Lecture 18: Projectivity

> **Thm.** *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*

Today's Outline

0) Recap of how we got here
1) Setup for $\overline{\mathcal{M}}_g$
2) Survey of projectivity methods
3) Nef vector bundles
4) The Ampleness Lemma
5) Application to $\overline{\mathcal{M}}_g$

References

* • Kollár, Projectivity of complete moduli
• Viehweg, Quasi-projective moduli for polarized manifolds

§0. Six steps toward projective moduli

Define

$\overline{\mathcal{M}}_g$ = stack of stable curves of genus g

$\cap$

$\mathcal{M}_g^{all}$ = stack of all curves of genus g

STEP 1 (Algebraicity)

$\mathcal{M}_g^{all}$ is algebraic locally of f.type

We used a Hilbert scheme to construct a smooth nbhd $Hilb \to \mathcal{M}_g^{all}$ around any curve

STEP 2 (Openness of stability)

$\overline{\mathcal{M}}_g \subset \mathcal{M}_g^{all}$ open substack ($\Rightarrow \overline{\mathcal{M}}_g$ algebraic)

Translates to: if $\mathcal{C} \to S$ family of arbitrary curves, $\{s \in S \mid \mathcal{C}_s \text{ stable}\} \subset S$ open

Broke this down

① Nodal locus is open (used local structure of nodes)

② Stable locus within nodal locus is open (C stable $\Leftrightarrow$ Aut(C) finite $\Leftrightarrow \omega_C$ ample

STEP 3 (Boundedness of stability)

$\overline{\mathcal{M}}_g$ is of f.type

We used: if $\mathcal{C} \to S$ stable family, $\omega_{\mathcal{C}/S}^{\otimes 3}$ rel. very ample & $\underbrace{\mathrm{Hilb}^P(\mathbb{P}^{5g-6})}_{\text{f. type}}$

STEP 4 (Existence of coarse moduli space)

$\exists\, \overline{\mathcal{M}}_g \to \overline{M}_g$ coarse mod. space (sep alg. space)

DM

We showed $\overline{\mathcal{M}}_g$ sep. DM stack & apply Keel-Mori thm

STEP 5 (Stable reduction)

$\overline{\mathcal{M}}_g$ is proper ($\Rightarrow \overline{M}_g$ proper)

We verified the val. crit for properness in char = 0 using birat'l geometry & MMP of surfaces

STEP 6 (Projectivity)

$\overline{M}_g$ is projective

TODAY!

§1. Setup

- Let $\mathcal{M}_g \xrightarrow{\pi} \overline{\mathcal{M}}_g$ be univ. family
- Define coherent sheaf

coherent sheaf

$$E_k := \pi_*\left(\omega_{\mathcal{M}_g/\overline{\mathcal{M}}_g}^{\otimes k}\right) \text{ on } \overline{\mathcal{M}}_g$$

- For $S \xrightarrow{f} \overline{\mathcal{M}}_g \longleftrightarrow$
$$\begin{array}{c} C \\ \downarrow \pi_S \\ S \end{array}$$

$$f^* E_k = \pi_{S,*}\left(\omega_{C/S}^{\otimes k}\right)$$

- E_k is a vector bdl by Coh & Base Change

Why consider $\pi_*\left(\omega_{\mathcal{M}_g/\overline{\mathcal{M}}_g}^{\otimes k}\right)$?

① Get line bdls

$$\lambda_k := \det \pi_*\left(\omega_{\mathcal{M}_g/\overline{\mathcal{M}}_g}^{\otimes k}\right) \text{ on } \overline{\mathcal{M}}_g$$

For $S \to \overline{\mathcal{M}}_g \longleftrightarrow C \xrightarrow{\pi} S$

$$\lambda_k|_S = \det \pi_*\left(\omega_{C/S}^{\otimes k}\right)$$

② Get multiplication maps

For $C \to S$,

$$\mathrm{Sym}^d \pi_*(\omega_{C/S}) \to \pi_*\left(\omega_{C/S}^{\otimes d}\right)$$

For $C \to \operatorname{Spec} k$

$$\mathrm{Sym}^d H^0(\omega_C) \to H^0\left(\omega_C^{\otimes d}\right)$$

$$\text{Kernel} = \left\{ \text{degree } d \text{ equations defining } C \xrightarrow{|\omega_C|} \mathbb{P}^{g-1} \right\}$$

More generally, given k & d

$$\mathrm{Sym}^d \pi_*\left(\omega_{C/S}^{\otimes k}\right) \to \pi_*\left(\omega_{C/S}^{\otimes dk}\right)$$

$$\text{Kernel} = \left\{ \deg d \text{ eqns cutting } C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^N \right\}$$

$k \geq 3$

recover C from mult. map

§2. Survey of projectivity methods

① Geometric Invariant Theory (GIT)

- Construction depends on 2 integers
  - $k \geq 5$
  - $d \gg 0$
- A stable curve C is pluri-canonically embedded

$$C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1} \qquad rk = h^0(\omega_C^{\otimes k}) = (2k-1)(g-1)$$

Hilbert poly $P(t)$

- $H' := \left\{ [C \hookrightarrow \mathbb{P}^{r(k)-1}] \;\middle|\; \begin{array}{l} \cdot\ C \text{ stable} \\ \cdot\ C \text{ embedded via } \omega_C^{\otimes k} \end{array} \right\}$

  $\subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1})$ locally closed

- $H := \overline{H'}$ projective

- For $d \gg 0$, $PGL_{r(k)-1} \circlearrowleft$ ↙ $P(d)$ dim'l quotients

$$(*)\ H \xhookrightarrow{\ L_d\ } Gr(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)))$$

$$[C \hookrightarrow \mathbb{P}^{r(k)-1}] \longmapsto [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))]$$

$$\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) = \mathrm{Sym}^d H^0(\omega_C^{\otimes k}) \xrightarrow{\text{mult map}} \Gamma(C, \mathcal{O}(d)) = H^0(\omega_C^{\otimes dk})$$

Setup for $\overline{M}_g$

- vector bdl $\pi_*(\omega_{\overline{C}_g/\overline{M}_g}^{\otimes k})$ of rank $r(k) = (2k-1)(g-1)$ $(k>1)$
- line bundle $\lambda_k := \det \pi_*(\omega_{\overline{C}_g/\overline{M}_g}^{\otimes k})$
- For $C \xrightarrow{\pi} S$, have mult map

$$\mathrm{Sym}^d(\pi_* \omega_{C/S}^{\otimes k}) \longrightarrow \pi_*(\omega_{C/S}^{\otimes dk})$$

- Easy: $\overline{M}_g = [H'/PGL_{r(k)-1}]$
- Hard (GIT) Given $h = [C \subset \mathbb{P}^{r(k)-1}] \in H$

$\underbrace{C \text{ stable}}_{\text{as a curve}} \iff h \in H$ GIT stable $\underbrace{\text{with respect } L_d}_{\exists s \in \Gamma(H, L_d^n)^{PGL} \text{ s.t. } s(h) \neq 0}$

Hilbert-Mumford $\Longrightarrow$ $\exists$ sections

(Equivalently, L_d semiample on $\overline{M}_g \subset [H/PGL_{r(k)}]$)

Conclusion: The cms of $\overline{M}_g$ is

$$\overline{M}_g = \mathrm{Proj} \underbrace{\bigoplus_{n \geq 0} \underbrace{\Gamma(H, L_d^n)^{PGL_{r(k)}}}_{\Gamma([H/PGL_{r(k)}], L_d^n)}}_{\text{grading is fin. gen!}}$$

$\Longrightarrow \overline{M}_g$ projective

Setup for $\overline{\mathcal{M}}_g$

- vector bdl $\pi_*(\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ of rank $r(k) = (2k-1)(g-1)$ $(k>1)$
- line bundle $\lambda_k := \det \pi_*(\omega_{\mathcal{C}_g/\overline{\mathcal{M}}_g}^{\otimes k})$
- For $\mathcal{C} \xrightarrow{\pi} S$, have mult map

$$\mathrm{Sym}^d(\pi_* \omega_{\mathcal{C}/S}^{\otimes k}) \to \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk})$$

For $k \geq 5$ and $d \gg 0$, the class of the ample line bdl is

$$r(k)\lambda_{dk} - r(dk)\lambda_k \text{ on } \overline{\mathcal{M}}_g$$

As $d \to \infty$, assymptotic limit is

$$\underset{\text{porportional}}{\sim} (12 - \tfrac{4}{k})\lambda_1 - \underbrace{\delta}_{\text{boundary divisor}}$$

Can get even more

<u>Cornalba-Harris:</u>

$$a\lambda_1 - \delta \text{ ample} \iff a > 11$$

(GIT w/ $k=5$ 11.2)

② Projectivity via Griffith's period maps

Main idea:

$$\underset{\text{smooth curve}}{C} \longmapsto \underset{\text{pol. Jacobian}}{\mathrm{Jac}(C)} = \overbrace{H^0(C, \omega_C)}^{\dim_{\mathbb{C}} = g} / \overbrace{H_1(C, \mathbb{Z})}^{\dim_{\mathbb{Z}} = 2g}$$

(Alternatively, can consider

$$C \longmapsto \underset{\text{pol. Hodge str.}}{[H^1(C, \omega_C) \subset \underset{\mathbb{C}^{2g}}{H^1(C, \mathbb{C})}]}$$

Gives

$$\mathcal{M}_g \longrightarrow \mathfrak{h}_g / Sp_g$$

Strategy: Show projectivity of $\overline{\mathfrak{h}_g / Sp_g}$ and infer projectivity of $\overline{\mathcal{M}}_g$

The sheaves

$$\pi_* \omega_{\mathcal{C}/S} \qquad R^1 \pi_* \mathbb{C}$$

play a role

③ Projectivity via positivity

We have a coarse moduli space

$$\begin{tikzcd} \overline{\mathcal{M}}_g \arrow[r, "\pi"] \arrow[dr, dashed, "\lambda_k"] & \overline{M}_g \arrow[d, dashed, "\bar\lambda_k"] \\ & {} \end{tikzcd}$$

$\overline{M}_g$ ← proper alg space

FACT For n suff. divisible, each λ_k descends to $\overline{M}_g$, i.e. $\lambda_k = \pi^*\bar\lambda_k$

GOAL: Show $\bar\lambda_k$ ample

Some approaches

- Suppose

(a) semiample (i.e. $\lambda_k^{\otimes n}$ basept free for $n \gg 0$) ← Tough to check

(b) For every $T \to \overline{\mathcal{M}}_g$ non-trivial, $\deg \lambda_k|_T > 0$

T ← proper curve

Then λ_k is ample

Reason

$$\begin{tikzcd} \overline{\mathcal{M}}_g \arrow[d, "\text{cms}"] \arrow[dr] & \\ \overline{M}_g \arrow[r, "\exists!"] & X \subset \mathbb{P}(H^0(\lambda_k^{\otimes n})) \end{tikzcd}$$

(a) ⟹ well-defined

(b) ⟹ map doesn't contract curves

$\overline{M}_g$ ← proper

q.finite & proper ⟹ finite ⟹ $\overline{M}_g$ proj

- Basepoint-free thms can imply semiampleness

Various statements

Example:

big
nef
$\bigoplus_{n \geq 0} \Gamma(\lambda_k^{\otimes n})$ fin gen

⟹ semiample

BCHM ⟹ fin gen

- **Nakai–Moishezon criterion**

X proper alg space & L line bdl

L ample ⟺ $\forall Z \subset X$ irred. closed $L^{\dim Z} \cdot Z > 0$

- **Kleiman's criterion**

closure of cone of curves

L ample ⟺ $\forall C \in \overline{NE}(X)$ $C \cdot L > 0$

- **Seshadri's criterion**

L ample ⟺ $\exists \varepsilon > 0$ s.t. $\forall$ curves C
$C \cdot L > \varepsilon\, m(C)$

§3. Nefness

Def A vector bundle E on a scheme X is **nef** (or semipositive) if

$\forall\, T \xrightarrow{f} X$ and $\forall\, f^*E \twoheadrightarrow L$, $\deg L \geq 0$

proper curve — line bdl

$\left(\Longleftrightarrow \mathcal{O}_{\mathbb{P}E}(1) \text{ nef on } \mathbb{P}E \right)$

Properties

① Quotients & extensions of nef vector bdls are nef

② Nefness is open in flat families

③ E nef $\Longrightarrow$ $\mathrm{Sym}^k E$ nef

If $\underbrace{\mathrm{Sym}^d(\pi_*\omega_{C/S})}_{\text{nef}} \twoheadrightarrow \underbrace{\pi_*(\omega_{C/S}^{\otimes d})}_{\text{nef}}$ surjective

nef $\Longrightarrow$ nef

$\boxed{\text{Thm 1}}$ Suppose we know that

- $\overline{M}_g$ proper Deligne-Mumford stack ✓
- $\exists\, k_0 > 0$ s.t. $\forall\, C \to T$ stable families (sm. proj curve)

$\pi_*(\omega_{C/T}^{\otimes k})$ nef for $k \geq k_0$

Then $\lambda_k = \det \pi_*(\omega_{C/T}^{\otimes k})$ ample for $k \gg 0$.

Remarks

- Generalizes to any moduli of polarized varieties.
  - [Kovacs–Patakfalvi '17] Moduli of stable varieties in any dim is projective.
  - [Codogni–Patakfalvi '20] & [Xu–Zhang '20] Moduli of K-polystable Fano ~~varieties~~ is projective
- Nefness of $\pi_*(\omega_{C/T})$ is easier & is classical
- Harder to show nefness for $\pi_*(\omega_{C/T}^{\otimes k})$ despite that they are more positive

§4. The ampleness lemma

Setup

- X proper alg. space
- W vector bdl of rank w with reductive structure group $G \to GL_w$
- $W \twoheadrightarrow Q$ quotient bdl of rk q

There is a classifying map

$$X \longrightarrow [Gr(q,w)/G]$$

$$x \longmapsto [W_x \twoheadrightarrow Q_x], \quad W_x \cong k^w, \quad Q_x \cong k^q$$

well-defined up to G

$\boxed{\text{Ampleness Lemma}}$ (char $=0$ version) If in addition

(a) W nef

(b) $X \longrightarrow [Gr(q,w)/G]$ quasi-finite

$\Longrightarrow$ det Q ample

Remarks

- Easy case: W trivial $\Rightarrow G = \{1\}$

$$\Longrightarrow X \xrightarrow[\text{proper}]{\text{proper, q.finite}} Gr(q,w) \Rightarrow \text{finite} \Rightarrow X \text{ proj}$$

- We are not assuming that the image of X lands in G-stable locus

But if it did,

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "\text{q.finite}"'] & {[Gr(q,w)^{ss}/G]} \arrow[d, "\text{proj, GIT quot}"] \\
& Gr(q,w)//G
\end{tikzcd}$$

q.finite $\Rightarrow$ finite $\Rightarrow X$ proj

Get $(\det Q)^{w} \otimes (\det W)^{-q}$ ample

Main idea of pf of Ampleness lemma is to use nefness to verify

Nakai–Moishezon criterion: $\forall\, Z \subset X$

$$(\det Q)^{\dim Z} \cdot Z > 0$$

§5. Application to $\overline{\mathcal{M}}_g$

$\boxed{\text{Ampleness Lemma}}$ (char 0 version) Let X proper alg space. Let $W \twoheadrightarrow Q$ surj. of vect. bdls of rk w & q s.t. structure gp G of W is reductive. If
(a) W nef
(b) $X \to [Gr(q,w)/G]$ quasi-finite

$\Longrightarrow$ $\det Q$ ample

$\boxed{\text{THM 1}}$ If we know that

- $\overline{\mathcal{M}}_g$ proper Deligne-Mumford stack
- $\exists k_0 > 0$ s.t. $\forall\, \mathcal{C} \to T$ stable families ($\uparrow$ sm. proj curve) $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ nef for $k \geq k_0$

Then $\lambda_k = \det \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ ample for $k \gg 0$.

Sketch of pf

Consider universal curve
$$\begin{array}{c} \mathcal{C} = \mathcal{U}_g \\ \downarrow \pi \\ S = \overline{\mathcal{M}}_g \end{array}$$

Choose k & d s.t.

- $\omega_{\mathcal{C}/S}^{\otimes k}$ rel very ample $\nearrow$ $\mathcal{O}_C(1) = \omega_C^{\otimes k}$
- $R^1\pi_*\omega_{\mathcal{C}/S}^{\otimes k} = 0$
- Every curve $C \xhookrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^w$ is cut out by deg $\leq d$ eqns.
- $\pi_*(\omega_{\mathcal{C}/S}^{\otimes k})$ nef
- $\underbrace{\operatorname{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k})}_{W \text{ rk } w} \twoheadrightarrow \underbrace{\pi_*(\omega_{\mathcal{C}/S}^{\otimes dk})}_{Q \text{ rk } q}$

To apply Ampleness Lemma $\to$ structure gp $G = PGL_{r(k)}$ (same G as in GIT)

Claim: $\overline{\mathcal{M}}_g \to [Gr(q,w)/G]$ injective

$$[C] \longmapsto \left[\underbrace{\operatorname{Sym}^d H^0(\omega_C^{\otimes k})}_{H^0(\mathbb{P}^n, \mathcal{O}(d))} \to \underbrace{H^0(\omega_C^{\otimes dk})}_{H^0(C, \mathcal{O}(d))}\right]$$

Know kernel determines C ✓

Apply Ampleness Lemma to
$$\underset{\nearrow\text{scheme}}{U} \xrightarrow{\text{fin}} \overline{\mathcal{M}}_g$$

Logical order of Kollár's argument

$\boxed{\text{Ampleness Lemma}}$ Let X proper alg space

Let $W \twoheadrightarrow Q$ surj. of vect. bdls of rk w & q s.t. structure gp of W is G. Suppose

(a) W nef

(b) $X \to [Gr(q,w)/G]$ quasi-finite

$\Longrightarrow$ det Q ample

Pf uses Nakai-Moishezon

$\boxed{\text{THM 1}}$ If we know that

- $\overline{M}_g$ proper Deligne-Mumford stack
- $\exists k_0 > 0$ s.t. $\forall\ \mathcal{C} \to T$ stable families

sm. proj curve

$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ nef for $k \geq k_0$

Then $\lambda_k = \det \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ ample for $k \gg 0$.

Pf used Ampleness Lemma

It remains to show:

smooth proj curve

$\boxed{\text{THM 2}}$ Let $\mathcal{C} \xrightarrow{\pi} T$ be a stable family of curves of genus $g \geq 2$ over a field k.

Then $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$

Thms 1 & 2 $\Longrightarrow$ $\overline{M}_g$ projective

(can also get $\overline{M}_{g,n}$ proj.)

§5. Application to $\overline{M}_g$

We need to show:

$\boxed{\text{THM 2}}$ Let $\mathcal{C} \xrightarrow{\pi} T$ be a stable family of curves of genus $g \geq 2$ over a field k. ← smooth proj curve

Then $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$

First reductions: Can assume

- $\mathcal{C}$ smooth & minimal surface
- (*) • $\mathcal{C} \to T$ gen. smooth
- genus of $T \geq 2$ ($\Rightarrow \mathcal{C}$ gen type)

Reduction to char p: Suppose char$(k)=0$

Since everything is of f.type, "spreading out"

$\exists\ A \subset k$ fin gen $\mathbb{Z}$-subalg (by adjoining all constants needed to define $\mathcal{C} \to T$)

& diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\mathcal{C}} \arrow[d, "\text{flat}"] \\
T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{T} \arrow[d, "\text{flat}"] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

After enlarging A, can arrange flatness & all fibers satisfy (*)

- The closed pts of Spec A are pos. char.
- Nefness is open in flat families

$\Longrightarrow$ nefness in char $=0$

$\boxed{\text{THM 2}}$ Let $\mathcal{C} \xrightarrow{\pi} T$ be a stable family of curves of genus $g \geq 2$ over a field k. Then $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$

smooth proj curve

We can assume

- $\mathcal{C}$ smooth & minimal surface
- $\mathcal{C} \to T$ gen. smooth
- genus of $T \geq 2$
- $\mathrm{char}(k) = p > 0$ $\quad p \neq 2$

Birational input (Ekedahl): In char $= p > 0$

- S smooth proj minimal surface of gen. type
- D effective divisor with $D^2 = 0$

Then $H^1(S, \omega_S^{\otimes n}(D)) = 0$ for $n \geq 2$

Ekedahl: $H^1(S, \omega_S^{\otimes(-n)}) = 0$ $\quad n \geq 1$

(Bombieri in char $= 0$)

Serre-Dual $\Rightarrow$ $H^1(S, \omega_S^{\otimes n}) = 0$ $\quad n \geq 2$

$0 \to \omega_S^{\otimes n} \to \omega_S^{\otimes n}(D) \to \omega_S^{\otimes n}|_D \to 0$

Proof If not nef,

$\exists\ \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$ with $d := \deg M > 0$

Consider Frobenius

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C} \\ \downarrow & & \downarrow \\ T & \xrightarrow{F} & T \end{array}$$

- $F^* \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) = \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$
- $\deg F^* M = p \cdot d$

$\rightsquigarrow$ Can arrange $d \gg 0$!

$\Rightarrow$ Can arrange $M = \omega_T^{\otimes k} \otimes L$ (L very ample)

$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee = (\omega_T^{\otimes k} \otimes L)^\vee$

$\Rightarrow$ $\underbrace{\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L}_{h^1 \geq 2} \twoheadrightarrow \mathcal{O}_T$ $\quad h^1 \geq 2$

Use Leray spectral sequence to relate

$H^1(\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L)$

$\underbrace{H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L)}_{\dim \geq 2}$

Contradicts Ekedahl!

Thank you!