particular, $\overline{\mathcal{H}}_{1,1}(9/11)$, $\overline{\mathcal{H}}_{1,2}(7/10)$, $\overline{\mathcal{H}}_{2,1}(2/3)$ each admit a good moduli space.

Proof. We describe the case of $\overline{\mathcal{H}}_{2,1}(2/3)$, as the other two are essentially identical. Consider the family of Weierstrass tails over $\mathbb{A}^4$ given by:

$$y^2 = x^5z + a_3x^3z^3 + a_2x^2z^4 + a_1xz^5 + a_0z^6,$$

where the Weierstrass section is given by $[1,0,0]$. Since $\mathbb{G}_m$ acts on the base and total space of this family by

$$x \to \lambda^2 x,\ y \to \lambda^5 y,\ a_i \to \lambda^{10-2i} a_i,$$

the family descends to $[\mathbb{A}^4/\mathbb{G}_m]$. One checks that the induced map $[\mathbb{A}^4/\mathbb{G}_m] \to \overline{\mathcal{H}}_{2,1}(2/3)$ is an isomorphism. □

Lemma 4.11 gives an explicit description of the stack of elliptic tails, elliptic bridges, and Weierstrass tails. In the case $\alpha_c = 7/10$ (resp., $\alpha_c = 2/3$), we will also need an explicit description of the stack of elliptic chains (resp., Weierstrass chains) of length r.

Lemma 4.12. *Let $r \geq 1$ be an integer, and let*

$$\mathcal{EC}_r \subset \overline{\mathcal{M}}_{2r-1,2}(7/10) \qquad (\textit{resp., } \mathcal{WC}_r \subset \overline{\mathcal{M}}_{2r,1}(2/3)\,)$$

denote the closure of the locally closed substack of elliptic chains (resp., Weierstrass chains) of length r. Then $\mathcal{EC}_r$ (resp., $\mathcal{WC}_r$) admits a good moduli space.

Proof. For elliptic chains, Lemma 4.11 handles the case $r = 1$ as $\mathcal{EC}_1 = \overline{\mathcal{H}}_{1,2}(7/10)$. By induction on r, we may assume that $\mathcal{EC}_{r-1}$ admits a good moduli space. By Proposition 4.9(3), there is a locally free sheaf $\mathcal{E}$ on $\mathcal{EC}_{r-1} \times \overline{\mathcal{H}}_{1,2}(7/10)$ such that $[V(\mathcal{E})/\mathbb{G}_m]$ is the base of the universal family of tacnodal sproutings over $\mathcal{EC}_{r-1} \times \overline{\mathcal{H}}_{1,2}(7/10)$. By Lemma 4.2, there is an induced morphism $\Psi\colon [\mathcal{V}(\mathcal{E})/\mathbb{G}_m] \to \overline{\mathcal{M}}_{2r-1,2}(7/10)$. The image of Ψ is $\mathcal{EC}_r$, and Ψ is finite by Proposition 4.7. Since $\mathcal{EC}_{r-1} \times \overline{\mathcal{H}}_{1,2}(7/10)$ admits a good moduli space, Proposition 1.4 implies that $\mathcal{EC}_r$ admits a good moduli space.

For Weierstrass chains, Lemma 4.11 again handles the case $r = 1$ as $\mathcal{WC}_1 = \overline{\mathcal{H}}_{2,1}(2/3)$. By induction, we may assume that $\mathcal{WC}_{r-1}$ admits a good moduli space. By Proposition 4.9(3), there is a locally free sheaf $\mathcal{E}$ on $\overline{\mathcal{H}}_{1,2}(7/10) \times \mathcal{WC}_{r-1}$ such that $[V(\mathcal{E})/\mathbb{G}_m]$ is the base of the universal family of tacnodal sproutings over $\overline{\mathcal{H}}_{1,2}(7/10) \times \mathcal{WC}_{r-1}$. Indeed, we may take $\mathcal{E}$ to be

$$\pi_*\left(\mathcal{O}_{\mathcal{C}}(-\sigma)/\mathcal{O}_{\mathcal{C}}(-2\sigma) \oplus \mathcal{O}_{\mathcal{C}}(-\tau)/\mathcal{O}_{\mathcal{C}}(-2\tau)\right),$$

where $\pi\colon \mathcal{C} \to \overline{\mathcal{H}}_{1,2}(7/10) \times \mathcal{WC}_{r-1}$ is the universal family, σ corresponds to one of the universal sections over $\overline{\mathcal{H}}_{1,2}(7/10)$, and τ corresponds to the universal section over $\mathcal{WC}_{r-1}$. If $\mathcal{V} \subset [V(\mathcal{E})/\mathbb{G}_m]$ is the open locus parameterizing sproutings which do not introduce an elliptic bridge, then $\mathcal{V}$ is the complement of the sub-bundle $[V(\pi_*\mathcal{O}_{\mathcal{C}}(-\tau))/\mathbb{G}_m] \subset [V(\mathcal{E})/\mathbb{G}_m]$. Since $\overline{\mathcal{H}}_{1,2}(7/10) \times \mathcal{WC}_{r-1}$ admits a good moduli space, and $V(\mathcal{E}) \setminus V(\pi_*\mathcal{O}_{\mathcal{C}}(-\tau))$ is affine over $\overline{\mathcal{H}}_{1,2}(7/10) \times \mathcal{WC}_{r-1}$, we have that $\mathcal{V}$ admits a good moduli space. By Lemma 4.2, there is an induced morphism $\Psi\colon \mathcal{V} \to \overline{\mathcal{M}}_{2r,1}(2/3)$. The image of Ψ is $\mathcal{WC}_r$ and Ψ is finite by Proposition 4.7. Thus Proposition 1.4 implies that $\mathcal{WC}_r$ admits a good moduli space. □

For higher (g,n), we can use gluing maps to decompose $\overline{\mathcal{H}}_{g,n}(\alpha_c)$ into products of lower-dimensional moduli spaces.

Lemma 4.13. *Let $\alpha_c \in \{9/11, 7/10, 2/3\}$. There exist finite gluing morphisms*

$$\Psi\colon \overline{\mathcal{M}}_{g_1,n_1+1}(\alpha_c) \times \overline{\mathcal{M}}_{g_2,n_2+1}(\alpha_c) \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}(\alpha_c)$$

obtained by identifying $(C, \{p_i\}_{i=1}^{n_1+1})$ and $(C', \{p'_i\}_{i=1}^{n_2+1})$ nodally at $p_{n_1+1} \sim p'_{n_2+1}$

Proof. Ψ is well-defined by Lemma 4.1. To see that Ψ is finite, first observe that Ψ is clearly representable and quasi-finite. Furthermore, since the limit of a disconnecting node is a disconnecting node in $\overline{\mathcal{M}}_{g,n}(\alpha_c)$ [AFSv14, Corollary 2.11], Ψ satisfies the valuative criterion for properness. □

In the case $\alpha_c = 7/10$, we will need two additional gluing morphisms.

Lemma 4.14. *There exist finite gluing morphisms*

$$\overline{\mathcal{M}}_{g,n+2}(7/10) \times \mathcal{EC}_r \to \overline{\mathcal{M}}_{g+2r,n}(7/10), \qquad \mathcal{EC}_r \to \overline{\mathcal{M}}_{2r}(7/10),$$

where the first map is obtained by nodally gluing $(C, \{p_i\}_{i=1}^{n+2})$ and an elliptic chain (Z, q_1, q_2) at $p_{n+1} \sim q_1$ and $p_{n+2} \sim q_2$, and the second map is obtained by nodally self-gluing an elliptic chain (Z, q_1, q_2) at $q_1 \sim q_2$.

Proof. These gluing maps are well-defined by Lemma 4.1, and finiteness follows as in Lemma 4.13. □

Proposition 4.15. *Let $\alpha_c \in \{9/11, 7/10, 2/3\}$ and suppose that $\overline{\mathcal{M}}_{g',n'}(\alpha_c)$ admits a proper good moduli space for all (g', n') satisfying $g' < g$. Then $\overline{\mathcal{H}}_{g,n}(\alpha_c)$ admits a proper good moduli space.*

Proof. Let $\alpha_c = 9/11$. By Lemma 4.11, we may assume $(g, n) \neq (1, 1)$. By Lemma 4.13, there exists a finite gluing morphism

$$\Psi\colon \overline{\mathcal{M}}_{g-1,n+1}(9/11) \times \overline{\mathcal{H}}_{1,1}(9/11) \to \overline{\mathcal{M}}_{g,n}(9/11),$$

whose image is precisely $\overline{\mathcal{H}}_{g,n}(9/11)$. Now $\overline{\mathcal{H}}_{g,n}(9/11)$ admits a proper good moduli space by Proposition 1.4.

Let $\alpha_c = 7/10$. For every r such that $\overline{\mathcal{M}}_{g-2r,n+2}(7/10)$ (resp., $\overline{\mathcal{M}}_{g-2r-1,n}(7/10)$) exists, Lemma 4.14 (resp., Lemma 4.13) gives a finite gluing morphism

$$\overline{\mathcal{M}}_{g-2r,n+2}(7/10) \times \mathcal{EC}_r \to \overline{\mathcal{H}}_{g,n}(7/10)$$
$$\big(\text{resp., } \overline{\mathcal{M}}_{g-2r-1,n}(7/10) \times \mathcal{EC}_r \to \overline{\mathcal{H}}_{g,n}(7/10)\big),$$

that identifies $(C, \{p_i\}_{i=1}^{n+2})$ (resp., $(C, \{p_i\}_{i=1}^{n})$) to (Z, q_1, q_2) at $p_{n+1} \sim q_1$, $p_{n+2} \sim q_2$ (resp., $p_n \sim q_1$). In addition, for every triple of integers (i, m, r) such that the stack $\overline{\mathcal{M}}_{i,m+1}(7/10) \times \overline{\mathcal{M}}_{g-i-2r+1,n-m+1}(7/10)$ is defined, Lemma 4.13 gives a finite gluing morphism

$$\overline{\mathcal{M}}_{i,m+1}(7/10) \times \overline{\mathcal{M}}_{g-i-2r+1,n-m+1}(7/10) \times \mathcal{EC}_r \to \overline{\mathcal{H}}_{g,n}(7/10),$$

which identifies $(C, \{p_i\}_{i=1}^{m+1})$, $(C', \{p_i'\}_{i=1}^{n-m+1})$, (Z, q_1, q_2) nodally at $p_{m+1} \sim q_1$, $p'_{n-m+1} \sim q_2$. Finally, if $(g, n) = (2r, 0)$, Lemma 4.14 gives a finite gluing morphism

$$\mathcal{EC}_r \to \overline{\mathcal{H}}_{2r}(7/10),$$

which nodally self-glues (Z, q_1, q_2) at $q_1 \sim q_2$. The union of these gluing morphisms covers $\overline{\mathcal{H}}_{g,n}(7/10)$. Thus, $\overline{\mathcal{H}}_{g,n}(7/10)$ admits a proper good moduli space by Proposition 1.4 and Lemma 4.12.

Let $\alpha_c = 2/3$. By Lemma 4.11, we may assume $(g, n) \neq (2, 1)$. For each $r = 1, \dots, \lfloor \frac{g}{2} \rfloor$, Lemma 4.13 provides a finite gluing morphism

$$\overline{\mathcal{M}}_{g-2r,n+1}(2/3) \times \mathcal{WC}_r(2/3) \to \overline{\mathcal{M}}_{g,n}(2/3)$$

(if $r = g/2$ and $n = 1$, we consider $\overline{\mathcal{M}}_{g-2r,n+1}(2/3)$ as the emptyset). The union of these gluing morphisms cover $\overline{\mathcal{H}}_{g,n}(2/3)$. Now $\overline{\mathcal{H}}_{g,n}(2/3)$ admits a proper good moduli space by Proposition 1.4 and Lemma 4.12. □

4.4. **Existence for $\overline{\mathcal{M}}_{g,n}(\alpha)$.**

Proof of Theorem 1.1. Fix $\alpha_c \in \{9/11, 7/10, 2/3\}$. Note that $\overline{\mathcal{M}}_{0,n}(\alpha_c) = \overline{\mathcal{M}}_{0,n}$, so $\overline{\mathcal{M}}_{0,n}(\alpha_c)$ admits a proper good moduli space for all n. By induction on g, we may assume that $\overline{\mathcal{M}}_{g',n'}(\alpha_c)$ admits a proper good moduli space for all (g', n') with $g' < g$. Note that $\overline{\mathcal{M}}_{g,n}(\alpha) = \overline{\mathcal{M}}_{g,n}$ for $\alpha > 9/11$. By descending induction on α, we may now assume that $\overline{\mathcal{M}}_{g,n}(\alpha)$ admits a good moduli space for all $\alpha \geq \alpha_c + \epsilon$. By [AFSv14, Theorem 3.17], the inclusions $\overline{\mathcal{M}}_{g,n}(\alpha + \epsilon) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\alpha) \hookleftarrow \overline{\mathcal{M}}_{g,n}(\alpha - \epsilon)$ arise from local VGIT with respect to $\delta - \psi$, and Propositions 4.15 and 4.10 imply that $\overline{\mathcal{H}}_{g,n}(\alpha_c) = \overline{\mathcal{M}}_{g,n}(\alpha_c) \setminus \overline{\mathcal{M}}_{g,n}(\alpha_c - \epsilon)$ and $\overline{\mathcal{S}}_{g,n}(\alpha_c) = \overline{\mathcal{M}}_{g,n}(\alpha_c) \setminus \overline{\mathcal{M}}_{g,n}(\alpha_c + \epsilon)$ admit proper good moduli spaces. Now Theorem 1.3 implies that $\overline{\mathcal{M}}_{g,n}(\alpha_c)$ and $\overline{\mathcal{M}}_{g,n}(\alpha_c - \epsilon)$ admit proper good moduli spaces fitting into the stated diagram. □

Appendix A.

In this appendix, we give examples of algebraic stacks including moduli stacks of curves which fail to have a good moduli space owing to a failure of conditions (1a), (1b), and (2) in Theorem 1.2. Note that there is an obviously necessary topological condition for a stack to admit a good moduli space, namely that every k-point has a unique isotrivial specialization to a closed point, and each of our examples satisfies this condition. The purpose of these examples is to illustrate the more subtle kinds of stacky behavior that can obstruct the existence of good moduli spaces.

We work over an algebraically closed field of characteristic 0.

A.1. **Failure of condition (1a) in Theorem 1.2.**

Example A.1. Let $\mathcal{X} = [X/\mathbb{Z}_2]$ be the quotient stack where X is the non-separated affine line and $\mathbb{Z}_2$ acts on X by swapping the origins and fixing all other points. The algebraic stack clearly satisfies condition (1b) and (2). Then there is an étale, affine morphism $\mathbb{A}^1 \to \mathcal{X}$ which is stabilizer preserving at the origin but is not stabilizer preserving in an open neighborhood. The algebraic stack $\mathcal{X}$ does not admit a good moduli space.

While the above example may appear entirely pathological, we now provide two natural moduli stacks similar to this example.

Example A.2. Consider the Deligne-Mumford locus $\mathcal{X} \subset [\operatorname{Sym}^4 \mathbb{P}^1 / \operatorname{PGL}_2]$ of unordered tuples (p_1, p_2, p_3, p_4) where at least three points are distinct. Consider the family $(0, 1, \lambda, \infty)$ with $\lambda \in \mathbb{P}^1$. When $\lambda \notin \{0, 1, \infty\}$, $\operatorname{Aut}(0, 1, \lambda, \infty) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; indeed, if $\sigma \in \operatorname{PGL}_2$ is the unique element such that $\sigma(0) = \infty$, $\sigma(\infty) = 0$ and $\sigma(1) = \lambda$, then $\sigma([x, y]) = [y, \lambda x]$ so that $\sigma(\lambda) = 1$ and therefore $\sigma \in \operatorname{Aut}(0, 1, \lambda, \infty)$. Similarly, there is an element which acts via $0 \leftrightarrow 1$, $\lambda \leftrightarrow \infty$ and an element which acts via $0 \leftrightarrow \lambda$, $1 \leftrightarrow \infty$. However, if $\lambda \in \{0, 1, \infty\}$, then $\operatorname{Aut}(0, 1, \lambda, \infty) \cong \mathbb{Z}/2\mathbb{Z}$.

Therefore, any étale neighborhood $f\colon [\operatorname{Spec} A/\mathbb{Z}_2] \to \mathcal{X}$ of $x = (0, 1, \infty, \infty)$ will be stabilizer preserving at x but not in any open neighborhood. This failure of condition (1a) here is due to the fact that automorphisms of the generic fiber do not extend to the special fiber. The algebraic stack $\mathcal{X}$ does not admit a good moduli space but we note that if one enlarges the stack $\mathcal{X}$ to $[(\operatorname{Sym}^4 \mathbb{P}^1)^{ss} / \operatorname{PGL}_2]$ by including the point $(0, 0, \infty, \infty)$, there does exist a good moduli space.

Example A.3. Let $\mathcal{V}_2$ be the stack of all reduced, connected curves of genus 2, and let $[C] \in \mathcal{V}_2$ denote a cuspidal curve whose pointed normalization is a generic 1-pointed smooth elliptic curve (E, p). We will show that any Deligne-Mumford open neighborhood $\mathcal{M} \subset \mathcal{V}_2$ of $[C]$ is non-separated and fails to satisfy condition (1a).

Note that $\mathrm{Aut}(C) = \mathrm{Aut}(E, p) = \mathbb{Z}/2\mathbb{Z}$. Thus, to show that no étale neighborhood

$$[\mathrm{Def}(C)/\,\mathrm{Aut}(C)] \to \mathcal{M}$$

can be stabilizer preserving where $\mathrm{Def}(C) = \operatorname{Spec} A$ is an $\mathrm{Aut}(C)$-equivariant algebraized miniversal deformation space, it is sufficient to exhibit a family $\mathcal{C} \to \Delta$ whose special fiber is C, and whose generic fiber has automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. To do this, let C' be the curve obtained by nodally gluing two identical copies of (E, p) along their respective marked points. Then C' admits an involution swapping the two components, and a corresponding degree 2 map $C' \to E$ ramified over the single point p. We may smooth C' to a family $\mathcal{C}' \to \Delta$ of smooth double covers of E, simply by separating the ramification points. By [Smy11a, Lemma 2.12], there exists a birational contraction $\mathcal{C}' \to \mathcal{C}$ contracting one of the two copies of E in the central fiber to a cusp. The family $\mathcal{C} \to \Delta$ now has the desired properties; the generic fiber has both a hyperelliptic and bielliptic involution while the central fiber is C.

A.2. **Failure of condition (1b) in Theorem 1.2.**

Example A.4. Let $\mathcal{X} = [\mathbb{A}^2 \setminus 0/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts via $t \cdot (x, y) = (x, ty)$. Let $\mathcal{U} = \{y \neq 0\} = [\operatorname{Spec} k[x, y]_y/\mathbb{G}_m] \subset \mathcal{X}$. Observe that the point $(0, 1)$ is closed in $\mathcal{U}$ and $\mathcal{X}$. Then the open immersion $f\colon \mathcal{U} \to \mathcal{X}$ has the property that $f(0, 1) \in \mathcal{X}$ is closed while for $x \neq 0$, $(x, 1) \in \mathcal{U}$ is closed but $f(x, 1) \in \mathcal{X}$ is not closed. In other words, $f\colon \mathcal{U} \to \mathcal{X}$ does not send closed points to closed points and, in fact, there is no étale neighborhood $\mathcal{W} \to \mathcal{X}$ of $(0, 1)$ which sends closed points to closed points. The algebraic stack $\mathcal{X}$ does not admit a good moduli space.

Example A.5. Let $\mathcal{M} = \overline{\mathcal{M}}_g \cup \mathcal{M}^1 \cup \mathcal{M}^2$, where $\mathcal{M}^1$ consists of all curves of arithmetic genus g with a single cusp and smooth normalization, and $\mathcal{M}^2$ consist of all curves of the form $D \cup E_0$, where D is a smooth curve of genus $g - 1$ and E_0 is a rational cuspidal curve attached to C nodally.

We observe that $\mathcal{M}$ has the following property: If $C = D \cup E$, where D is a curve of genus $g-1$ and E is an elliptic tail, then $[C] \in \mathcal{M}$ is a closed point if and only if D is singular. Indeed, if D is smooth, then C admits an isotrivial specialization to $D \cup E_0$, where E_0 is a rational cuspidal tail. Now consider any curve of the form $C = D \cup E$ where D is a singular curve of genus $g - 1$ and E is a smooth elliptic tail, and, for simplicity, assume that D has no automorphisms. We claim that there is no étale neighborhood of the form $[\mathrm{Def}(C)/\,\mathrm{Aut}(C)] \to \mathcal{M}$, which sends closed points to closed points. Indeed, curves of the form $D' \cup E$ where D' is smooth will appear in any such neighborhood and will obviously be closed in $[\mathrm{Def}(C)/\,\mathrm{Aut}(C)]$ (since this is a Deligne-Mumford stack), but are not closed in $\mathcal{M}$.

A.3. **Failure of condition (2) in Theorem 1.2.**

Example A.6. Let $\mathcal{X} = [X/\mathbb{G}_m]$ where X is the nodal cubic curve with the $\mathbb{G}_m$-action given by multiplication. Observe that $\mathcal{X}$ is an algebraic stack with two points – one open and one closed. But $\mathcal{X}$ does not admit a good moduli space; if it did, $\mathcal{X}$ would necessarily be cohomologically affine and consequently X would be affine, a contradiction. However, there is an étale and affine (but not finite) morphism $\mathcal{W} = [\operatorname{Spec}(k[x,y]/xy)/\mathbb{G}_m] \to \mathcal{X}$ where $\mathbb{G}_m = \operatorname{Spec} k[t,t^{-1}]$ acts on $\operatorname{Spec} k[x,y]/xy$ via $t \cdot (x,y) = (tx, t^{-1}y)$ which is stabilizer preserving and sends closed points to closed points; however, the two projections $\mathcal{W} \times_{\mathcal{X}} \mathcal{W} \rightrightarrows \mathcal{W}$ do not send closed points to closed points.

To realize this étale local presentation concretely, we may express $X = Y/\mathbb{Z}_2$ where Y is the union of two $\mathbb{P}^1$'s with coordinates $[x_1,y_1]$ and $[x_2,y_2]$ glued via nodes at $p = 0_1 = 0_2$ and $q = \infty_1 = \infty_2$ by the action of $\mathbb{Z}/2\mathbb{Z}$ where -1 acts via $[x_1,y_1] \leftrightarrow [y_2,x_2]$. There is a $\mathbb{G}_m$-action on Y given by $t \cdot [x_1,y_1] = [tx_1,y_1]$ and $t \cdot [x_2,y_2] = [x_1,ty_1]$ which descends to the $\mathbb{G}_m$-action on X. There is a finite étale morphism $[Y/\mathbb{G}_m] \to \mathcal{X}$, but $[Y/\mathbb{G}_m]$ is not cohomologically affine. If we instead, consider the open substack $\mathcal{W} = [(Y \setminus \{p\})/\mathbb{G}_m]$, then $\mathcal{W} \cong [\operatorname{Spec}(k[x,y]/xy)/\mathbb{G}_m]$ is cohomologically affine and there is an étale representable morphism $f\colon \mathcal{W} \to \mathcal{X}$. It is easy to see that

$$\mathcal{W} \times_{\mathcal{X}} \mathcal{W} \cong [(Y \setminus \{p\})/\mathbb{G}_m] \coprod [(Y \setminus \{p,q\})/\mathbb{G}_m]$$

But $[(Y \setminus \{p,q\})/\mathbb{G}_m] \cong \operatorname{Spec} k \coprod \operatorname{Spec} k$ and the projections $p_1, p_2\colon \mathcal{W} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{W}$ correspond to the inclusion of the two open points into $\mathcal{W}$ which clearly don't send closed points to closed points.

Example A.7. Consider the algebraic stack $\mathcal{M}_g^{ss,1}$ of Deligne-Mumford semistable curves C where any rational subcurve connected to C at only two points is smooth. Let D_0 be the Deligne-Mumford semistable curve $D' \cup \mathbb{P}^1$, obtained by gluing a $\mathbb{P}^1$ to a smooth genus $g-1$ curve D' at two points p,q. For simplicity, let us assume that $\operatorname{Aut}(D', \{p,q\}) = 0$, so $\operatorname{Aut}(D_0) = \mathbb{G}_m$. There is a unique isomorphism class of curves which isotrivially specializes to D_0, namely the nodal curve D_1 obtained by gluing D at p and q. Thus, $\overline{\{[D_1]\}}$ has two points—one open and one closed. In fact, $\overline{\{[D_1]\}}$ is isomorphic to the quotient stack $[X/\mathbb{G}_m]$ considered in Example A.6.

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